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Local Fixed Point Theorems for Graphic Contractions in Generalized Metric Spaces

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Abstract

In this paper, we will present some local fixed point theorems for graphic contractions on a generalized metric space in the sense of Perov.

Keywords: vector-valued metric, fixed point, graphic contraction, local fixed point theorem.
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1. Introduction

The classical Banach contraction principle was extended for single-valued contraction on spaces endowed with vector-valued metrics by Perov (Perov, 1964). Other fixed point results, given in the framework of a set endowed with a complete vector-valued metric, are given in (Agarwal, 1983), (Filip & Petruşel, 2009), (O'Regan *et al.*, 2007), (Petruşel *et al.*, 2015), (Precup, 2009), ...

On the other hand, the concept of graphic contraction is more general that that of contraction mapping, since the contraction condition is assumed to be satisfied only for pairs $(x, y) \in Graph(f) := \{(x, f(x)) : x \in X\}$. In this case, existence of the fixed point can be established under some additional continuity assumption on f. In this sense, several fixed point results for graphic contractions were proved in (Rus, 1972) (see also (Rus *et al.*, 2008), page 29), (Subrahmanyam, 1974) and (Hicks & Rhoades, 1979).

An existence and uniqueness result for graphic contractions in complete metric spaces was recently proved in (Chaoha & Sudprakhon, 2017).

For a synthesis and new results concerning fixed point theory for graphic contractions in complete metric spaces see (Petruşel & Rus, 2018).

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The purpose of this paper is to give some local fixed point theorems for graphic contractions in the context of complete vector-valued metric spaces. Our results extend, to the case of vector-valued metric spaces, a local variant of Banach's contraction principle, which was proved for the first time (to our best knowledge) by M.A. Krasnoselskii. Our results also extend some local fixed point theorems for graphic contractions in complete metric spaces given in (Petruşel, 2019).

2. Main results

We first some preliminary notions and results.

We denote by $M_{mm}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, By O_m the null $m \times m$ matrix and by I_m the identity $m \times m$ matrix. If $x, y \in \mathbb{R}^m$, $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$, then, by definition:

$$x \leq y \text{ if and only if } x_i \leq y_i \text{ for } i \in \{1, 2, ..., m\}.$$

Notice that, through this paper, we will make an identification between row and column vectors in $\mathbb{R}^m$.

Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

(a) $d(x, y) \geq O$ for all $x, y \in X$; if $d(x, y) = O$, then $x = y$; (where $O := \underbrace{(0, 0, \cdots, 0)}_{m-times}$)

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$.

A nonempty set X endowed with a vector-valued metric d is called a generalized metric space in the sense of Perov (or a vector-valued metric space) and it will be denoted by (X, d). In this context, if $x_0 \in X$ and $r \in \mathbb{R}^m$ with $r_i > 0$ for every $i \in \{1, 2, \cdots, m\}$, then we denote

$$B(x_0, r) := \{x \in X : d(x_0, x) < r\}, \ \tilde{B}(x_0, r) := \{x \in X : d(x_0, x) \leq r\}.$$

The notions of convergent sequence, Cauchy sequence, completeness, open, closed, bounded and compact subset are similar to those for usual metric spaces. Notice also that in Precup (Precup, 2009) are pointed out the advantages of working with vector-valued metrics with respect to the usual scalar ones.

Definition 2.1. ((Varga, 2000)) A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(A)$ is strictly less than 1. In other words, this means that all the eigenvalues of A are in the open unit disc, i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$, where I denotes the unit matrix of $\mathcal{M}_{m,m}(\mathbb{R})$.

A classical result in matrix analysis is the following theorem (see (Varga, 2000)).

Theorem 2.1. *Let $A \in M_{mm}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) A is convergent to zero;

(ii) $A^n \to O_m$ as $n \to \infty$;

(iii) The matrix $(I_m - A)$ is nonsingular and

$$(I_m - A)^{-1} = I_m + A + ... + A^n + ... \tag{2.1}$$

(iv) The matrix $(I_m - A)$ is nonsingular and $(I_m - A)^{-1}$ has nonnegative elements.

We recall now some contraction conditions in vector-valued metric spaces.

Definition 2.2. Let (X, d) be a generalized metric space in the sense of Perov and $f : X \to X$ be an operator. Then, f is called:

(i) an A-contraction if $A \in M_{mm}(\mathbb{R}_+)$ converges to zero and

$$d(f(x), f(y)) \leq Ad(x, y), \text{ for every } x, y \in X.$$

(ii) a graphic A-contraction if $A \in M_{mm}(\mathbb{R}_+)$ converges to zero and

$$d(f(x), f^2(x)) \leq Ad(x, f(x)), \text{ for every } x \in X.$$

Notice that any A-contraction is a graphic A-contraction, but not reversely.

The following local fixed point theorem in generalized metric space in the sense of Perov is an extension of a result proved by R. Agarwal in (Agarwal, 1983).

Theorem 2.2. *Let (X, d) be a complete generalized metric in the sense of Perov. Let $x_0 \in X$, $r = (r_1, \cdots, r_m) \in \mathbb{R}^m$ with $r_i > 0$ for every $i \in \{1, 2, \cdots, m\}$ and let $f : \tilde{B}(x_0, r) \to X$ be an operator which has closed graph with respect to d. We suppose:*

(i) f is a graphic A-contraction on $\tilde{B}(x_0, r)$;

(ii) $(I_m - A)^{-1} d(x_0, f(x_0)) \leq r$.

Then:

(a) $Fix(f) \neq \emptyset$;

(b) $f^n(x_0) \in \tilde{B}(x_0, R)$ for each $n \in \mathbb{N}$ (where $R := (I_m - A)^{-1} d(x_0, f(x_0))$) and the sequence of successive approximations $(f^n(x_0))_{n\in\mathbb{N}}$ converges to a fixed point of f;

(c) if $x^* := \lim_{n\to\infty} f^n(x_0)$, then the following apriori estimation holds

$$d(f^n(x_0), x^*) \leq A^n (I_m - A)^{-1} d(x_0, f(x_0)), \textit{ for each } n \in \mathbb{N}.$$

Proof. We can prove, by mathematical induction, that

$$d(x_0, f^n(x_0)) \leq (I_m + A + \cdots + A^{n-1}) d(x_0, f(x_0)), \text{ for each } n \in \mathbb{N}, n \geq 2. \tag{2.2}$$

Indeed, we have

$$d(x_0, f^2(x_0)) \leq d(x_0, f(x_0)) + d(f(x_0), f^2(x_0)) \leq$$
$$d(x_0, f(x_0)) + Ad(x_0, f(x_0)) = (I + A)d(x_0, f(x_0)).$$

Next, for the general case of (2.2), we have

$$d(x_0, f^n(x_0)) \leq d(x_0, f^{n-1}(x_0)) + d(f^{n-1}(x_0), f^n(x_0)) \leq$$
$$(I_m + A + \cdots + A^{n-2}) d(x_0, f(x_0)) + A^{n-1} d(x_0, f(x_0)) =$$
$$(I_m + A + \cdots + A^{n-1}) d(x_0, f(x_0)).$$

Thus, by (2.2), we obtain that

$$d(x_0, f^n(x_0)) \leq (I_m - A)^{-1} d(x_0, f(x_0)) := R, \text{ for each } n \in \mathbb{N}, n \geq 2. \tag{2.3}$$

Hence, $f^n(x_0) \in \tilde{B}(x_0, R)$ for each $n \in \mathbb{N}$. Then, by the graphic contraction condition, we obtain that $d(f^n(x_0), f^{n+1}(x_0)) \leq A^n d(x_0, f(x_0))$, for each $n \in \mathbb{N}$. Using this relation, we immediately obtain, for every $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, that

$$d(f^n(x_0), f^{n+p}(x_0)) \leq A^n(I_m + A + \cdots + A^{p-1})d(x_0, f(x_0)) \leq A^n(I_m - A)^{-1}d(x_0, f(x_0)). \quad (2.4)$$

The relation (2.4) shows that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is Cauchy and, by the completeness of the space, there exists $x^* \in \tilde{B}(x_0, R)$ such that $x^* := \lim_{n\to\infty} f^n(x_0)$. The conclusions follow now by the closed graph condition of the operator f. The apriori evaluation follows by (2.4) letting $p \to \infty$. □

Remark. In particular, if f is an A-contraction, we get Theorem 2.1 in (Agarwal, 1983).

A more general result can be proved using the framework of a complete metric space endowed with a partial order relation. Our next theorem result extends the main result given in (Ran & Reurings, 2004).

Theorem 2.3. *Let X be a nonempty set endowed with a partial order relation "$\preceq$" and let $d : X \times X \to \mathbb{R}^m_+$ be a complete generalized metric in the sense of Perov on X. Let $x_0 \in X$, $r = (r_1, \cdots, r_m) \in \mathbb{R}^m$ with $r_i > 0$ for every $i \in \{1, 2, \cdots, m\}$ and $f : \tilde{B}(x_0, r) \to X$ be an operator which has closed graph with respect to d and is increasing with respect to "$\preceq$". We suppose:*

(i) there exists $A \in M_{mm}(\mathbb{R}_+)$ convergent to zero such that

$$d(f(x), f^2(x)) \leq Ad(x, f(x)), \text{ for every } x \in X \text{ with } x \preceq x_0;$$

(ii) $f(x_0) \preceq x_0$;

(iii) $(I_m - A)^{-1}d(x_0, f(x_0)) \leq r$.

Then $Fix(f) \neq \emptyset$ and the sequence of successive approximations $(f^n(x_0))_{n\in\mathbb{N}}$ converges to a fixed point of f. Moreover, if $x^* := \lim_{n\to\infty} f^n(x_0)$, then the following apriori estimation holds

$$d(f^n(x_0), x^*) \leq A^n(I_m - A)^{-1}d(x_0, f(x_0)), \text{ for each } n \in \mathbb{N}.$$

Proof. By (ii) and the monotonicity assumption on f we get that

$$x_0 \succeq f(x_0) \succeq f^2(x_0) \succeq \cdots \succeq f^n(x_0) \succeq \cdots$$

Next, as before, we can prove that

$$d(x_0, f^n(x_0)) \leq (I_m + A + \cdots + A^{n-1})d(x_0, f(x_0)), \text{ for each } n \in \mathbb{N}, n \geq 2. \quad (2.5)$$

Thus, by (2.5), we obtain that

$$d(x_0, f^n(x_0)) \leq (I_m - A)^{-1}d(x_0, f(x_0)) := R, \text{ for each } n \in \mathbb{N}, n \geq 2. \quad (2.6)$$

Hence, $f^n(x_0) \in \tilde{B}(x_0, R)$ for each $n \in \mathbb{N}$. Then, by the graphic contraction condition, we obtain that $d(f^n(x_0), f^{n+1}(x_0)) \leq A^n d(x_0, f(x_0))$, for each $n \in \mathbb{N}$. Using this relation, we immediately obtain, for every $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, that

$$d(f^n(x_0), f^{n+p}(x_0)) \leq A^n(I_m + A + \cdots + A^{p-1})d(x_0, f(x_0)) \leq A^n(I_m - A)^{-1}d(x_0, f(x_0)). \quad (2.7)$$

The relation (2.7) shows that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is Cauchy and, thus, it converges to an element $x^* \in \tilde{B}(x_0, R)$. We notice that $x^* \in Fix(f)$, by the closed graph condition of the operator f. The apriori evaluation follows again letting $p \to \infty$ in (2.7). □

Remark. It is an open question to obtain the convergence (to a fixed point) of the sequence of successive approximations $(f^n(x))_{n\in\mathbb{N}}$ for each $x \in \tilde{B}(x_0; R)$. Another open question to extend the above results to the multi-valued case.

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Coefficient Inequalities for Some Subclasses of Analytic Functions Associated with Conic Domains Involving q-calculus

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Abstract

Main purpose of this paper is to define and study some new classes of analytic functions associated with conic type regions. By using Salagean q-differential operator we investigate several interesting properties of these newly defined classes. Comparison of new results with those that were obtained in earlier investigation are given as Corollaries.

Keywords: q-differential operator, Salagean q-differential operator, Janowski functions, k-uniformly convex functions, k-starlike functions, close-to-convex functions, conic domain.
2010 MSC: 30C45, 30C50.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f analytic in the open unit disc $E = \{z : z \in \mathbb{C}, |z| < 1\}$ and satisfying the normalization condition $f(0) = f'(0) - 1 = 0$. Thus, the functions in $\mathcal{A}$ are represented by the Taylor-Maclaurin series expansion given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E. \tag{1.1}$$

Let $\mathcal{S}$ be the subset of $\mathcal{A}$ consisting of the functions that are univalent in E. The convolution (Hadamard product) of functions $f, g \in \mathcal{A}$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in E,$$

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where $f(z)$ is given by (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \qquad z \in E.$$

For two functions $f, g \in \mathcal{A}$, we say that f is subordinate to g in E, denoted by

$$f(z) \prec g(z) \qquad (z \in E),$$

if there exists a function w where

$$w(0) = 0, \ |w(z)| < 1, \qquad (z \in E),$$

such that

$$f(z) = g(w(z)), \qquad (z \in E).$$

If g is univalent in E, then it follows that

$$f(z) \prec g(z) \qquad (z \in E), \ \Rightarrow \ f(0) = 0 \text{ and } f(E) \subset g(E).$$

For more detail see (Miller & Mocanu, 2000). A function p analytic in E and of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}[A, B] \Leftrightarrow p(z) \prec \frac{1 + Az}{1 + Bz}$$

where $-1 \leq B < A \leq 1$. This class was introduced and investigated by Janowski (Janowski, 1973). In particular, if $A = 1$ and $B = -1$, we obtain the class $\mathcal{P}$ of functions with a positive real part (see (Goodman, 1983)). The classes $\mathcal{P}$ and $\mathcal{P}[A, B]$ are connected by the relation

$$p(z) \in \mathcal{P} \Leftrightarrow \frac{(A+1)\,p(z) - (A-1)}{(B+1)\,p(z) - (B-1)} \in \mathcal{P}[A, B].$$

Now consider, for $k \geq 0$, the classes $k - CV$ and $k - ST$ of k-uniformly convex functions and corresponding k-starlike functions, respectively, introduced by Kanas and Wisniowska. For some details, see (Kanas, 2003), (Kanas & Wisniowska, 2000), (Kanas & Wisniowska, 1999).
Kanas and Wisniowska (Kanas & Wisniowska, 2000), (Kanas & Wisniowska, 1999) introduced the conic domain $\Omega_k, k \geq 0$ as

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}.$$

We note that Ω_k represents the conic region bounded successively by the imaginary axis ($k = 0$), the right branch of hyperbola ($0 < k < 1$), a parabola for $k = 1$, and ellipse for $k > 1$. The extremal functions for these conic regions are

$$p_k(z) = \begin{cases} \frac{1+z}{1-z} & k = 0, \\ 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 & k = 1, \\ 1 + \frac{2}{1-k^2} \sinh^2 \left\{ \frac{2}{\pi} (\arccos k) \arctan h \sqrt{z} \right\} & 0 < k < 1, \\ 1 + \frac{1}{k^2-1} \sin\left(\frac{\pi}{2R(t)} \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}} \right) + \frac{1^2}{k^2-1} & k > 1, \end{cases} \tag{1.2}$$

where

$$u(z) = \frac{z - \sqrt{t}}{1 - \sqrt{t}z}, \quad z \in E,$$

and $t \in (0, 1)$ is chosen such that $k = \cosh(\pi R'(t)/(4R(t)))$. Here $R(t)$ is Legender's complete elliptic integral of first kind and $R'(t) = R(\sqrt{1-t^2})$ and $R'(t)$ is the complementary integral of $R(t)$ for details see (Ahiezer, 1970), (Hussain *et al.*, 2017), (Kanas & Wisniowska, 2000), (Kanas & Wisniowska, 1999). If $p_k(z) = 1 + Q_1(k)z + Q_2(k)z^2 + ..., z \in E$. Then it was shown in (Kanas & Wisniowska, 2000) that for (1.2) one can have

$$Q_1 := Q_1(k) = \begin{cases} \frac{2A^2}{1-k^2} & 0 \le k < 1, \\ \frac{8}{\pi^2} & k = 1, \\ \frac{\pi^2}{4(k^2-1)\sqrt{t}(1+t)R^2(t)} & k > 1, \end{cases} \tag{1.3}$$

with $A = \frac{2}{\pi} \arccos t$.
The classes $k - UCV$ and $k - ST$ are defined as follows.
A function $f(z) \in \mathcal{A}$ is said to be in the class $k - UCV$, if and only if,

$$\frac{(zf'(z))'}{f'(z)} \prec p_k(z), \quad z \in E, \ k \ge 0.$$

A function $f(z) \in \mathcal{A}$ is said to be in the class $k - ST$, if and only if,

$$\frac{zf'(z)}{f(z)} \prec p_k(z), \quad z \in E, \ k \ge 0.$$

For more study (see (Srivastava *et al.*, 2012), (Srivastava *et al.*, 2009), (Srivastava *et al.*, 2007)). These classes were then generalized to $KD(k, \alpha)$ and $SD(k, \alpha)$ respectively by Shams et al. (Shams *et al.*, 2004) subject to the conic domain $G(k, \alpha), k \ge 0, 0 \le \alpha < 1$, which is

$$G(k, \alpha) = \{w : \Re(w) > k|w - 1| + \alpha\}.$$

Now using the concepts of Janowski functions and the conic domain, Noor and Malik (Noor & Malik, 2011) define the following

Definition 1.1. (See (Noor & Malik, 2011)) A function $p(z)$ is said to be in the class $k - \mathcal{P}[A, B]$, if and only if,

$$p(z) \prec \frac{(A+1)p_k(z) - (A-1)}{(B+1)p_k(z) - (B-1)}, \quad k \ge 0,$$

where $p_k(z)$ is defined in (1.2) and $-1 \le B < A \le 1$. Geometrically, the function $p \in k - \mathcal{P}[A, B]$ takes all values from the domain $\Omega_k[A, B]$, $1 \le B < A \le 1$, $k \ge 0$ which is defined as:

$$\Omega_k[A, B] = \left\{ w : \Re\left(\frac{(B-1)w - (A-1)}{(B+1)w - (A+1)}\right) > k\left|\frac{(B-1)w - (A-1)}{(B+1)w - (A+1)} - 1\right| \right\},$$

or equivalently $\Omega_k[A, B]$ is a set of numbers $w = u + iv$ such that

$$\begin{aligned}&\left[\left(B^2-1\right)\left(u^2+v^2\right)-2\left(AB-1\right)u+\left(A^2-1\right)\right]^2\\ >\ & k^2\left[\left(-2\left(B+1\right)\left(u^2+v^2\right)+2\left(A+B+2\right)u-2\left(A+1\right)\right)^2+4\left(A-B\right)^2v^2\right].\end{aligned}$$

This domain represents the conic type regains for detail see (Noor & Malik, 2011), (Noor *et al.*, 2017). It can be easily seen that $0-\mathcal{P}[A,B]=\mathcal{P}[A,B]$ introduced in (Janowski, 1973) and $k-\mathcal{P}[1,-1]=\mathcal{P}(p_k)$ introduced in (Kanas & Wisniowska, 1999).
For any non-negative integer n, the $q-$integer number n, $[n]_q$ is defined by:

$$[n]_q=\frac{1-q^n}{1-q}=1+q+...+q^{n-1},\qquad [0]_q=0.$$

The $q-$number shifted factorial is defined by $[0]!=1$ and $[n]_q!=[1]_q[2]_q...[n]_q$. Clearly, $\underset{q\to1^-}{Lim}[n]_q=n$ and $\underset{q\to1^-}{Lim}[n]_q!=n!$. In general we will denote $[t]_q=\frac{1-q^t}{1-q}$ also for a non-integer number.

Definition 1.2. Let $f\in\mathcal{A}$, and let the $q-$derivative operator or $q-$difference operator be defined by

$$\partial_q f(z)=\frac{f(qz)-f(z)}{(q-1)z}\qquad (z\in E).$$

It is easy to check that for $n\in\mathbb{N}:=\{1,2,...\}$ and $z\in E$

$$\partial_q z^n=[n]_q z^{n-1}.$$

In the field of Geometric Function Theory, various subclasses of the normalized analytic function class $\mathcal{A}$ have been studied from different viewpoints. The q-calculus as well as the fractional q-calculus provide important tools that have been used in order to investigate various subclasses of $\mathcal{A}$. Moreover, in recent years, such q-calculus operators as the fractional q-integral and fractional q-derivative operators were used to construct several subclasses of analytic functions (see, for example, (Altınkaya & Yalçın, 2017), (Magesh *et al.*, 2018), (Purohit & Raina, 2013), (Srivastava, 1989)).

Throughout this paper we assume q to be a fixed number between 0 and 1.

Definition 1.3. (See (Govindaraj & Sivasubramanian, 2018)) For $f\in\mathcal{A}$, let Salagean q-differential operator be defined as follows:

$$S_q^0 f(z)=f(z),\ S_q^1 f(z)=z\partial_q f(z),...,S_q^m f(z)=z\partial_q\left(S_q^{m-1}f(z)\right).\tag{1.4}$$

A simple calculation implies

$$S_q^m f(z)=f(z)*F_{m,q}(z),\ z\in E,\ m\in\mathbb{N}U\{0\}=\mathbb{N}_0.$$

where

$$F_{m,q}(z) = z + \sum_{n=2}^{\infty} [n]_q^m z^n. \tag{1.5}$$

Making use of (1.4) and (1.5), the power series of $S_q^m f(z)$ for f of the form (1.1) is given by

$$S_q^m f(z) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n.$$

Note that

$$\lim_{q\to 1-} F_{m,q}(z) = z + \sum_{n=2}^{\infty} n^m z^n$$

and

$$\lim_{q\to 1-} S_q^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n$$

which is the familiar Salagean derivative (Salagean, 1983).
Motivated by the recent work presented by Noor and Malik (Noor & Malik, 2011) and (Mahmood *et al.*, 2017), we define some classes of analytic functions associated with conic domains and by using Salagean q-differential operator.

Definition 1.4. A function $f(z) \in \mathcal{A}$ is said to be in the class $k - \mathcal{ST}_q(m, C, D)$, $k \geq 0$, $-1 \leq D < C \leq 1$, if and only if

$$\Re\left(\frac{(D-1)G_{m,q}(z) - (C-1)}{(D+1)G_{m,q}(z) - (C+1)}\right) > k\left|\frac{(D-1)G_{m,q}(z) - (C-1)}{(D+1)G_{m,q}(z) - (C+1)} - 1\right|,$$

where

$$G_{m,q}(z) = \frac{S_q^{m+1} f(z)}{S_q^m f(z)},$$

or equivalently

$$G_{m,q}(z) \in k - P[C, D].$$

Definition 1.5. A function $f(z) \in \mathcal{A}$ is said to be in the class $k - \mathcal{CV}_q(m, C, D)$, $k \geq 0$, $-1 \leq D < C \leq 1$, if and only if

$$\Re\left(\frac{(D-1)H_{m,q}(z) - (C-1)}{(D+1)H_{m,q}(z) - (C+1)}\right) > k\left|\frac{(D-1)H_{m,q}(z) - (C-1)}{(D+1)H_{m,q}(z) - (C+1)} - 1\right|,$$

where

$$H_{m,q}(z) = \frac{z\partial_q S_q^{m+1} f(z)}{S_q^{m+1} f(z)},$$

or equivalently,

$$H_{m,q}(z) \in k - P[C, D].$$

It can be easily seen that

$$f(z) \in k - \mathcal{CV}_q(m, C, D) \iff z\partial_q f(z) \in k - \mathcal{ST}_q(m, C, D). \tag{1.6}$$

Special cases:

(i) For $q \to 1^-$, and $m = 0$, then the classes $k - \mathcal{ST}_q(m, C, D)$ and $k - \mathcal{CV}_q(m, C, D)$ reduce into the classes $k - \mathcal{ST}(C, D)$ and $k - \mathcal{CV}(C, D)$ introduced by Noor and Malik in (Noor & Malik, 2011).

(ii) For $q \to 1^-$, $C = 1$, $D = -1$, and $m = 0$, then the classes $k - \mathcal{ST}_q(m, C, D)$ and $k - \mathcal{CV}_q(m, C, D)$ reduce into the classes $k - \mathcal{ST}$ and $k - \mathcal{UCV}$ introduced by Kanas and Wisniowska in (Kanas & Wisniowska, 2000), (Kanas & Wisniowska, 1999).

(iii) For $q \to 1^-$, $C = 1 - 2\alpha$, $D = -1$, and $m = 0$, then the classes $k - \mathcal{ST}_q(m, C, D)$ and $k - \mathcal{CV}_q(m, C, D)$ reduce into the classes $SD(k, \alpha)$ and $KD(k, \alpha)$ introduced by Shams et al. in (Shams *et al.*, 2004).

(iv) For $q \to 1^-$, $k = 0$, and $m = 0$, then the classes $k - \mathcal{ST}_q(m, C, D)$ and $k - \mathcal{CV}_q(m, C, D)$ reduce into the classes $\mathcal{S}^*(C, D)$ and $C(C, D)$ introduced by Janowski (Janowski, 1973).

Definition 1.6. A function $f(z) \in \mathcal{A}$ is said to be in the class $k - \mathcal{UK}_q(m, A, B, C, D)$, if and only if, for $k \geq 0$, $-1 \leq D < C \leq 1$, $-1 \leq B < A \leq 1$, there exists $g(z) \in k - \mathcal{ST}_q(m, C, D)$, such that

$$\Re\left(\frac{(B-1)L_{m,q}(z) - (A-1)}{(B+1)L_{m,q}(z) - (A+1)}\right) > k\left|\frac{(B-1)L_{m,q}(z) - (A-1)}{(B+1)L_{m,q}(z) - (A+1)} - 1\right|,$$

where

$$L_{m,q}(z) = \frac{S_q^{m+1} f(z)}{S_q^m g(z)},$$

or equivalently

$$L_{m,q}(z) \in k - \mathcal{P}[A, B].$$

Definition 1.7. A function $f(z) \in \mathcal{A}$ is said to be in the class $k - \mathcal{UQ}_q(m, A, B, C, D)$, if and only if, for $k \geq 0$, $-1 \leq D < C \leq 1$, $-1 \leq B < A \leq 1$, there exists $g(z) \in k - \mathcal{CV}_q(m, C, D)$, such that

$$\Re\left(\frac{(B-1)K_{m,q}(z) - (A-1)}{(B+1)K_{m,q}(z) - (A+1)}\right) > k\left|\frac{(B-1)K_{m,q}(z) - (A-1)}{(B+1)K_{m,q}(z) - (A+1)} - 1\right|,$$

where

$$K_{m,q}(z) = \frac{z\partial_q S_q^{m+1} f(z)}{S_q^{m+1} g(z)},$$

or equivalently,

$$K_{m,q}(z) \in k - \mathcal{P}[A, B].$$

It can be easily seen that

$$f(z) \in k - \mathcal{UQ}_q(m, A, B, C, D) \iff z\partial_q f(z) \in k - \mathcal{UK}_q(m, A, B, C, D). \tag{1.7}$$

Special cases:

(i) For $q \to 1^-$, and $m = 0$, then the classes $k - \mathcal{UK}_q(m, A, B, C, D)$ and $k - \mathcal{UQ}_q(m, A, B, C, D)$ reduce into the classes $k - \mathcal{UK}(A, B, C, D)$ and $k - \mathcal{UQ}(A, B, C, D)$ introduced by Mahmood at al. in (Mahmood *et al.*, 2017).

(ii) For $q \to 1^-$, $A = 1 - 2\beta$, $B = -1, C = 1 - 2\gamma, D = -1$ and $m = 0$, then the classes $k - \mathcal{UK}_q(m, A, B, C, D)$ and $k - \mathcal{UQ}_q(m, A, B, C, D)$ reduce into the classes $k - \mathcal{UK}(\beta, \gamma)$ and $k - \mathcal{UQ}(\beta, \gamma)$ introduced by AghalaryAghalary and Azadi in (Aghalary & Azadi, 2015).

(iii) For $q \to 1^-$, $A = 1 - 2\beta$, $B = -1, C = 1 - 2\gamma, D = -1$, $k = 0$ and $m = 0$, then the classes $k - \mathcal{UK}_q(m, A, B, C, D)$ and $k - \mathcal{UQ}_q(m, A, B, C, D)$ reduce into the classes $\mathcal{K}(\beta, \gamma)$ and $\mathcal{Q}(\beta, \gamma)$ introduced by Libera and Noor in (Libera, 1964), (Noor, 1987).

(iv) For $q \to 1^-$, $k = 0$, and $m = 0$, then the class $k - \mathcal{UK}_q(m, A, B, C, D)$ reduce into the class $\mathcal{K}(A, B, C, D)$ introduced by Silvia in (Silvia, 1983).

(v) For $q \to 1^-$, $k = 0$, $C = 1$, $D = -1$, and $m = 0$, then the class $k - \mathcal{UQ}_q(m, A, B, C, D)$ reduce into the class $\mathcal{Q}(A, B)$ introduced by Noor in (Noor, 1989).

(vi) For $q \to 1^-, A = 1, B = -1, C = 1, D = -1$, and $m = 0$, then the classes $k - \mathcal{UK}_q(m, A, B, C, D)$ and $k - \mathcal{UQ}_q(m, A, B, C, D)$ reduce into the classes $k - \mathcal{UK}$ and $k - \mathcal{UQ}$ introduced by Acu in (Acu, 2006).

(vii) For $q \to 1^-$, $k = 0$, $A = 1$, $B = -1$, $C = 1$, $D = -1$, and $m = 0$, then the classes $k - \mathcal{UK}_q(m, A, B, C, D)$ and $k - \mathcal{UQ}_q(m, A, B, C, D)$ reduced into the classes $\mathcal{K}$ and $\mathcal{Q}$ introduced by Kaplan and Noor et al. in (Kaplan, 1952), (Noor *et al.*, 2009).

Lemma 1.1. *(See (Rogosinski, 1943)) Let* $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *be subordinate to* $H(z) = 1 + \sum_{n=1}^{\infty} C_n z^n$. *If* $H(z)$ *is univalent in* E *and* $H(E)$ *is convex, then*

$$|c_n| \leq |C_1|, \quad n \geq 1.$$

Lemma 1.2. *(See (Noor & Malik, 2011)) Let* $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in k - \mathcal{P}[A, B]$, *then*

$$|c_n| \leq |Q_1(k, A, B)|, |Q_1(k, A, B)| = \frac{A - B}{2} |Q_1(k)|,$$

where $|Q_1(k)|$ *is given by (1.3).*

2. Main Results

Theorem 2.1. *A function* $f \in \mathcal{A}$ *and of the form (1.1) is in the class* $k - \mathcal{ST}_q(m, C, D)$, *if it satisfies the condition*

$$\sum_{n=2}^{\infty} \left\{ 2(k+1)(q[n-1]_q) + \left| [n]_q (D+1) - (C+1) \right| \right\} [n]_q^m |a_n| \leq C - D, \tag{2.1}$$

where $-1 \leq D < C \leq 1$, $k \geq 0$.

Proof. Assuming that (2.1) holds, then it suffices to show that

$$k\left|\frac{(D-1)G_{m,q}(z)-(C-1)}{(D+1)G_{m,q}(z)-(C+1)}-1\right|-\Re\left\{\frac{(D-1)G_{m,q}(z)-(C-1)}{(D+1)G_{m,q}(z)-(C+1)}-1\right\}<1.$$

We have

$$\begin{aligned}
& k\left|\frac{(D-1)G_{m,q}(z)-(C-1)}{(D+1)G_{m,q}(z)-(C+1)}-1\right|-\Re\left\{\frac{(D-1)G_{m,q}(z)-(C-1)}{(D+1)G_{m,q}(z)-(C+1)}-1\right\}\\
\leq\ & (k+1)\left|\frac{(D-1)G_{m,q}(z)-(C-1)}{(D+1)G_{m,q}(z)-(C+1)}-1\right|\\
=\ & (k+1)\left|\frac{(D-1)S_q^{m+1}f(z)-(C-1)S_q^m f(z)}{(D+1)S_q^{m+1}f(z)-(C+1)S_q^m f(z)}-1\right|\\
=\ & 2(k+1)\left|\frac{S_q^m f(z)-S_q^{m+1}f(z)}{(D+1)S_q^{m+1}f(z)-(C+1)S_q^m f(z)}\right|\\
=\ & 2(k+1)\left|\frac{\sum_{n=2}^{\infty}([n]_q-1)[n]_q^m a_n z^n}{(D-C)z+\sum_{n=2}^{\infty}\left\{(D+1)[n]_q-(C+1)\right\}[n]_q^m a_n z^n}\right|\\
\leq\ & 2(k+1)\left\{\frac{\sum_{n=2}^{\infty}q[n-1]_q[n]_q^m|a_n|}{C-D-\sum_{n=2}^{\infty}\left|(D+1)[n]_q-(C+1)\right|[n]_q^m|a_n|}\right\}.
\end{aligned}$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}\left\{2(k+1)q[n-1]_q+\left|[n]_q(D+1)-(C+1)\right|\right\}[n]_q^m|a_n|\leq C-D.$$

This completes the proof. □

When $q\to 1^-$, $m=0$, we have the following known result, proved by Noor and Malik in (Noor & Malik, 2011).

Corollary 2.1. *A function $f\in\mathcal{A}$ and of the form (1.1) is in the class $k-\mathcal{ST}(C,D)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{2(k+1)(n-1)+|n(D+1)-(C+1)|\}|a_n|\leq|D-C|.$$

When $q\to 1^-$, $m=0$, $C=1-2\alpha$, $D=-1$ with $0\leq\alpha<1$, then we have the following known result, proved by Shams et al. in (Shams *et al.*, 2004).

Corollary 2.2. *A function $f\in\mathcal{A}$ and of the form (1.1) is in the class $SD(k,\alpha)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{n(k+1)-(k+\alpha)\}|a_n|\leq 1-\alpha,$$

where $0\leq\alpha<1$ and $k\geq 0$.

When $q \to 1^-$, $k = 0$, $m = 0$, $C = 1 - 2\alpha$, $D = -1$ with $0 \leq \alpha < 1$, then we have the following known result, proved by Silverman in (Silverman, 1975).

Corollary 2.3. *A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $S^*(\alpha)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{n - \alpha\} |a_n| \leq 1 - \alpha, \qquad 0 \leq \alpha < 1.$$

Theorem 2.2. *A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $k - \mathcal{CV}_q(m, C, D)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} \left\{2(k+1)(q\,[n-1]_q) + \left|[n]_q\,(D+1) - (C+1)\right|\right\} [n]_q^{m+1}\,|a_n| \leq C - D,$$

where $-1 \leq D < C \leq 1$, $k \geq 0$.

The proof follows immediately by using Theorem 2.1 and (1.6).

When $q \to 1^-$, $m = 0$, then, we have the following known result, proved by Noor and Malik in (Noor & Malik, 2011).

Corollary 2.4. *A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $k - \mathcal{UCV}(C, D)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} n\,\{2(k+1)(n-1) + |n(D+1) - (C+1)|\}\,|a_n| \leq C - D.$$

Theorem 2.3. *If $f(z) \in k - \mathcal{ST}_q(m, C, D)$ and is of the form (1.1). Then*

$$|a_n| \leq \prod_{j=0}^{n-2} \left(\frac{\left|Q_1(k)(C-D) - 2q[j]_q\,[j+1]_q^m\,D\right|}{2q\,[j+1]_q\,[j+2]_q^m} \right), \qquad n \geq 2, \tag{2.2}$$

where $|Q_1(k)|$ is defined by (1.3).

Proof. By definition, for $f(z) \in k - \mathcal{ST}_q(m, C, D)$, we have

$$\frac{S_q^{m+1} f(z)}{S_q^m f(z)} = p(z), \tag{2.3}$$

where

$$p(z) \in k - P[C, D].$$

Now from (2.3), we have

$$S_q^{m+1} f(z) = S_q^m f(z) p(z),$$

which implies that

$$\begin{aligned}
z+\sum_{n=2}^{\infty}[n]_q^{m+1}a_nz^n &= \left(1+\sum_{n=1}^{\infty}c_nz^n\right)\left(z+\sum_{n=2}^{\infty}[n]_q^{m}a_nz^n\right)\\
z+\sum_{n=2}^{\infty}[n]_q^{m+1}a_nz^n &= \left(1+\sum_{n=1}^{\infty}c_nz^n\right)\left(\sum_{n=1}^{\infty}[n]_q^{m}a_nz^n\right)\\
z+\sum_{n=2}^{\infty}[n]_q^{m+1}a_nz^n &= \sum_{n=1}^{\infty}[n]_q^{m}a_nz^n+\left(\sum_{n=1}^{\infty}[n]_q^{m}a_nz^n\right)\left(\sum_{n=1}^{\infty}c_nz^n\right)\\
\sum_{n=2}^{\infty}\left\{[n]_q-1\right\}[n]_q^{m}a_nz^n &= \left(\sum_{n=1}^{\infty}[n]_q^{m}a_nz^n\right)\left(\sum_{n=1}^{\infty}c_nz^n\right)\\
\sum_{n=2}^{\infty}q\,[n-1]_q\,[n]_q^{m}a_nz^n &= \left(\sum_{n=1}^{\infty}[n]_q^{m}a_nz^n\right)\left(\sum_{n=1}^{\infty}c_nz^n\right).
\end{aligned} \tag{2.4}$$

By using Cauchy product formula on R.H.S of (2.4), we have

$$\sum_{n=2}^{\infty}q\,[n-1]_q\,[n]_q^{m}a_nz^n=\sum_{n=1}^{\infty}\left[\sum_{j=1}^{n-1}[j]_q^{m}a_jc_{n-j}\right]z^n. \tag{2.5}$$

Equating coefficients of z^n on both sides of (2.5), we have

$$q\,[n-1]_q\,[n]_q^{m}a_n=\sum_{j=1}^{n-1}[j]_q^{m}a_jc_{n-j},\quad [1]_q^m=1,\ a_1=1.$$

This implies that

$$|a_n|\le\frac{1}{q\,[n-1]_q\,[n]_q^{m}}\sum_{j=1}^{n-1}[j]_q^{m}\left|a_j\right|\left|c_{n-j}\right|,\ [1]_q^m=1,\ a_1=1.$$

Using lemma (1.2), we have

$$|a_n|\le\frac{|Q_1(k)|\,(C-D)}{2q\,[n-1]_q\,[n]_q^{m}}\sum_{j=1}^{n-1}[j]_q^{m}\left|a_j\right|,\qquad [1]_q^m=1,a_1=1. \tag{2.6}$$

Now we prove that

$$\begin{aligned}
&\frac{|Q_1(k)|\,(C-D)}{2q\,[n-1]_q\,[n]_q^{m}}\sum_{j=1}^{n-1}[j]_q^{m}\left|a_j\right|\\
&\le\ \prod_{j=1}^{n-1}\left(\frac{\left|Q_1(k)(C-D)-2q[j-1]_q\,[j]_q^{m}D\right|}{2q\,[j]_q\,[j+1]_q^{m}}\right),\\
&\frac{|Q_1(k)|\,(C-D)}{2q\,[n-1]_q\,[n]_q^{m}}\sum_{j=1}^{n-1}[j]_q^{m}\left|a_j\right|\\
&\le\ \prod_{j=0}^{n-2}\left(\frac{\left|Q_1(k)(C-D)-2q[j]_q\,[j+1]_q^{m}C\right|}{2q\,[j+1]_q\,[j+2]_q^{m}}\right).
\end{aligned}$$

For this, we use the induction method.
For $n=2$ from (2.6), we have

$$|a_2|\le\frac{|Q_1(k)|\,(C-D)}{2q\,[2]_q^{m}}.$$

From (2.2), we have

$$|a_2| \leq \frac{|Q_1(k)|\,(C-D)}{2q\,[2]_q^m}.$$

For $n = 3$ from (2.6), we have

$$\begin{aligned}|a_3| &\leq \frac{|Q_1(k)|\,(C-D)}{2q[2]_q\,[3]_q^m}\left\{1+[2]_q^m\,|a_2|\right\}\\ &\leq \frac{|Q_1(k)|\,(C-D)}{2q[2]_q\,[3]_q^m}\left\{1+\frac{|Q_1(k)|\,(C-D)}{2q}\right\}.\end{aligned}$$

From (2.2), we have

$$\begin{aligned}|a_3| &\leq \frac{(C-D)\,|Q_1(k)|}{2q\,[2]_q^m}\left\{\frac{\left|Q_1(k)(C-D)-2q\,[2]_q^m\,D\right|}{2q[2]_q\,[3]_q^m}\right\},\quad [1]_q=1,\\ &\leq \frac{(C-D)\,|Q_1(k)|}{2q\,[2]_q^m}\left\{\frac{|Q_1(k)|\,(C-D)+2q\,[2]_q^m\,|D|}{2q[2]_q\,[3]_q^m}\right\}\\ &\leq \frac{(C-D)\,|Q_1(k)|}{2q[2]_q\,[3]_q^m}\left\{1+\frac{|Q_1(k)|\,(C-D)}{2q\,[2]_q^m}\right\}.\end{aligned}$$

Let the hypothesis be true for $n = t$.
From (2.6), we have

$$|a_t| \leq \frac{|Q_1(k)|\,(C-D)}{2q\,[t-1]_q\,[t]_q^m}\sum_{j=1}^{t-1}[j]_q^m\left|a_j\right|,\quad a_1=1,\ \ [1]_q^m.$$

From (2.2), we have

$$\begin{aligned}|a_t| &\leq \prod_{j=0}^{t-2}\left(\frac{\left|Q_1(k)(C-D)-2q[j]_q\,[j+1]_q^m\,D\right|}{2q\,[j+1]_q\,[j+2]_q^m}\right)\\ &\leq \prod_{j=0}^{t-2}\left(\frac{\left|Q_1(k)(C-D)+2q[j]_q\,[j+1]_q^m\right|}{2q\,[j+1]_q\,[j+2]_q^m}\right).\end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned}&\frac{|Q_1(k)|\,(C-D)}{2q\,[t-1]_q\,[t]_q^m}\sum_{j=1}^{t-1}[j]_q^m\left|a_j\right|\\ &\leq \prod_{j=0}^{t-2}\left(\frac{\left|Q_1(k)(C-D)+2q[j]_q\,[j+1]_q^m\right|}{2q\,[j+1]_q\,[j+2]_q^m}\right).\end{aligned}\tag{2.7}$$

Multiplying both sides by

$$\frac{\left|Q_1(k)(C-D)+2q[t-1]_q\,[t]_q^m\right|}{2q\,[t+1]_q\,[t+2]_q^m},$$

we have

$$\begin{aligned}
&\frac{\left|Q_1(k)(C-D)+2q[t-1]_q\,[t]_q^m\right|}{2q\,[t+1]_q\,[t+2]_q^m}\times\prod_{j=0}^{t-2}\left(\frac{\left|Q_1(k)(C-D)+2q[j]_q\,[j+1]_q^m\right|}{2q\,[j+1]_q\,[j+2]_q^m}\right)\\
\geq\;&\left\{\frac{\left|Q_1(k)(C-D)+2q[t-1]_q\,[t]_q^m\right|}{2q\,[t+1]_q\,[t+2]_q^m}\right\}\times\frac{|Q_1(k)|\,(C-D)}{2q\,[t-1]_q\,[t]_q^m}\sum_{j=1}^{t-1}[j]_q^m\left|a_j\right|,\\
&\prod_{j=0}^{t-2}\left(\frac{\left|Q_1(k)(C-D)+2q[j]_q\,[j+1]_q^m\right|}{2q\,[j+1]_q\,[j+2]_q^m}\right)\\
\geq\;&\left\{\begin{array}{l}\frac{|Q_1(k)|(C-D)}{2q[t+1]_q[t+2]_q^m}\left\{\frac{|Q_1(k)|(C-D)}{2q[t-1]_q[t]_q^m}\sum_{j=1}^{t-1}[j]_q^m\left|a_j\right|\right\}\\[2ex]
+\frac{2q[t-1]_q[t]_q^m}{2q[t+1]_q[t+2]_q^m}\left\{\frac{|Q_1(k)|(C-D)}{2q[t-1]_q[t]_q^m}\sum_{j=1}^{t-1}[j]_q^m\left|a_j\right|\right\},\end{array}\right.\\
\geq\;&\frac{|Q_1(k)|\,(C-D)}{2q\,[t+1]_q\,[t+2]_q^m}\left\{\frac{|Q_1(k)|\,(C-D)}{2q\,[t-1]_q\,[t]_q^m}\sum_{j=1}^{t-1}[j]_q^m\left|a_j\right|+\sum_{j=1}^{t-1}[j]_q^m\left|a_j\right|\right\},\\
&\prod_{j=0}^{t-2}\left(\frac{\left|Q_1(k)(C-D)+2q[j]_q\,[j+1]_q^m\right|}{2q\,[j+1]_q\,[j+2]_q^m}\right)\\
\geq\;&\frac{|Q_1(k)|\,(C-D)}{2q\,[t+1]_q\,[t+2]_q^m}\left\{|a_t|+\sum_{j=1}^{t-1}[j]_q^m\left|a_j\right|\right\},\\
=\;&\frac{|Q_1(k)|\,(C-D)}{2q\,[t+1]_q\,[t+2]_q^m}\sum_{j=1}^{t}[j]_q^m\left|a_j\right|.
\end{aligned}$$

That is,

$$\begin{aligned}
&\frac{|Q_1(k)|\,(A-B)}{2q\,[t+1]_q\,[t+2]_q^m}\sum_{j=1}^{t}[j]_q^m\left|a_j\right|\\
\leq\;&\prod_{j=0}^{t-2}\left(\frac{\left|Q_1(k)(C-D)+2q[j]_q\,[j+1]_q^m\right|}{2q\,[j+1]_q\,[j+2]_q^m}\right).
\end{aligned}$$

which shows that inequality (2.7) is true for $n=t+1$. Hence the required result. □

When $m=0$, $q\to 1^-$, we have the following known result, proved by Noor and Malik in (Noor & Malik, 2011).

Corollary 2.5. *A function $f\in\mathcal{A}$ and of the form (1.1) is in the class $k-\mathcal{ST}[C,D]$, if it satisfies the condition*

$$|a_n|\leq\prod_{j=0}^{n-2}\left(\frac{|Q_1(k)(C-D)-2jD|}{2\,(j+1)}\right).$$

When $m=0$, $q\to 1^-$, $C=1$, $D=-1$, then we have the following known result, proved by Kanas and Wisniowska in (Kanas & Wisniowska, 2000).

Corollary 2.6. *A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $k - \mathcal{ST}$, if it satisfies the condition*

$$|a_n| \leq \prod_{j=0}^{n-2} \left(\frac{|Q_1(k) + j|}{(j+1)} \right).$$

When $m = 0$, $q \to 1^-$, $k = 0$, then $Q_1(k) = 2$ and we get the following known result, proved in (Janowski, 1973).

Corollary 2.7. *A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $\mathcal{ST}[C, D]$, if it satisfies the condition*

$$|a_n| \leq \prod_{j=0}^{n-2} \left(\frac{|(C-D) - jD|}{(j+1)} \right), \quad -1 \leq D < C \leq 1.$$

When $m = 0$, $q \to 1^-$, $C = 1 - 2\alpha$, $D = -1$, with $0 \leq \alpha < 1$, then we have the following known result, proved by Shams et al. in (Shams et al., 2004).

Corollary 2.8. *A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $SD(k, \alpha)$, if it satisfies the condition*

$$|a_n| \leq \prod_{j=0}^{n-2} \left(\frac{|Q_1(k)(1-\alpha) + j|}{(j+1)} \right), \quad -1 \leq D < C \leq 1.$$

Theorem 2.4. *If $f(z) \in k - \mathcal{CV}_q(m, C, D)$ and is of the form (1.1). Then*

$$|a_n| \leq \frac{1}{[n]_q} \prod_{j=0}^{n-2} \left(\frac{\left|Q_1(k)(C-D) - 2q[j]_q\,[j+1]_q^m\,D\right|}{2q\,[j+1]_q\,[j+2]_q^m} \right), \qquad (n \geq 2).$$

The proof follows immediately by using Theorem (2.3) and the relation (1.6).

When $m = 0$, $q \to 1^-$, we have the following known result, proved by Noor and Sarfaraz in (Noor & Malik, 2011).

Corollary 2.9. *A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $k - \mathcal{UCV}[C, D]$, if it satisfies the condition*

$$|a_n| \leq \frac{1}{n} \prod_{j=0}^{n-2} \left(\frac{|Q_1(k)(C-D) - 2jD|}{2\,(j+1)} \right).$$

Theorem 2.5. *A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $k - \mathcal{UK}_q(m, A, B, C, D)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} \left\{ 2(k+1)\left|b_n - [n]_q\,a_n\right| + \left|(B+1)\,[n]_q\,a_n - (A+1)b_n\right| \right\} [n]_q^m \leq A - B, \tag{2.8}$$

where $-1 \leq D < C \leq 1$, $-1 \leq B < A \leq 1$, $k \geq 0$.

Proof. Assuming that (2.8) holds, then it suffices to show that

$$k \left| \frac{(B-1)L_{m,q}(z) - (A-1)}{(B+1)L_{m,q}(z) - (A+1)} - 1 \right| - \Re \left\{ \frac{(B-1)L_{m,q}(z) - (A-1)}{(B+1)L_{m,q}(z) - (A+1)} - 1 \right\} < 1.$$

We have

$$\begin{aligned} & k\left|\frac{(B-1)L_{m,q}(z)-(A-1)}{(B+1)L_{m,q}(z)-(A+1)}-1\right|-\Re\left\{\frac{(B-1)L_{m,q}(z)-(A-1)}{(B+1)L_{m,q}(z)-(A+1)}-1\right\} \\ \leq\ & (k+1)\left|\frac{(B-1)L_{m,q}(z)-(A-1)}{(B+1)L_{m,q}(z)-(A+1)}-1\right| \\ =\ & (k+1)\left|\frac{(B-1)S_q^{m+1}f(z)-(A-1)S_q^m g(z)}{(B+1)S_q^{m+1}f(z)-(A+1)S_q^m g(z)}-1\right| \\ =\ & 2(k+1)\left|\frac{S_q^m g(z)-S_q^{m+1}f(z)}{(B+1)S_q^{m+1}f(z)-(A+1)S_q^m g(z)}\right| \\ =\ & 2(k+1)\left|\frac{\sum_{n=2}^{\infty}\left\{b_n-[n]_q a_n\right\}[n]_q^m z^n}{(B-A)z+\sum_{n=2}^{\infty}\left\{(B+1)[n]_q a_n-(A+1)b_n\right\}[n]_q^m z^n}\right| \\ \leq\ & 2(k+1)\left\{\frac{\sum_{n=2}^{\infty}\left|b_n-[n]_q a_n\right|[n]_q^m}{A-B-\sum_{n=2}^{\infty}\left|(B+1)[n]_q a_n-(A+1)b_n\right|[n]_q^m}\right\}. \end{aligned}$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}\left\{2(k+1)\left|b_n-[n]_q a_n\right|+\left|(B+1)[n]_q a_n-(A+1)b_n\right|\right\}[n]_q^m \leq A-B.$$

This completes the proof. □

When $q \to 1^-$, $m = 0$, we have the following known result, proved by Mahmood at al. (Mahmood *et al.*, 2017).

Corollary 2.10. *A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $k-\mathcal{UK}(A,B,C,D)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\left\{2(k+1)\left|b_n-na_n\right|+\left|(B+1)na_n-(A+1)b_n\right|\right\} \leq A-B.$$

Theorem 2.6. *A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $k-\mathcal{UQ}_q(m,A,B,C,D)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}[n]_q^{m+1}\left\{2(k+1)\left|b_n-[n]_q a_n\right|+\left|(B+1)[n]_q a_n-(A+1)b_n\right|\right\} \leq A-B,$$

where $-1 \leq D < C \leq 1$, $-1 \leq B < A \leq 1, k \geq 0$.

The proof follows immediately by using Theorem 2.1 and (1.7).

When $q \to 1^-$, $m = 0$, we have the following known result, proved by Mahmood at al. (Mahmood *et al.*, 2017)

Corollary 2.11. *A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $k-\mathcal{UQ}(A,B,C,D)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} n\left\{2(k+1)\left|b_n - na_n\right| + \left|(B+1)na_n - (A+1)b_n\right|\right\} \leq A - B.$$

When $q \to 1^-$, $m = 0$, $A = 1-2\beta$, $B = -1$, $C = 1$, $D = -1$ with $0 \leq \beta < 1$, then we have the following known result, proved by Subramanian et al. in (Subramanian *et al.*, 2003).

Corollary 2.12. *A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $\mathcal{UQ}(\beta)$, $g(z) = z$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} n^2 \left|a_n\right| \leq 1 - \beta.$$

Theorem 2.7. *If $f(z) \in k-\mathcal{UK}_q(m,A,B,C,D)$ and is of the form (1.1). Then*

$$|a_n| \leq \begin{cases} \frac{1}{[n]_q} \prod_{i=0}^{n-2} \left(\frac{\left|Q_1(k)(C-D)-2q[i]_q[i+1]_q^m D\right|}{2q[i+1]_q[i+2]_q^m}\right) \\ + \frac{|Q_1(k)|(A-B)}{2[n]_q[n]_q^m} \sum_{j=1}^{n-1} [j]_q^m \prod_{i=0}^{j-2} \left(\frac{\left|Q_1(k)(C-D)-2q[i]_q[i+1]_q^m D\right|}{2q[i+1]_q[i+2]_q^m}\right), \quad n \geq 2. \end{cases}$$

where $|Q_1(k)|$ is defined by (1.3).

Proof. Let us take

$$\frac{S_q^{m+1} f(z)}{S_q^m g(z)} = p(z), \tag{2.9}$$

where

$$p(z) \in k-\mathcal{P}[A,B] \text{ and } g(z) \in k-\mathcal{ST}_q(m,C,D).$$

Now from (2.9), we have

$$S_q^{m+1} f(z) = S_q^m g(z) p(z),$$

which implies that

$$\begin{aligned} z + \sum_{n=2}^{\infty} [n]_q^{m+1} a_n z^n &= \left(1 + \sum_{n=1}^{\infty} c_n z^n\right)\left(z + \sum_{n=2}^{\infty} [n]_q^m b_n z^n\right), \\ z + \sum_{n=2}^{\infty} [n]_q^{m+1} a_n z^n &= \left(1 + \sum_{n=1}^{\infty} c_n z^n\right)\left(\sum_{n=1}^{\infty} [n]_q^m b_n z^n\right), \\ z + \sum_{n=2}^{\infty} [n]_q^{m+1} a_n z^n &= \sum_{n=1}^{\infty} [n]_q^m b_n z^n + \left(\sum_{n=1}^{\infty} [n]_q^m b_n z^n\right)\left(\sum_{n=1}^{\infty} c_n z^n\right), \\ \sum_{n=2}^{\infty} \left\{[n]_q a_n - b_n\right\} [n]_q^m z^n &= \left(\sum_{n=1}^{\infty} [n]_q^m b_n z^n\right)\left(\sum_{n=1}^{\infty} c_n z^n\right), \end{aligned} \tag{2.10}$$

By using Cauchy product formula on R.H.S of (2.10), we have

$$\sum_{n=2}^{\infty} \left\{[n]_q a_n - b_n\right\} [n]_q^m z^n = \sum_{n=1}^{\infty} \left[\sum_{j=1}^{n-1} [j]_q^m b_j c_{n-j}\right] z^n. \tag{2.11}$$

Equating coefficients of z^n on both sides of (2.11), we have

$$\left\{[n]_q a_n - b_n\right\}[n]_q^m = \sum_{j=1}^{n-1}[j]_q^m b_j c_{n-j}, \quad , a_0 = 1,$$

$$[n]_q^{m+1} a_n = [n]_q^m b_n + \sum_{j=1}^{n-1}[j]_q^m b_j c_{n-j}.$$

This implies that

$$[n]_q^{m+1}|a_n| \leq [n]_q^m |b_n| + \sum_{j=1}^{n-1}[j]_q^m \left|b_j\right|\left|c_{n-j}\right|, \quad a_1 = 1. \tag{2.12}$$

Since $p(z) \in k - \mathcal{P}[A,B]$, therefore by using lemma 1.2 on (2.12), we have

$$[n]_q^{m+1}|a_n| \leq [n]_q^m |b_n| + \sum_{j=1}^{n-1}\frac{|Q_1(k)|(A-B)}{2}[j]_q^m \left|b_j\right|. \tag{2.13}$$

Again $g(z) \in k - \mathcal{ST}_q(m,C,D)$, therefore by using Theorem 2.3 on (2.13), we have

$$[n]_q^{m+1}|a_n| \leq \left\{\begin{array}{c} [n]_q^m \prod_{i=0}^{n-2}\left(\frac{\left|Q_1(k)(C-D)-2q[i]_q[i+1]_q^m D\right|}{2q[i+1]_q[i+2]_q^m}\right) \\ +\frac{|Q_1(k)|(A-B)}{2}\sum_{j=1}^{n-1}[j]_q^m \prod_{i=0}^{j-2}\left(\frac{\left|Q_1(k)(C-D)-2q[i]_q[i+1]_q^m D\right|}{2q[i+1]_q[i+2]_q^m}\right), \end{array}\right.$$

which implies that

$$|a_n| \leq \left\{\begin{array}{c} \frac{1}{[n]_q}\prod_{i=0}^{n-2}\left(\frac{\left|Q_1(k)(C-D)-2q[i]_q[i+1]_q^m D\right|}{2q[i+1]_q[i+2]_q^m}\right) \\ +\frac{|Q_1(k)|(A-B)}{2[n]_q[n]_q^m}\sum_{j=1}^{n-1}[j]_q^m \prod_{i=0}^{j-2}\left(\frac{\left|Q_1(k)(C-D)-2q[i]_q[i+1]_q^m D\right|}{2q[i+1]_q[i+2]_q^m}\right). \end{array}\right.$$

□

When $q \to 1^-$, $m = 0$, we have the following known result, proved by Mahmood et al. (Mahmood *et al.*, 2017).

Corollary 2.13. *If $f(z) \in k - \mathcal{UK}(m,A,B,C,D)$ and is of the form (1.1). Then*

$$|a_n| \leq \left\{\begin{array}{c} \frac{1}{n}\prod_{i=0}^{n-2}\left(\frac{|Q_1(k)(C-D)-2iD|}{2(i+1)}\right) \\ +\frac{|Q_1(k)|(A-B)}{2n}\sum_{j=1}^{n-1}\prod_{i=0}^{j-2}\left(\frac{|Q_1(k)(C-D)-2iD|}{2(i+1)}\right), \; n \geq 2, \end{array}\right.$$

where $Q_1(k)$ is defined by (1.3).

When $q \to 1^-$, $m = 0$, $A = 1$, $B = -1$, $C = 1$, $D = -1$, we have the following known result, proved by Noor et al. (Noor *et al.*, 2009).

Corollary 2.14. *If $f(z) \in k - \mathcal{UK}(0,1,-1,1,-1)$ and is of the form (1.1). Then*

$$|a_n| \leq \frac{(|Q_1(k)|)_{n-1}}{n!} + \frac{|Q_1(k)|}{n}\sum_{j=0}^{n-1}\frac{(|Q_1(k)|)_{j-1}}{(j-1)!}, \quad n \geq 2.$$

When $q \to 1^-$, $m = 0$, $k = 0$, $A = 1$, $B = -1$, $C = 1$, $D = -1$, we have the following known result, proved by Kaplan (Kaplan, 1952).

Corollary 2.15. *If* $f(z) \in 0 - \mathcal{UK}(0, 1, -1, 1, -1) = \mathcal{K}$ *and is of the form* (1.1). *Then*

$$|a_n| \le n, \quad n \ge 2.$$

Theorem 2.8. *If* $f(z) \in k - \mathcal{UQ}_q(m, A, B, C, D)$ *and is of the form* (1.1). *Then*

$$|a_n| \le \begin{cases} \frac{1}{[n]_q^2} \prod_{i=0}^{n-2} \left(\frac{\left|Q_1(k)(C-D)-2q[i]_q[i+1]_q^m D\right|}{2q[i+1]_q[i+2]_q^m} \right) \\ + \frac{|Q_1(k)|(A-B)}{2[n]_q^2[n]_q^m} \sum_{j=1}^{n-1} [j]_q^m \prod_{i=0}^{j-2} \left(\frac{\left|Q_1(k)(C-D)-2q[i]_q[i+1]_q^m D\right|}{2q[i+1]_q[i+2]_q^m} \right), \end{cases}$$

where $|Q_1(k)|$ *is defined by* (1.3).

Proof. The proof follows immediately by using Theorem 2.7 and (1.7). □

When $q \to 1^-$, $m = 0$, we have the following known result, proved by Mahmood at al. (Mahmood *et al.*, 2017).

Corollary 2.16. *If* $f(z) \in k - \mathcal{UK}(m, A, B, C, D)$ *and is of the form* (1.1). *Then*

$$|a_n| \le \begin{cases} \frac{1}{n^2} \prod_{i=0}^{n-2} \left(\frac{|Q_1(k)(C-D)-2iD|}{2(i+1)} \right) \\ + \frac{|Q_1(k)|(A-B)}{2n^2} \sum_{j=1}^{n-1} \prod_{i=0}^{j-2} \left(\frac{|Q_1(k)(C-D)-2iD|}{2(i+1)} \right), \quad n \ge 2. \end{cases}$$

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Domain Complexity in Corrective Maintenance Tasks' Complexity: An Empirical Study in a Micro Software Company

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Abstract

Corrective maintenance is very important in software engineering practice since it enables correction of problems identified in operational use of software applications. Therefore, modeling complexity of maintenance tasks is essential for estimation and planning activities in software organizations that spend majority of resources on maintenance tasks. The article presents a study aimed at developing a model for maintenance task complexity by considering specific parameters of domain complexity associated to each software application. The study was conducted in a micro software company. The model enables analysis of trends for maintenance task complexity and correlation between task complexity and time spent for completing tasks. Implication and benefits of the presented research for the selected software company, for managers in software industry and researchers are discussed. The article concludes with challenging research directions.

Keywords: Task complexity, Domain complexity, Mathematical model, Corrective maintenance.
2010 MSC: 68N30 Mathematical aspects of software engineering (specification, verification, metrics, requirements, etc.), 68Q25 Analysis of algorithms and problem complexity.

1. Introduction

Software maintenance relates to post delivery activities aimed at ensuring efficient use of software systems without significant changes in software design. Software maintenance includes planned activities, such as bug fixing or enhancing functionality, and unplanned activities, such as adapting a system to new business conditions (Tripathy & Naik, 2015). Maintenance activities involve tight cooperation of software engineers engaged in maintaining software systems and clients that use software systems, both of them with different views of software maintenance, which strongly emphasizes managerial issues as the biggest problem in software maintenance

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(April & Abran, 2008). Recognized technical and organizational complexity of maintenance activities resulted with high costs, which are usually between 50% and 90% of total costs in software life cycle (Grubb & Takang, 2003; Junio *et al.*, 2011; Pino *et al.*, 2012; Bourque & Fairley, 2014). Despite recognized complexity and high costs of software maintenance activities, it is much less investigated compared to software development (Banker & Slaughter, 1997; Jones, 2010). In addition, software maintenance activities are mainly short term tasks, in many cases completed on day-to-day basis in order to keep software operational (Tan & Mookerjee, 2005; Junio *et al.*, 2011). Software maintenance activities are more difficult and complex comparing to software development activities (Jones, 2010).

Software maintenance tasks are performed in order to sustain software systems useful and operable for users. Maintenance tasks are performed on already developed software, and in many cases require not only technical skills but also organizational skills related to estimation of costs and risks, and evaluation of necessary tasks to be performed. The complexity of software maintenance is reflected in existence of 23 types of work that might be performed on software systems, which are systematized by (Jones, 2010) in a book Software Engineering Best Practices. In many cases, intensive maintenance of software systems results with degradation of their structure and characteristics, making them more difficult to maintain (April & Abran, 2008).

Software maintenance tasks have been researched for over 40 years, which resulted with several classifications and typologies of maintenance types. The first and the most influential typology of maintenance types was proposed by (Swanson, 1976), in which corrective, adaptive and perfective maintenance can be distinguished. This typology was later used and interpreted in variety of ways by many researchers, resulting with several typologies and definitions of maintenance types. (Chapin *et al.*, 2001) proposed a refined classification of software evolution and maintenance types aligned to clusters related to software systems, suggesting that different software organization can use different types of software maintenance. In this refined typology with 12 types of software maintenance activities, corrective maintenance occurs in the cluster related to business rules. In standard ISO 14764:2006 Software Engineering - Software Life Cycle Processes - Maintenance(ISO, 2006), there are four maintenance categories: corrective, adaptive, perfective and preventive. Corrective maintenance is in ISO 14764:2006 defined as "*the reactive modification of a software product performed after delivery to correct discovered problems*". According to (April & Abran, 2008), corrective maintenance is reactive since it is performed after a problem or a failure is identified in a software, requiring work to solve the problem and bring the software into a usable state. In *Guide to the Software Engineering Body of Knowledge (SWEBOK)* (Bourque & Fairley, 2014) is stated that emergency maintenance is a special type of corrective maintenance aimed at mitigating identified problems without scheduling maintenance request in a classical maintenance process defined in software organizations. (Tripathy & Naik, 2015) distinguished intention-based classification of software maintenance activities which is aligned with ISO 14764:2006 (ISO, 2006), but also proposed activity-based classification with corrections and enhancements as the main types of activities. In addition, maintenance types usually overlap in industrial practice, and it is common case that adaptive and perfective work hide corrective work in software maintenance (Hatton, 2007). In any of proposed classification of maintenance types, corrective maintenance attracted significant attention since it intends to fix discovered problems and bring software to operational state for end users, and usually has priority over other types of work (April & Abran, 2009)

Task is a central concept in studying human behavior in various situations and includes a set of activities performed by humans in a given context in order to achieve proposed goals. Studying tasks is necessary part of organizational research lading towards improvement of practice. The main directions of researching task complexity relates to complexity of information processing, decision making and goal setting (Wood, 1986; Campbell, 1988), in which the complexity has been treated as a psychological experience, as an interaction between task and person characteristics, and as a function of tack properties. The most commonly used definition of task complexity defines a task as set of products, required acts and information ques (Wood, 1986). In organizational research of human performed tasks, objective task complexity depends only on task intrinsic characteristics, while subjective task complexity depends on task solver's experience and knowledge (Maynard & Hakel, 1997; Braarud, 2001; Parkes, 2017). Since software engineering has been considered as a complex discipline that includes technical, organizational and human factors, investigation of human performed tasks is essential for understanding and improving everyday practice (Dybå *et al.*, 2011; Capretz, 2014).

Complexity is very important issue to consider in software evolution and maintenance since it influences quality of software products and all activities in software life cycle (Keshavarz *et al.*, 2011). Complexity assessment assumes the use of carefully selected metrics for measuring trends and relations in historical data about software evolution and maintenance (Suh & Neamtiu, 2010). (Sun *et al.*, 2015) stated that selecting relevant information from maintenance repositories is essential for improving maintenance tasks. Complexity has been in software engineering mostly associated to structural complexity of software systems (Lilienthal, 2009; Lu *et al.*, 2016), but (Li & Delugach, 1997) suggested that traditional software complexity metrics cannot be effectively implemented for measuring complexity of application domains. In addition, (Shaft & Vessey, 1998) indicated the importance of domain knowledge (knowledge of the problem area) on program comprehension activities, which are essential in software maintenance practice. However, due to the evolving nature of domain knowledge (evolving nature of business), (Mendes-Moreira & Davies, 1993) suggested regular update of domain knowledge for efficient maintenance of software systems.

Based on the stated observations and the authors experience in researching software maintenance processes in small software companies the proposed objective of this study is to examine the influence of software domain complexity on corrective maintenance tasks. The empirical study was organized in a local micro company with majority of resources dedicated to maintenance activities. The rest of the article is structured as follows. The second section provides a literature review of studies dealing with corrective maintenance tasks. The third section presents the study conducted in a micro software company, followed with the sections in which limitations and validity, as well as implications of the presented research are discussed. The last section contains concluding remarks and outlines future research directions.

2. Related work

It has been recognized in software industry, and reported in empirical studies, that software maintenance tasks are complex and require skilled maintainers (Jones, 2010; Bourque & Fairley, 2014). The complexity of maintenance tasks is the consequence of the following facts (Podnar & Mikac, 2001): (1) they are implemented on complex software systems, (2), they involve people who have different roles in the maintenance process, and (3) they contain several feedback loops that ensure the flow of information between participants in the process. (Ko *et al.*, 2006) emphasized the importance of collecting and tracking information relevant for each specific maintenance task, while (Vasilev, 2012) pointed out the importance of information related to processes for reduction of costs and enterprise practice improvement. Proposed or required maintenance task is usually defined in a textual field, which contains maintenance request description that is essential for efficient performance of a maintenance task (Mockus & Votta, 2000). Understanding of maintenance requests' descriptions requires both technical knowledge specific for software systems and domain knowledge specific for the domain of software use. According to (Boehm & Basili, 2001) understanding of context dependent-factors (e.g. the level of data coupling and cohesion, data size and complexity) can positively contribute to corrective maintenance tasks

(Vans *et al.*, 1999) conducted a field study with four professional software maintainers engaged in maintaining large-scale software systems, aimed at investigating program understanding behavior in corrective maintenance tasks. During the study, the authors observed maintainers while they were solving corrective maintenance tasks. Data analysis revealed that maintainers work at three levels of abstractions: code, algorithm and application domain. In addition, maintainers regularly switch between these abstraction levels based on the current problem they solve. Maintainers need information about domain concepts and connect this information to software being maintained during corrective tasks.

(De Lucia *et al.*, 2005) presented an empirical study aimed at assessing and improving the effort estimation models for corrective maintenance in an international software enterprise. The study contained two phases. In the first phase was constructed a multiple linear regression models that were validated against real data from five corrective maintenance project. In the second phase the authors replicated the assessment of the constructed models from the first phase on a new corrective maintenance project. The results enable prediction of trends for corrective maintenance tasks, while early estimates of the average number of corrective tasks contribute to practice improvement in the selected company.

(Wang & Arisholm, 2009) investigated the difficulty level of maintenance tasks based on the number and complexity of classes that would be affected by the change. The study was based on two controlled experiments with 3rd to 5th year software engineering students without prior knowledge of the software being maintained. Results revealed that solving easier tasks (less complex) before harder (more complex) is more appropriate for inexperienced programmers, and that task order influence correctness of performed maintenance tasks.

(Li *et al.*, 2010) presented an empirical study aimed at analyzing around 1400 corrective maintenance activities associated to defect reports in two large software companies in Norway. The most important cost drivers for corrective maintenance tasks identified in the first company are: size of the system to be maintained, complexity of the system to be maintained and maintainers experience. In the second company the most important cost driver is domain knowledge. These results indicate that: (1) models resulted from empirical research studies should be customized for each company based on its specific characteristics, and (2) maintainers experience and domain knowledge significantly influence corrective task performance.

(Nguyen *et al.*, 2011) conducted a controlled experiment to assess the productivity and effort distribution of three different maintenance types: enhancive, corrective, and reductive. As the metrics were used three independent LOC (lines of code) metrics (added, modified, and deleted). Results revealed that: (1) the productivity of corrective maintenance is significantly lower than that of the other types of maintenance, and (2) task comprehension activity is the most complex task in maintenance. Based on these results it is evident that corrective maintenance tasks are more complex than other maintenance tasks, which requires highly skilled maintainers.

(Lee *et al.*, 2015) organized a qualitative study aimed at identifying factors that impact effort in corrective maintenance tasks. By using causal mapping methodology, the authors identified and ranked a set of 17 factors that contribute to corrective maintenance tasks implementation. Among all identified factors *High code complexity* (structural complexity) was ranked as the most critical with the weighted score of 0.8027, while *High version/deployment complexity* (management of multiple versions of software systems) was ranked at the 12th place with the weighted score of 0.6508. Regarding the influence of maintenance request description, the important identified factor is *Low clarity or availability of defect documentation*, which is ranked at the 10th place with weighted score of 0.6729. The identified factors suggest that complexity factors related to software structure and domain of problem should be taken into account in corrective maintenance tasks.

(Lenarduzzi *et al.*, 2018) presented an industrial case study aimed at prioritizing corrective maintenance tasks caused by crash reports and the exceptions for android applications for the period of four years. The applications were developed by an Italian public transportation company, while crash reports were collected directly from the Google Play Store. The study results indicate that six exceptions caused over 70% corrective tasks, and that most of the exceptions were generated by bad development practices. Results are useful for the selected company for improving its corrective maintenance efforts.

To summarize, there has been a large number of empirical studies addressing corrective maintenance tasks. Some studies were organized at universities (Wang & Arisholm, 2009; Nguyen *et al.*, 2011), while some of them were organized in industrial settings (De Lucia *et al.*, 2005; Li *et al.*, 2010; Lee *et al.*, 2015; Lenarduzzi *et al.*, 2018). Although these studies address different aspects of corrective maintenance in different settings, there is significant space for researching this important segment of industrial practice. Our study differs from the outlined studies because it deals with the subjective assessment of domain complexity influence on corrective maintenance tasks, which has not been addressed in previous research.

3. Case study

The study was conducted in an indigenous software company, which can be classified as a micro enterprise according to European Commission for Enterprise and industry publications (Commission, 2015). The company has 7 employees: 3 senior programmers, 3 junior programmers and 1 administrative worker. The company develops business software applications for local clients in Serbia. Totally 48 software applications are used by over 100 client companies in Serbia.

Data analysis is based on historical data extracted from the company internal repository of tasks, which is common practice in empirical software engineering studies aimed at investigating

Table 1. Distribution of software maintenance tasks according to the typology proposed in (Stojanov *et al.*, 2017)

Maintenance task type	Number of tasks	Share [%]
Adaptation	22	1.08
Correction	489	24.02
Enhancement	1050	51.57
Preventive	8	0.39
Support	467	22.94
TOTAL	**2036**	**100.00**

what happens in everyday practice (Dit *et al.*, 2013; Stojanov *et al.*, 2013*b*). The study is a continuation of the research on maintenance trends in the selected company (Stojanov *et al.*, 2013*a*), but with the improved typology of software maintenance tasks introduced in (Stojanov *et al.*, 2017). The data set consists of totally 2293 tasks solved in 2013 and 2014 years, where 2036 tasks were categorized as maintenance tasks (88.79% of all tasks). The classification of software maintenance tasks according to the typology presented in (Stojanov *et al.*, 2017) is presented in table 1.

Maintenance tasks are created in order to solve maintenance requests (MR) submitted by software users. Submission of a MR assumes that a user should provide a textual description of a request and indicate a software application to which a MR relates to. Each request is assigned to one of the programmers who is responsible for maintaining a target software application. Maintenance task record contains the fields that enable tracking of all relevant data for processing associated MR and calculating costs of implemented work.

Corrective maintenance tasks account for almost one quarter of all maintenance tasks (24.02%), and since these tasks relates to direct solving of client problems with software, they deserve attention to be analyzed. The aim of the study is to analyze the influence of domain complexity on corrective maintenance tasks complexity, where domain complexity reflects the complexity of a business domain in which software is used.

3.1. Maintenance tasks

Maintenance tasks were recorded in a local repository of tasks in the company. For each task, the following parameters recorded in the repository are interesting for the data analysis:

- *Worker ID*. The identification number of a programmer engaged in solving a task.
- *Application*. The name of the application to which a maintenance task relates to.
- *Maintenance request description*. The description of a maintenance request to which the task is associated to.
- *Working Hours Company [WHC]*. Working hours spend in the company on solving the task.

Table 2. Distribution of corrective maintenance tasks on software applications

Software application	Number of tasks	Share [%]
Application 1	89	18.20
Application 2	48	9.82
Application 3	97	19.84
Application 4	28	5.73
Application 5	97	19.84
Other 29 applications	130	26.58
TOTAL	**489**	**100.00**

- *Working Hours Internet [WHI]*. Working hours spend at Internet on solving the task. Programmer works in the company and uses Internet to access software application at client side.
- *Working Hours Client Side [WHCS]*. Working hours spend at the client side (in the client's organization) on solving the task.
- *Working Hours TOTAL [WHT]* Total number of working hours spent on solving the task

Total number of working hours is the cum of three types of working hours, which is expressed in the following way

$$WHT = WHC + WHI + WHCS. \tag{3.1}$$

The values for WHC, WHI, and WHCS enters a programmer assigned to the task after completing it. The value for WHT is calculated and stored in the repository.

3.2. Corrective maintenance trends in the company

Corrective maintenance tasks were performed on 34 software applications. The data about corrective maintenance tasks were extracted from the local repository for tracking all tasks in the company. Each maintenance task is associated to a maintenance request (MR) received from a user, and assigned to a programmer in the company.

Initial data analysis based on descriptive statistical methods revealed that only 5 software applications have more than 5% of the total number of tasks. Other 29 software applications together consume 26.58% of all tasks, which is approximately less than 1% per software application, which can be treated as insignificant for further statistical data analysis. Based on this fact, further data analysis is focused on the selected 5 software applications. Distribution of corrective maintenance tasks for all software applications is presented in table 2.

For the selected five software applications, average number of working hours spent on corrective tasks is presented in table 3.

Average values of working hours spent on corrective maintenance tasks presented in table 3 revealed the following interesting trends: (1) Average time spent on tasks is usually between 1 and 1.5 hours, (2) tasks associated to applications 2 and 3 last longer than tasks for applications 1,4

Table 3. Average number of working hours for 5 selected software applications

Software application	Average WHC	Average WHI	Average WHCS	Average WHT
Application 1	0.26	0.49	0.45	1.21
Application 2	0.48	0.58	0.35	1.42
Application 3	0.41	0.45	0.66	1.53
Application 4	0.25	0.52	0.54	1.30
Application 5	0.14	0.61	0.49	1.25

Table 4. Scale for subjective rating of domain complexity

Level	Abbreviation	Value
Very Low	VL	1
Low 2	L	2
Medium	M	3
High	H	4
Very High	VH	5

and 5, and (3) programmers usually access clients information system via Internet (WHI) or work at client side (WHCS) since the average values for WHC are the lowest for all applications.

3.3. Domain complexity model

Software is used to solve problems in specific domains of business or living, which influences all requirements and activities related to software. Maintenance activities are triggered by maintenance requests submitted by software users, who describe requests by using unstructured text with domain terminology. These requests should be understandable for programmers who are engaged to solve reported problems by correcting identified faults. Therefore, it is important to describe domain complexity, and develop a model that can be used for modeling complexity of corrective maintenance tasks. Domain complexity reflects intellectual effort required for understanding a domain of software use, and how the domain influences complexity of maintenance tasks performed on that software.

In this study, domain complexity for all software applications was rated by the company manager by using predefined scale with the values presented in table 4.

Since the majority of time and effort in maintenance consume activities related to comprehension of a maintenance request and software to be modified (Von Mayrhauser & Vans, 1995; O'Brien *et al.*, 2004), complexity in this study relates to understanding a maintenance task for the given domain of a software application based on the description available in the maintenance request. For that purpose, a set of subjective measures (parameters) for domain complexity for all software applications is defined:

- *Terminology Complexity (TC)*. Complexity of terminology used for defining and describing entities, relations and processes in a domain.

Table 5. Subjective ratings of domain complexity for selected software applications

Software application	TC	RC	BPC	HFC
Application 1	4	5	5	5
Application 2	4	5	4	4
Application 3	5	5	4	5
Application 4	4	3	3	3
Application 5	5	5	4	5

- *Relations complexity (RC)*. Complexity of relations between entities and processes in a domain.
- *Business processes complexity (BPC)*. Complexity of business processes in a domain (process flow, sub-processes, constraints, inputs and outputs).
- *Human Factor Complexity (HFC)*. Complexity of humans who perform business processes in a domain, including the number and roles of people engaged in business processes.

The model of domain complexity assumes subjective rating of each specific parameter of domain complexity obtained from the company manager (the most experienced programmer). Subjective ratings of domain complexity TC, RC, BPC and HFC for selected 5 software applications are presented in table 5.

3.4. *Task complexity model*

Task is the basic unit of work in software maintenance in the company, aimed at solving a maintenance request. Each task is performed by one programmer, and always is associated to a specific software application. The task is defined with the description of a maintenance request, which is in the form of unstructured text submitted by a user. The description is implemented as a text field in each task record stored in a local repository of tasks. For each task description complexity measures are defined as a subjective ratings of TC, RC, BPC and HFC parameters expressed with values from the table 4. These values present subjective ratings of a task complexity provided by a programmer engaged in solving the task. Maintenance task complexity is expressed as

$$TaskCompl = TC * mtTC + RC * mtRC + BPC * mtBPC + HFC * mtHFC, \quad (3.2)$$

where coefficients TC, RC, BPC and HFC presents specific domain complexities for the selected application, while mtTC, mtRC, mtBPC and mtHFC presents specific complexities for a selected maintenance task.

For each of the five selected applications, overall maintenance task complexity is calculated for all maintenance tasks by using formula 3.2.

For the extracted data and defined subjective specific domain complexity measures for all software applications, the arithmetic mean (MEAN), standard deviation (STDEV) and coefficient

Table 6. Measures of spread for corrective maintenance task complexity affected by domain complexity parameters for selected software applications

Software application	MEAN	STDEV	CV [%]
Application 1	50.79	11.00	21.66
Application 2	40.63	8.46	20.82
Application 3	39.43	10.63	26.95
Application 4	31.50	6.96	22.10
Application 5	29.99	6.47	21.56

Table 7. Correlation coefficients between calculated corrective maintenance task complexity and working hours spent on solving task

	WHC	WHI	WHCS	WHT
Task complexity of Application 1	0.10	0.19	0.05	0.35
Task complexity of Application 2	0.74	0.18	0.02	0.85
Task complexity of Application 3	0.61	0.17	0.48	0.81
Task complexity of Application 4	-0.08	-0.06	0.63	0.71
Task complexity of Application 5	0.05	-0.13	0.59	0.61

of variance (CV) for all tasks are calculated (Buglear, 2001). Calculated values are presented in table 6.

Data presented in table 6 revealed that Application 1 has the highest complexity of maintenance tasks (50.79 in average), followed with Application 2 with maintenance task complexity of 40.63 in average, while the simplest tasks are tasks related to Application 5 (29.99 in average). Tasks are almost twice as complex for Application 1 than for Application 5.

Variance coefficient analysis for selected applications revealed that the spread of task complexity for each application is acceptable (between 20.82 for Application 2 and 26.95 for application 3), which indicates small variances of task complexity. This enables more reliable predictions of task complexity for further maintenance activities. Based on data presented in table 6, the most reliable prediction of maintenance task complexity can be given for Application 2, while the most unreliable predictions of maintenance task complexity are for Application 3.

3.5. The task complexity and working hours correlation

Table 7 presents correlation coefficients between corrective maintenance tasks complexity and working hours spent on solving tasks. Correlation coefficients are calculated between task complexity and each type of working hours: in the company (WHC), at Internet (WHI), in the client company (WHCS) and total working hours WHT calculated by using formula 3.1.

Data presented in table 7 revealed that the correlation between task complexity and total working hours (WHT) vary from 0.35 for the Application 1 to 0.85 for the Application 2. Calculated values indicate strong correlations for Application 2, Application 3 and Application 4, which means that based on subjective evaluation of task complexity provided by a programmer, reliable assessment of total working hours can be given. For Application 1 (r=0.35) and Application 5 (r=0.61) it is not possible to reliably estimate working hours.

Although estimates of total working hours based on task complexity are reliable for Application 2, Application 3 and Application 4, correlation between task complexity and specific types of working hours (WHC, WHI and WHCS) are weak, which means that it is not possible to estimate these specific working hours. The only exception is correlation between task complexity and WHC for the Application 2 with value of 0.74, which means that only working hours in the company can be estimated for the Application 2.

4. Limitations and validity

Despite the clear and useful results obtained through empirical data analysis, this study certainly has some limitations that influence the results and conclusions. The first limitation is quite simple mathematical model for calculating task complexity. The model resulted with results that enable reliable estimates in some segments of maintenance practice, but it will be incrementally improved in order to increase reliability of decisions based on obtained results.

The next limitation relates to initial examination and proper preprocessing of data that deviate from typical values in empirical data set (outliers) (Chatfield, 1985; Cousineau & Chartier, 2010). Data analysis with appropriate treatments of outliers could provide more reliable results and estimates, which will be used for assessment and improvement of task complexity models, and finally better planning and decision making in the company. This limitation will be addressed in further research, and results will be compared with results obtained in this study.

Internal and external validity are commonly used for judging quality and reliability of empirical studies in software engineering (Kitchenham *et al.*, 2002; Shull *et al.*, 2008). Internal validity relates to selection and definition of used parameters (variables) and proper use of selected data analysis methods that leads to reliable results. The main threat to internal validity relates to data set used for modeling domain complexity of maintenance tasks, which is collection of subjective measures provided by programmers for each maintenance tasks. The improvement of the presented model will include more objective measures based on data extracted directly from maintenance request descriptions and data related to technical details of maintained software applications (e.g. number of lines and modules affected by maintenance request). This improvement of task complexity model requires more accurate data in the company repository of tasks, and will be addressed in future research after improving recording of maintenance tasks in the company.

(Briand *et al.*, 2017) discussed context-driven aspect of empirical research in software engineering and suggested that there is no need to force external validity issue related to generalizability of study results. However, generalizability can be viewed from the aspect of used research methods that may produce specific, but different, findings in other settings. Therefore, it is possible to use the presented methods for analyzing complexity of tasks in other software (or engineering) organizations and get context-specific results that can be of benefit for these organizations.

5. Implications and benefits

Despite limitations stated in the previous section, this research has significant benefits and implications for practice and research in the field of software engineering. Study design and results can be of benefits to the selected company, software industry in general, and software engineering research community.

The benefits for the selected company are: (1) The presented model of task complexity enables calculation of complexity for corrective maintenance tasks by considering subjective evaluation of maintainers that solve these tasks, which further enables identification of trends for task complexity for each software application, (2) Based on the calculated task complexity, the company staff can estimate required time for solving maintenance tasks based on the interpretation of the correlation between task complexity and spent working hours, and (3) Based on calculated task complexity and estimated time for solving the task, the company management can design more reliable and effective organization of maintenance activities in the company (e.g. scheduling of maintenance tasks among programmers in order to accelerate processing of maintenance requests).

Managers and experts from software industry can find useful directions how to collect and use field data in their organizations for developing models for task complexity by considering some specific characteristics of the practice in their organizations. In addition, they can find some directions for correlating task complexity with elements of planning in their organizations

Researchers can find lessons how to organize an empirical study aimed at assessing complexity of tasks in real industrial settings by considering subjective evaluations of specific parameters that affects specific types of tasks. Presented study shows how to identify attributes that influence task complexity, how to define a scale for evaluating specific attributes of task complexity, and how to identify the correlation between the complexity of tasks and organizational parameters that are essential for planning activities in a selected organization.

6. Concluding remarks

As the task complexity is significant factor that affects software maintenance, presented study contributes to software maintenance practice and research. The study presents the model for calculating the complexity of corrective maintenance tasks, which is based on subjective evaluations of domain specific factors provided by programmers engaged in handling maintenance tasks. The model enables calculation of corrective maintenance task complexity, which can be further used for estimating the time needed to solve the tasks. The results can be useful for planning in everyday maintenance practice in the selected software company, but the study design can be implemented in other software companies by considering their specificity.

Several further work directions can be distinguished. The first direction relates to including other factors that influence software maintenance tasks in the analysis of task complexity. These factors might be characteristics of human factor involved in maintenance tasks (experience, knowledge of specific software technologies, domain knowledge, communication skills) and objective (quantitative) attributes of software applications such as structural complexity of maintained software systems. The second direction relates to developing more accurate model for task complexity by including preliminary analysis of empirical data and excluding from the analysis all data that significantly variate from a typical set of values. The third direction relates to implementation of the proposed model on other types of maintenance tasks in the selected company (enhancement and support tasks) which will enable more reliable planning and scheduling of all maintenance activities. And finally, adaptation of the presented model to other software organizations by considering their specificity is also challenging research direction that will provide further evaluation of the model usefulness.

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Maximal Centroidal Vortices in Triangulations. A Descriptive Proximity Framework in Analyzing Object Shapes

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Abstract

This paper introduces a framework for approximating visual scene object shapes captured in sequences of video frames. To do this, we consider the hyper-connectedness of image object shapes by extending the Smirnov proximity measure to more than two sets. In this context, a *shape* is a finite, bounded planar region with a nonempty interior. The framework for this work is encapsulated in descriptive frame recurrence diagrams, introduced here. These diagrams offer a new approach in tracking the appearance and eventual disappearance of shapes in studying the persistence of object shapes in visual scenes. This framework is ideally suited for a machine intelligence approach to tracking the lifespans of visual scene structures captured in sequences of images in videos. A practical application of this framework is given in terms of the analysis of vehicular traffic patterns.

Keywords: Hyper-connectedness, Object shape, Proximity, Recurrence, Vortices
2010 MSC No: Primary 54E05 (Proximity), Secondary 68U05 (Computational Geometry).

1. Introduction

A grasp of the persistence and wearout patterns of object shapes in visual scenes is important from a machine intelligence perspective, especially in terms of the increasing need for analytic methods to cope with the high volume of object shape data obtained by video capture devices that monitor our environment. This paper introduces a framework for approximating visual scene surface shapes captured in single digital images and in sequences of video frames. To do this,

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we consider the hyper-connectedness of image objects by extending the Smirnov proximity measure (Smirnov, 1964, §1, pp. 8-10) to more than two sets. It is the topology of cellular complexes introduced by P. Alexandroff (Alexandroff, 1965; Alexandroff & Hopf, 1935; Alexandroff, 1928, 1926), (extended and elaborated by G.E. Cooke and R.L. Finney (Cooke & Finney, 1967)), K. Borsuk (Borsuk, 1948, 1975; Borsuk & Dydak, 1980), a recent formulation of this topology by H. Edelsbrunner and J.L. Harer (Edelsbrunner & Harer, 2010) and the work on persistence homology by E. Munch (Munch, 2013) that provide a solid basis for this study of the object shapes and structures in visual scenes.

2. Preliminaries

A nonempty set K, such that each element in K is contained in a disjoint open set is termed a *Hausdorff space*. Every subset in the partion of K is a *cell*. The *boundary* of a cell A is denoted by $bdy\,A$, and its *closure* by $cl\,A$. The *interior* of a cell is defined as $int\,A = cl\,A - bdy\,A$. A *complex* σ is a collection of subsets in K. σ^n denotes a complex with n cells in K. The closure of a complex is its image under a continuous homomorphic map $f : \sigma^n \to \mathrm{cl}\,\sigma^n$. An *n-skeleton* K^n is the union of all $\sigma^j \in K$ such that $j \leq n$. A CW space is Hausdorff and satisfies the following two conditions:

1^o The closure of each cell $cl\,A\ s.t.\, A \in K$, intersects only a finite number of other cells (**Finite Closure**).

2^o A cell $A \in K$ is closed, provided $A \cap cl\,\sigma^n \neq \emptyset$ is also closed (**Weak Topology**).

Next, consider the structures inherent in a triangulated digital image. In a triangulation of a finite, bounded, planar region K, a collection of triangles A with a common vertex is called a *nerve* (denoted by Nrv$A$). The nerve with the highest number of triangles is called a *maximal nuclear cluster*(MNC). The intersection of an MNC is called the *nucleus*. Each of the triangles in an MNC is called a *1-spoke* (denoted by sk_1). The notion of a 1-spoke can be extended to a *k-spoke*,sk_k, using a recursive definition. All the sets that are not a sk_{k-1}, but have a nonempty intersection with a sk_{k-1} are a sk_k. The nucleus is a 0-spoke (denoted by sk_0). The union of all the $sk_k \in K$ forms a *k-spoke complex* denoted by $skcx_k$.

Nerve spokes lead to two new structures that are useful. A *maximal k-cycle* is a simple closed path connecting the centroids of all the $sk_k \in K$. A closed path has the same start and end points. A simple path has no self-intersections. As a triangualtion can have multiple MNCs, each one has its own maximal k-cycles. Let $mcyc_k(d)$ denote a maximal k-cycle associated with the MNC $d \in K$. A *maximal k-vortex* is the union of all the maximal j-cycles for an MNC $d \in K$,$mcyc_j(d)$, such that $j \leq k$. Let $mvort_k(d)$ be the maximal k-vortex for the MNC $d \in K$.

In a CW topology on a triangulated finite bounded region, two topological spaces are homotopic, provided they can be transformed into one another by means of continuous functions (no tearing and gluing incolved). A classical example is the transformation of a coffee cup to a doughnut and vice versa. An important result linking *nerves* with *homotopy* is the Edelsbrunner-Harer nerve theorem.

Theorem 1. *(Edelsbrunner & Harer, 2010, p. 59). A finite collection of closed, convex sets in Euclidean space, then nerve of the collection is homotopy equivalent to the union of sets in the nerve*

The notion of proximity can be extended from a relation on two as defined previously (Naimpally & Warrack, 1970)(Peters, 2013), to a binary valued function on $n > 2$ sets. This extended notion is termed **hyper-connectedness**. Suppose $A, B \in X$, then $A\delta B$, $A \overset{\wedge\wedge}{\delta} B$, $A\delta_\Phi B$ represent that A, B are spatial Lodato(δ), strongly($\overset{\wedge\wedge}{\delta}$) and as descriptively near ($A\delta_\Phi B$), respectively. Similarly, $A \not\delta B$, $A \overset{\wedge\wedge}{\not\delta} B$, $A\not\delta_\Phi B$ represent that the sets are spatial Lodato, strongly and descriptivly far respectivley. Proximity can also be quantified using the Smirnov proximity measure, defined as $\delta(A, B) = 0$, if the sets A and B are close and $\delta(A, B) = 1$ if the sets A and B are far from each other.

Recall the notation for hyper-connectedness in (Ahmad & Peters, 2017), by extending the Smirnov proximity measure to more than two sets. Suppose $X_1, \cdots, X_n \in X$, then $\delta^n(X_1, \cdots, X_n) = 0$ if they are near and $\delta^n(X_1, \cdots, X_n) = 1$ if they are far. The corresponding hyper-connectedness notions for the proximity relations discussed above are spatial Lodato δ^n, strong $\overset{\wedge\wedge}{\delta}{}^n$, and descripticve δ^n_Φ hyper-connectedness. The superscript n represents the number of sets regarding which the notion of proximity is being formulated. For the strong hyper-connectedness $\overset{\wedge\wedge}{\delta}{}^k$, the super-script $\wedge\wedge$ signifies intersection of the interiors required to satisfy this particular relation.

Let $\{A\}_i, B, C \in X$, where $i \in \mathbb{Z}$ is an index set. We define the hyper-connectedness as a function on set X. Moreover, if F is a set then $S(F)$ is the set of all the n-permutations of the elements in F, where $n = |F|$. As an example suppose $F = \{a, b, c\}$, then $S(F) = \{\{a, b, c\}, \{a, c, b\}, \{b, a, c\}, \{b, c, a\}, \{c, a, b\}, \{c, b, a\}\}$. Different types of hyper-connectedness require conformity to varying axioms. The spatial Lodato hyper-connectedness(δ^k) on k sets, requires the following axioms:

(hP1) $\forall A_k \subset X, \delta^k(A_1, \cdots, A_k) = 1,\ if\ any\ A_1, \cdots, A_k = \emptyset.$

(hP2) $\delta^k(A_1, \cdots, A_k) = 0 \Leftrightarrow \delta^k(Y) = 0,\ \forall Y \in S(\{A_1, \cdots, A_k\}).$

(hP3) $\bigcap_{i=1}^{k} A_i \neq \emptyset \Rightarrow \delta^k(A_1, \cdots, A_k) = 0.$

(hP4) $\delta^k(A_1, \cdots, A_{k-1}, B \cup C) = 0 \Leftrightarrow \delta^k(A_1, \cdots, A_{k-1}, B) = 0\ or\ \delta^k(A_1, \cdots, A_{k-1}, C) = 0.$

(hP5) $\delta^k(A_1, \cdots, A_{k-1}, B) = 0\ and\ \forall b \in B,\ \delta^2(\{b\}, C) = 0 \Rightarrow \delta^k(A_1, \cdots, A_{n-1}, C) = 0.$

(hP6) $\forall A \subset X,\ \delta^1(A) = 0$, a constant map.

Next, the definition of strong hyper-connectedness($\overset{\wedge\wedge}{\delta}{}^k$) on k sets, requires the following axioms:

(snhN1) $\forall A_k \subset X, \overset{\wedge\wedge}{\not\delta}{}^k(A_i, \cdots, A_k) = 1\ if\ any\ A_1, \cdots, A_k = \emptyset\ and\ \overset{\wedge\wedge}{\delta}{}^k(X, A_1, \cdots, A_{k-1}) = 0, \forall A_i \subset X.$

(snhN2) $\overset{\wedge\wedge}{\delta}{}^k(A_1, \cdots, A_k) = 0 \Leftrightarrow \overset{\wedge\wedge}{\delta}{}^k(Y) = 0,\ \forall Y \in S(\{A_1, \cdots, A_k\}).$

(snhN3) $\overset{\wedge\!\!\wedge}{\delta}^k(A_1, \cdots, A_k) = 0 \Rightarrow \bigcap_{i=1}^{k} A_i \neq \emptyset$.

(snhN4) If $\{B_i\}_{i\in I}$ is an arbitrary family of subsets of X and $\overset{\wedge\!\!\wedge}{\delta}^k(A_1, \cdots, A_{k-1}, B_{i^*}) = 0$ for some $i^* \in I$ such that $int(B_{i*}) \neq \emptyset$, then $\overset{\wedge\!\!\wedge}{\delta}^n(A_1, \cdots, A_{k-1}, (\bigcup_{i\in I} B_i)) = 0$.

(snhN5) $\bigcap_{i=1}^{k} intA_i \neq \emptyset \Rightarrow \overset{\wedge\!\!\wedge}{\delta}^k(A_1, \cdots, A_k) = 0$.

(snhN6) $x \in \bigcap_{i=1}^{k-1} int(A_i) \Rightarrow \overset{\wedge\!\!\wedge}{\delta}^k(x, A_1, \cdots, A_{k-1}) = 0$.

(snhN7) $\overset{\wedge\!\!\wedge}{\delta}^k(\{x_1\}, \cdots, \{x_k\}) = 0 \Leftrightarrow x_1 = x_2 = \cdots = x_n$.

(snhN8) $\forall A \in X, \overset{\wedge\!\!\wedge}{\delta}^1(A) = 0$ is a constant map.

Let us define the notion of a descriptive intersection, $A \underset{\Phi}{\cap} B = \{x \in A \cup B : \phi(x) \in \phi(A)\, and\, \phi(x) \in \phi(B)\}$. Here $\phi : K \to \mathbb{R}^n$ is a probe function whic can be seen as a feature extractor. Using these notions, the descriptive hyper-connectedness(δ_Φ^k) on k sets, has the underlying axioms:

(dhP1) $\forall A_i \subset X$, $\delta_\Phi^k(A_1, \cdots, A_k) = 1$ if any of the $A_1, \cdots, A_k = \emptyset$.

(dhP2) $\delta_\Phi^k(A_1, \cdots, A_k) = 0 \Leftrightarrow \delta_\Phi^k(Y) = 0\, \forall Y \in S(\{A_1, \cdots, A_k\})$.

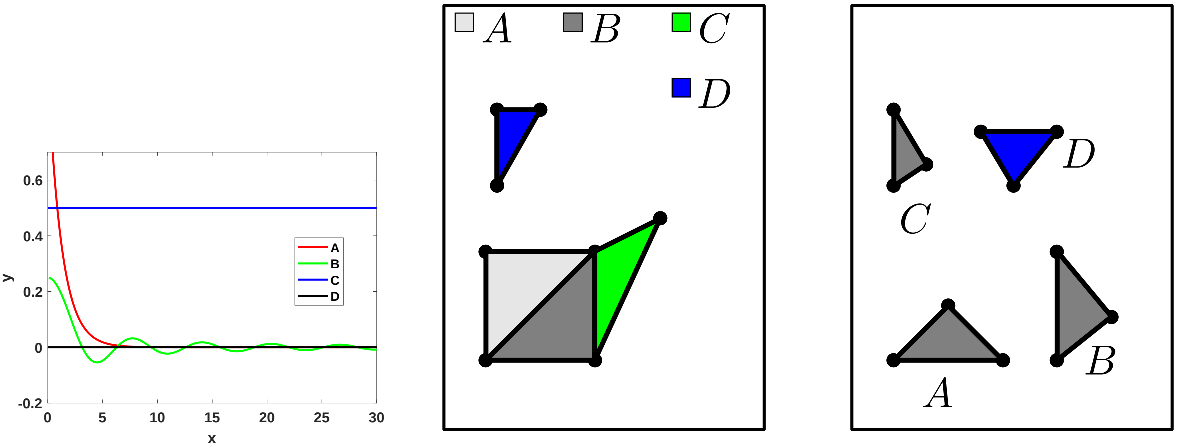
(dhP3) $\underset{\Phi}{\bigcap} A_i \neq \emptyset \Rightarrow \delta_\Phi^k(A_1, \cdots, A_k) = 0$.

(dhP4) $\delta_\Phi^k(A_1, \cdots, A_{k-1}, B) = 0\, and\, \forall b \in B$, $\delta_\Phi^2(\{b\}, C) = 0 \Rightarrow \delta_\Phi^k(A_1, \cdots, A_{k-1}, C) = 0$.

(dhP5) $\forall A \subset X, \delta_\Phi^1(A) = 0$ a constant map.

The distinctions between different notions of hyper-connectedness are important. The spatial Lodato(δ^k) version allows k sets to be near, provided the sets overlap or asymptotically approach each other. Strong hyper-connectedness($\overset{\wedge\!\!\wedge}{\delta}^k$) requires that the sets have non-empty intersection. The descriptive(δ_Φ^k) version allows for the sets to be near, provided the sets contain elements with matching descriptions under the probe functionϕ, regardless of their spatial proximity.

Example 1. *We begin with the notion of Lodato hyper-connectedness(δ^k). The most important thing to note here is that although sharing points implies δ^k, it is not necessary. In addition, asymptotic equality can also qualify sets for Lodato hyper-connectedness. Consider, for example, Fig. 1.1, where A, B, C, D are sets defined by $e^{-0.8x}$, $\frac{\sin x}{4x}$, 0.5, and 0. It must be noted that A approaches D asypmtotically, but B and D intersect at many points. Moreover, B does not share any elements with C, but, A and C intersect. Thus, we can write $\delta^2(A, C) = 0$, $\delta^3(A, B, C) = 0$ and $\delta^4(A, B, C, D) = 1$.*



1.1: Lodato hyperconnectedness 1.2: Strong hyperconnectedness 1.3: Descriptive hyperconnectedness

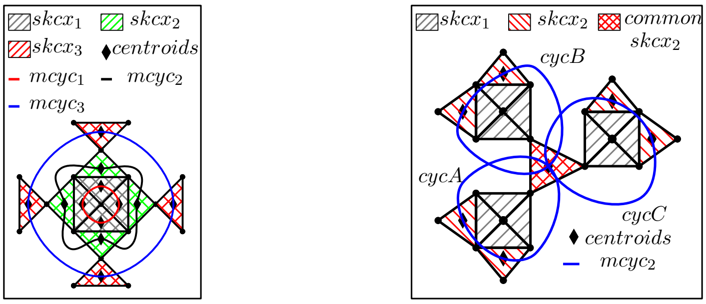
Figure 1: This figure illustrates the different variants of hyperconnectedness. Fig. 1.1 depicts Lodato hyperconnectedness δ^k. Fig.1.2 illustrates the strong hyperconnectedness $\overset{\wedge\!\wedge}{\delta^k}$. Fig. 1.3 shows descriptive hyperconnectedness δ^k_Φ.

The sets in Fig. 1.2 illustrate the notion of strong hyper-connectedness($\overset{\wedge\!\wedge}{\delta^k}$). Observe that A, B, C share a common vertex while D is disjoint from the remaining sets. We can hence conclude that $\overset{\wedge\!\wedge}{\delta^3}(A, B, C) = 0$ and $\overset{\wedge\!\wedge}{\delta^4}(A, B, C, D) = 1$. The sets in Fig. 1.3 illustrate descriptive hyperconnectedness. We can see that the filled triangles in Fig. 1.3 are spatially disjoint. Triangles A, B, C are coloured gray, while D is blue. This means that $\delta^3_\Phi(A, B, C) = 0$ and $\delta^4(A, B, C, D) = 1$.
■

Next, consider hyper-connectedness relationships in terms of spoke complexes and maximal centroidal cycles.

Example 2. *Let us first illustrate the idea of spoke complexes($skcx_k$) and the associated maximal centroidal cycles($mcyc_k$), using Fig. 2.1. The notion of maximal nuclear cluster(MNC) is identical to the concept of $skcx_1$ as defined in this section. The common intersection of the triangles in $skcx_1$ is the nucleus. It is shown as the four triangles shaded gray. The $skcx_2$ are represented as green, and have a non-empty intersection with the $skcx_1$ but not with the nucleus. Similarly, the $skcx_3$ are the red triangles that have a non-empty intersection with $skcx_2$ and an empty interesection with $skcx_1$. It can be seen that the closed simple path constructed by connecting the centroids of the triangles in a spoke complex is the corresponding maximal centroidal cycle. We can see that $mcyc_1$ is shown in red, $mcyc_2$ in black and $mcyc_3$ is in blue.*

Next, consider proximity and hyper-connectedness of maximal centroidal cycles associated with multiple MNCs in a triangulation. For this consider the illustration in Fig. 2.2. In this figure we have threee different MNCs with three disjoint $skcx_1$ represented by gray triangles. The red triangles(with slanted lines) are the $skcx_2$, but in this case the three MNCs share a triangle which is represented as a red triangle with a crosshatch pattern. We only consider the $mcyc_2$ for each of the three different MNCs A, B, C, represented as cycA,cycB and cycC.



2.1: Maximal Centroidal Cycles 2.2: Hyper-connectedness of *mcyc*

Figure 2: This figure illustrates the concept of maximal centroidal vortex. Fig. 2.1 displays the different maximal centroidal k-cycles, $mcyc_k$ in relation to the corresponding spoke complexes, $skcx_k$. Fig. 2.2 illustrates the notion of hyper-connectedness of *mcyc* for different MNCs.

From the proximity relations we have introduced, we can make a number observations. Observe that $skcx_2A,\ \delta\ skcx_2B$, $skcx_2A\ \delta\ skcx_2C$, $skcx_2B\ \delta\ skcx_2C$ *and* $\delta^3(skcx_2A, skcx_2B, skcx_2C)$. *Similarly, we can say* $mcyc_2A\ \delta\ mcyc_2B$, $mcyc_2A\ \delta\ mcyc_2C$, $mcyc_2B\ \delta\ mcyc_2C$ *and* $\delta^3(mcyc_2\ A, mcyc_2B, mcyc_2C)$. *As we have seen that the spoke complexes and the cycles share a triangle and centroid respectively, it can be concluded that they share the same description. This leads to* $skcx_2A\ \delta_\Phi\ skcx_2B$, $skcx_2A\ \delta_\Phi\ skcx_2C$, $skcx_2B\ \delta_\Phi\ skcx_2C$, $\delta^3_\Phi(skcx_2A, skcx_2B, skcx_2C)$, $mcyc_2A\ \delta_\Phi\ mcyc_2B$, $mcyc_2A\ \delta_\Phi\ mcyc_2C$, $mcyc_2B\delta_\Phi\ mcyc_2C$ *and* $\delta^3_\Phi(mcyc_2A, mcyc_2B, mcyc_2C)$. ■

3. Main Theoretical Results

Proximity and topology are two ways of talking about how a space is constructed from its subspaces. In this section, we introduce some proximity-related results regarding spoke complexes. Consider first a result for spatial Lodato hyper-connectedness(δ^n) on spoke complexes.

Theorem 2. *Let K a cell complex equipped with a Lodato hyper-connectedness relation. Let* $skcx_{k-1}, skcx_k, skcx_{k+1} \in K$ *be spoke complexes in K. Then*
1^o $skcx_k \cap skcx_{k+1} \neq \emptyset \Rightarrow \delta^2(skcx_k, skcx_{k+1}) = 0.$
2^o $skcx_k \cap skcx_{k-1} \neq \emptyset \Rightarrow \delta^2(skcx_k, skcx_{k-1}) = 0.$

Proof.

1^o It can be established that $skcx_k \cap skcx_{k+1} \neq \phi$ by definition of a spoke complex. Which implies $\delta^2(skcx_k, skcx_{k+1}) = 0$ as per axiom (**hP3**).

2^o It can be establised that $skcx_{k-1} \cap skcx_k \neq \phi$ by the definition of a spoke complex. Which implies $\delta^2(skcx_{k-1}, skcx_k) = 0$ as per axiom (**hP3**)

□

Next, consider a result pertaining to the descriptive Lodato hyper-connectedness(δ^n_Φ).

Theorem 3. *Let K a cell complex equipped with a descriptive hyper-connectedness relation. Let* $skcx_{k-1}, skcx_k, skcx_{k+1} \in K$ *be spoke complexes in K. Then*

1^o $skcx_k \cap skcx_{k+1} \neq \emptyset \Rightarrow \delta^2_\Phi(skcx_k, skcx_{k+1}) = 0$.
2^o $skcx_k \cap skcx_{k-1} \neq \emptyset \Rightarrow \delta^2_\Phi(skcx_k, skcx_{k-1}) = 0$.

Proof. 1^o It can be established that $skcx_k \cap skcx_{k+1} \neq \emptyset$ by definition of a spoke complex. Suppose $x \in skcx_k \cap skcx_{k+1}$, then $x \in skcx_k \cup skcx_{k+1}$ and $x \in \phi(skcx_k)$, $x \in \phi(skcx_{k+1})$. Hence, $skcx_k \underset{\Phi}{\bigcap} skcx_{k+1}$. From axiom (**hdP3**), this implies $\delta^2_\Phi(skcx_k, skcx_{k+1}) = 0$.

2^o It can be established that $skcx_k \cap skcx_{k-1} \neq \emptyset$ by definition of a spoke complex. Suppose $x \in skcx_k \cap skcx_{k-1}$, then $x \in skcx_k \cup skcx_{k-1}$ and $x \in \phi(skcx_k)$, $x \in \phi(skcx_{k-1})$. Hence, $skcx_k \underset{\Phi}{\bigcap} skcx_{k-1}$. From axiom (**hdP3**), this implies $\delta^2_\Phi(skcx_k, skcx_{k-1}) = 0$.

□

Theorems 2 and 3 give results for spatial($\delta^k = 0$) and descriptive($\delta^k_\Phi = 0$) hyper-connectedness, respectively. Consider next results for sub-complexes that are far either spatially($\delta^k = 1$) or descriptively($\delta^k_\Phi = 1$).

Theorem 4. *Let $(K, \{\delta^n, \delta^n_\Phi\})$ be a relator space equipped with two hyper-connectedness relations and let $skcx_j, skcx_k \in K$ be spoke complexes in the K, where $j, k \in \mathbb{Z}^+$. Then*
1^o $\|j - k\| \geq 2 \Leftrightarrow \delta^2(skcx_j, skcx_k) = 1$.
2^o $\|j - k\| \geq 2 \nRightarrow \delta^2_\Phi(skcx_j, skcx_k) = 1$.

Proof. 1^o Since this is a biconditional, we need to prove the implication in both directions. By the definition of spoke complex it can be established that $skcx_k \cap skcx_j \neq \emptyset \Leftrightarrow \|j - k\| \geq 2$. Using a result from (Naimpally & Warrack, 1970, pg. 7), if a space is equipped with a pseudo-metric and $D(A, B) = inf\{d(a, b) : a \in A, b \in B\}$, then $A\delta B$ iff $D(A, B) = 0$. Suppose K is a Euclidean space equipped with the Euclidean metric. It follows that in this triangulation $D(A, B) = 0 iff A \cap B \neq \emptyset$. Thus, $D(skcx_k, skcx_j) = 0$ iff $\|j - k\| < 2$ and $D(skcx_k, skcx_j) \neq 0$ otherwise. Hence, we can conclude that $\|j - k\| \geq 2 \Leftrightarrow \delta^2(skcx_j, skcx_k) = 1$.

2^o It can be established from the definition of the spoke complex that any $\sigma^2 \in K$, that is not in $skcx_k$ but has a non-empty intersection with $skcx_k$ is an element of $skcx_{k+1}$. Thus, it can be established that $skcx_j \cap skcx_k \neq \emptyset$ iff $\|j - k\| \leq 1$. Which means $j \in \{k - 1, k, k + 1\}$. From this we can establish that $skcx_j \cap skcx_k = \emptyset$ if $\|j - k\| \geq 2$. From the definition of $\underset{\Phi}{\cap}$, it follows that $A \cap B \Rightarrow A \underset{\Phi}{\cap} B$, and $A \underset{\Phi}{\cap} B \nRightarrow A \cap B$. Thus, it is possible that $A \underset{\Phi}{\cap} B$ eventhough $A \cap B = \phi$. Thus, it is still possible that $skcx_j \underset{\Phi}{\cap} skcx_k \neq \emptyset$ even if $\|j - k\| \geq 2$. From axiom (**hdP3**), we can conclude that, $\delta^2_\Phi(skcx_j, skcx_k) = 0$ even if $\|j - k\|$. Hence proved.

□

The notion of proximity in a space can be defined using a function. Let us cosntruct a function which quantifies the proximity of a subset to the nucleus of a MNC in the triangulation.

Definition 1. *Let K be a triangulation, $d \subset K$ be the nucleus and $A, skcx_k \subset K$. Then, $\mu_d(A) : K \to \mathbb{Z}^+$ is a function satisfying*

$$A \subset skcx_k \Leftrightarrow \mu_d(A) = k.$$

We can use this function to quantify the proximity between pairs $\sigma^2 \in K$ by using the MNC as a reference point.

Definition 2. *Let A, B are two $\sigma^2 \in K$, μ_d is a function as defined in def. 1. We can define*

$$\mu_d(A, B) = ||\mu_d(B) - \mu_d(A)||.$$

It can be shown that $\mu_d(x, y)$ is a pseudo-metric.

Theorem 5. *Let K be a triangulation of a finite, bounded planar region, $A, B, C \in K$ be σ^2, and let $d \subset K$ be the MNC. We can define a function $\mu_d(A, B)$ as per def. 2. Then*
1^o $\mu_d(A, A) = 0$.
2^o $\mu_d(A, B) = \mu_d(B, A)$.
3^o $\mu_d(A, C) \leq \mu_d(A, B) + \mu_d(B, C)$.
Hence, $\mu_d(A, B)$ is a pseudo-metric.

Proof. 1^o is true by definition, since $\mu_d(A, A) = ||\mu_d(A) - \mu_d(A)||= 0$.
2^o is true by definition, since $\mu_d(A, B) = ||\mu_d(B) - \mu_d(A)||= ||\mu_d(A) - \mu_d(B)||= \mu_d(B, A)$
3^o To prove this, consider the following two cases. First, let B, C are in the same spoke complex and A in a different complex, $B, C \in skcx_k$, and $A \in skcx_j$. Then it is easy to see that $\mu_d(A, C) = \mu_d(A, B) + \mu_d(B, C)$, as $\mu_d(B, C) = 0$, $\mu_d(A, B) = \mu_d(B, A) = ||j - k||$.
For the second case, let A, B, C be in different spoke complexes, where $A \in skcx_j$,$B \in skcx_k$ and $C \in skcx_l$. For simplicity, let us divide this into two subcases. Let us begin with $j < k < l$. It can be seen that $\mu_d(A, C) = \mu_d(A, B) + \mu_d(B, C)$, as $\mu_d(A, C) = ||l - j||$, $\mu_d(A, B) = ||k - j|$ and $\mu_d(B, C) = ||l - k||$. The second subcase is $j < l < k$. It can be seen that $\mu_d(A, C) < \mu_d(A, B) + \mu_d(B, C)$, as $\mu_d(A, C) = ||l - j||$, $\mu_d(A, B) = ||k - j|$ and $\mu_d(B, C) = ||l - k||$. Hence, we have proved that $\mu_d(A, C) \leq \mu_d(A, B) + \mu_d(B, C)$.

□

It can be established that $\mu_d(A, B)$ is not a metric, since $\mu_d(A, B) = 0$ for any two distinct σ^2 in the same spoke complex $A, B \in skcx_k$, as per def. 1. A spoke complex is a subcomplex of the original triangulation K. It is important to note that as per the def. 1, every $\sigma^2 \in skcx_k$ has the same proximity, as quantified by μ_d, to the MNC. Moreover, the $\sigma^2 \in skcx_k$ have $\mu_d = 0$ with each other. Let us define some new notions using the function μ_d.

Definition 3. *Let K be a CW complex, μ_d be the function as per def. 1. We only relax the codition that d is a nucleus of an MNC and let it be any arbitrary $\sigma \in K$. Then*

$$K^n_{\mu,d} = \{\sigma^2 : \sigma^2 \in K \, and \, \mu_d(\sigma^2) = n\}$$

is a n-proximal subcomplex of K w.r.t base point d.

We can term all the n-proximal subcomplexes as *iso-proximal complexes*, since all of their elements the same proximity, as quantified by μ_d to the base point. It is also important to note the following result.

Theorem 6. *Let K be a triangulation, $d \subset K$ be the nucleus and $skcx_k \in K$ be the k-spoke complex. Then,*

$$skcx_k \iff K^k_{\mu,d}$$

Proof. This follows directly from defs. 1 and 3. □

We specify the notation for a cycle in a triangulation. Suppose $a, b, c \in K$ are three vertices, then (abc) is a cycle such that there is a path from a back to a, passing through b and c. Moreover, $cntr(a)$ represents the centroid of set a. Let us define the notion of an iso-proximal cycle using the function μ_d.

Definition 4. *Let K be a CW complex, μ_d be the function from def. 1. We relax the condition that d is the nucleus of a MNC and let it be an arbitrary $\sigma \in K$. Let us first generate an index set of the $\sigma^2 \in K^k_{\mu,d}$ as follows*

$$\mathcal{K} = \{k_i : k_i = \sigma^2 \in K^k_{\mu,d}, k_i = k_j \iff i = j, i \in \mathbb{Z}^+\}.$$

Now, using this index set we can define the notion

$$cyc^n_{\mu,d} = \{(a_1 a_2 \cdots a_n) : a_i = cntr(k_i)\ s.t.\ k_i \in \mathcal{K}\}.$$

Here, $cyc^n_{\mu,d}$ is a $n-$proximal cycle of K relative to the base point d.

Similar to the notion of n-proximal complex, a n-proximal cycle can be termed an iso-proximal cycle. Next, consider the following result.

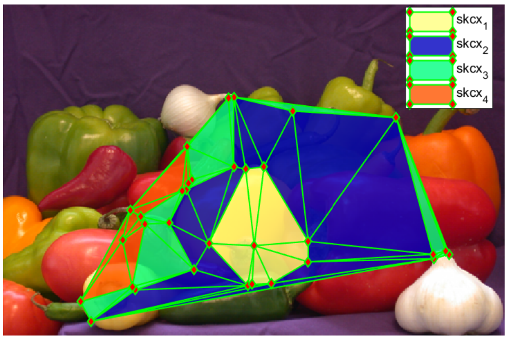
Theorem 7. *Let K be a triangulation, $d \subset K$ be the nucleus and $mcyc_k d$ be the maximal k-cycle. Then*

$$mcyc_k d \iff cyc^k_{\mu,d}.$$

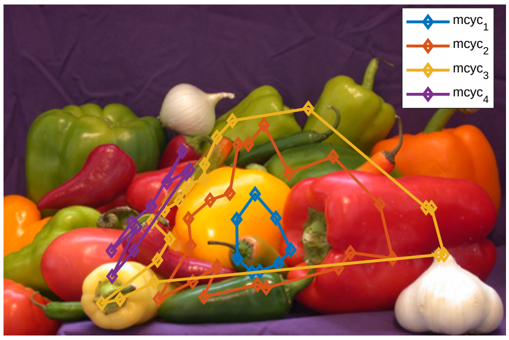
Proof. A maximal k-cycle, $mcyc_k$, is a simple closed path connecting the centroids of all the $sk_k \in K$. Comparing this with Def. 4, it follows that maximal k-cycle is a $k-$proximal cycle. □

To help clarify the concepts we have introduced, we give an example.

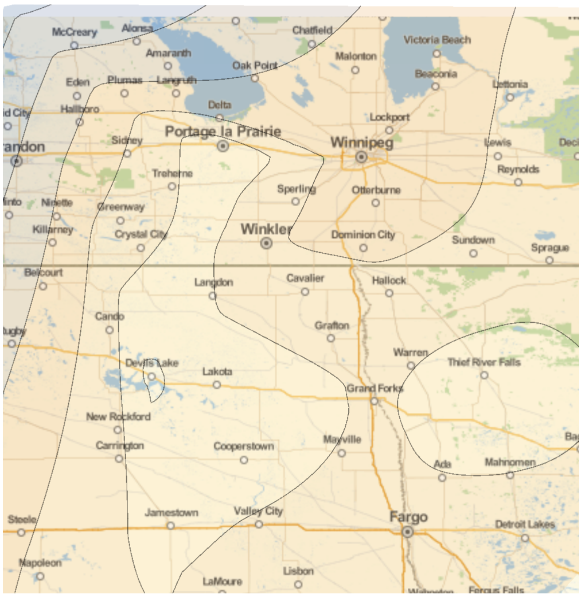
Example 3. *We illustrate the spke complexes in a stock digital image peppers.png in MATLAB®. Now consider theorem 6 stating the equivalence between the k^{th} spoke complex and $k-$proximal cycle w.r.t. nucleus as the base point. Thus, $skcx_1$ shown in yellow is $K^1_{\mu,d}$, $skcx_2$ (blue) is $K^2_{\mu,d}$, $skcx_3$ (green) is $K^3_{\mu,d}$ and $skcx_4$ (orange) is $K^4_{\mu,d}$. Let us now move on to theorem 7, which states that k^{th} maximal centroidal cycle is the $k-$proximal cycle w.r.t. nucleus as the base point. Thus, $mcyc_1$ shown in blue is the $cyc^1_{\mu,d}$, $mcyc_2$(red) is $cyc^2_{\mu,d}$, $mcyc_3$(orange) is $cyc^3_{\mu,d}$ and $mcyc_4$(indigo) is $cyc^4_{\mu,d}$. Now that we have seen how the iso-proximal complexes and cycles are defined in digital images, let us look at a closely associated concept in meteorology. It is the conept of an isobar, which is a line connecting points with the same barometric pressure.*



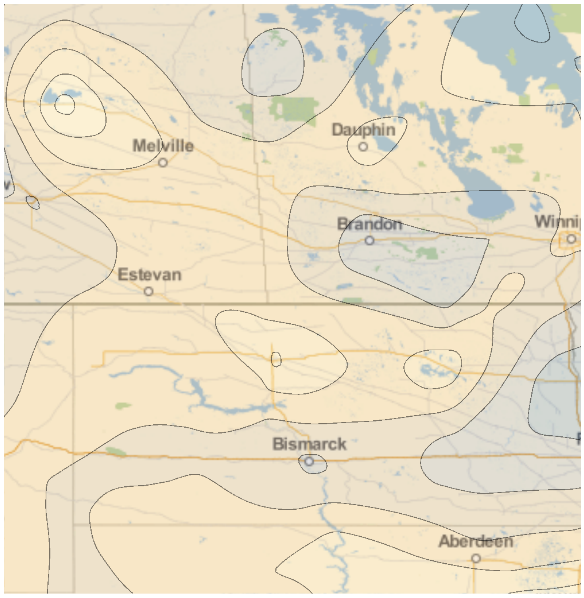
3.1: $skcx_k$ in digital image



3.2: $mcyc_k$ in digital image



3.3: Isobars in Weather Maps



3.4: Isotherms in Weather Maps

Figure 3: This figure illustrates the concepts of iso-proximal complexes and cycles. Fig. 3.1 shows the $skcx_k$, each of which is an iso-proximal complex. Fig. 3.2 shows $mcyc_k$, each of which is an iso-proximal cycle. Fig. 3.3 is an example of isobars in weather map, generated using Mathematica. Fig. 3.4 is an example of isotherms in weather map, generated using Mathematica.

Replace μ *in the definition of* $cyc^k_{\mu,d}$, *which is a function that measures proximity w.r.t. point d, with the function that defines barometeric pressure over the surface of earth. We can then get isobars in a region around Winnipeg using Mathematica as shown in Fig. 3.3. Each of the black lines connects regions with similar pressure. The value of barometric pressure is different for each line. Again in the case of isotherms, we have lines connecting regions with identical tempratures. Next, we can obtain a similar effect by replacing* μ *with a function that defines air temperature over the surface of earth. Isotherms in the region around Winnipeg are shown in Fig. 3.4. These are also generated using Mathematica. Similarly, each of the black lines corresponds to a different temprature. The regions on each line have same value of air temprature.* ■

4. Descriptve Frame Recurrence Diagrams

This section introduces what are known as descriptive frame recurrence diagrams derived from triangulated video frames. For construction of the Delaunay trianglation of video frames, we use hole based keypoints. A hole is defined as a region of constant intensity in a digital image. For this purpose, we use the gradient magnitude for detection and filter out small holes based on the number of pixels contained in them. The process is detailed in Alg. 1. These keypoints are used

Algorithm 1: Hole based Keypoints

Input : digital image img, Horizontal filter radius r_x, Vertical filter radius r_y, Hole threshold t, Number of holes n_{hole}

Output: Hole locations $\mathscr{K}_{holes}$

$g := empty\ matrix$;

foreach $pixel \in img$ **do**

  $g(i, j) \leftarrow Gradient\ Magnitude\ at\ (i, j)$;

$g :=$ *set all values of* $g < t$ *to* 1 *and rest to* 0;

$g \longmapsto$ *connected components*;

connected components $\longmapsto$ *size in terms of pixels*;

/* arrange in descending order w.r.t. size in terms of pixels */;

connected components $\longmapsto$ *arranged connected components*;

hole $\leftarrow$ *first* n_{hole} *arranged connected components*;

hole $\longmapsto$ *centroids*;

$\mathscr{K}_{holes} \leftarrow centroids$;

to generate a triangulation, in which the spoke complexes and the maximal centroidal vortices are identified. The process of detecting the spoke complexes starts with the identification of MNCs. The nucleus($skcx_0$) in a triangulation are the vertices that are common to the greatest number of triangles. The nuclei along with the repective triangles containing them are the MNCs($skcx_1$). The triangles that are excluded in $skcx_i$ for $i = 0, 1$, and share interesections(vertices, edges) with $skcx_1$ are included in the $skcx_2$. $skcx_k$ for any $k > 2$ can be constructed in a similar fashion. The process stops when all the triangles in the triangulation have been assigned to a particular level k in the spoke complex. It is to be noted that we have separate $skcx_k$ for each of the nuclei.

The construction of centroidal maximal centroidal cycles follows from this. The $mcyc_k$ is a closed simple path that connects the centroids of triangles included in $skcx_k$. There is a slight problem regarding the arrangements of centroids so as to form a non-intersecting cycle. For this we caluclate the centroid of vertices in a particular $mcyc_k$, represented as $cntr_{cyck} = (cntr^x_{cyck}, cntr^y_{cyck})$. Then for each vertex $v_i = (v^x_i, v^y_i) \in mcyc_k$ calculate arctan $\frac{v^y_i - cntr^y_{cyck}}{v^x_i - cntr^x_{cyck}}$. Going through the vertices in order of ascending values of this quantity results in a simple closed path. The collection of all the $mcyc_k$ is the maximal centroidal vortex.

In Theorems 6 and 7, we have established that each of the spoke complexes is a cell complex is an iso-proximal complex and maximal centroidal cycles are iso-proximal cycles. These structures encode the spatial proximity of sub-regions in a triangulated finite, bounded planar region. This is an alternate way to study the topology or the shape of a space. Traditionaly, the interior of a shape is considered to have binary nature. It is either empty or nonempty, which paves the way for shape interiors with subregions that are holes. This is a narrow view which is ill-suited to the study of digital images which have a rich interior that is instrumental in understanding and analysis. An earlier attempt at overhauling the classical methods of homology(classification of shapes based on the holes) was taken up in (Ahmad & Peters, 2018). Using the notion of iso-proximal cycles and complexes the description of a shape interior can be fused with a consideration of spatial proximities.

Next, we introduce structures to integrate proximity structure of triangulation with the description of interiors. For this purpose, we introduce descriptive maximal centroidal cycles. A fibre bundle is a structure (E, B, π, F), where $\pi : E \to B$ is a continuous surjection, E is the *total space*, B is the *base space* and $F \subset E$ is the *fiber*. Using this framework we can define two different structures. The first of these is

Definition 5. *Let $mcyc_k$ be a maximal k-cycle, $skcx_k$ a $k-$spoke complex in a triangulation CW complex K. Let us define a function,*

$$\phi_{vrt} : 2^K \to \mathbb{R}^n,$$
$$a \mapsto \begin{cases} \phi_{vrt}(a) & if\, a = cntr(\triangle \in skcx_k), \\ 0 & otherwise, \end{cases}$$

Then the descriptive maximal k-cycle (denoted by $mcyc^{\phi}_k$) is a fiber bundle $(mcyc^{\phi}_k, mcyc_k, \pi, \phi_{vrt}(U))$, where $U \subset mcyc_k$.

A variant of the above is

Definition 6. *Let $mcyc_k$ be a maximal $k-$ cycle, $skcx_k$ be a $k-$spoke complex in a triangulation K. Let us define a function,*

$$\phi_{avg} : 2^K \to \mathbb{R}^n$$
$$a \mapsto \begin{cases} \frac{\sum \phi(a)}{|a|} & if\, a = cntr(\triangle \in skcx_k), \\ 0 & otherwise, \end{cases}$$

Then the region descriptive maximal k-cycle ($mcyc_k^{\bar{\phi}}$), is a fiber bundle ($mcyc_k^{\bar{\phi}}, mcyc_k, \pi, \phi_{vrt}(U)$), where $U \subset mcyc_k$.

The two proposed structures, namely the descriptive($mcyc_k^{\phi}$) and region descriptive maximal cycle($mcyc_k^{\bar{\phi}}$), differ only slightly. The basic idea behind these structures is that each vertex in a $mcyc_k$ corresponds to a trinagle in the $skcx_k$. We have established via Thm. 6 that each of the triangles in a $skcx_k$ has the same proximity to the nucleus. We have also established vis Thm. 7 that each of the vertices in a $mcyc_k$ is the centroid of a triangle in $skcx_k$. Thus, both $skcx_k$ and $mcyc_k$ encode the spatial proximity of the triangulation.

Next, we choose $mcyc_k$ as the base space and then assign to each vertex a description using the probe functions ϕ_{vrt} or ϕ_{avg}. The only differennce between the two probes is that ϕ_{vrt} assigns to a vertex the description at centroid of the corresponding triangle in $skcx_k$, while ϕ_{avg} assigns the average description of all the subregions(pixels) of the triangle. These descriptions can be arranged in the form of a vector for each of the $mcyc_k$ or combine all into a single vector. The basic feature that we will be using in this study is wavelength,λ. It is a nonlinear function of hue. The first step in the calculation of λ is the conversion of RGB image to HSV (Hue Saturation Value). Then we transform the hue channel(h) according to the following equation which is an approximation of the nonlinear mapping.

$$\lambda(i,j) = \begin{cases} 435nm & \text{if } h(i,j) > 0.7483, \\ \frac{-(h(i,j)-2.60836)}{0.004276} & otherwise. \end{cases} \tag{4.1}$$

Here we assume that the hue values are scaled between $[0, 1]$. The wavelengths caluclated by this equation (measured in nanometers nm, *i.e.*, 10^{-9} meter) are limited to the range $[435nm, 610nm]$.

A consideration of hue wavelength gives a useful feature vector useful in the study of shapes in video frames. The i^{th} image(frame) is represented as $\mathcal{V}_i$. Tracking similar frames in a video is important. We use the framework of descriptive similarity with the two probe functions defined in Defs. 5,6. To discuss regarding similarity of frames we define a feature vector,

$$\eta_\phi(\mathcal{V}_i) = \{\frac{\sum\limits_{\forall \triangle \in skcx_k} \phi(mcyc_k(\mathcal{V}_i))}{|skcx_k|} : \text{for } k \in \mathbb{Z}^+, \text{and } \phi = \phi_{avg} \text{ or } \phi_{vrt}\}, \tag{4.2}$$

where $|skcx_k|$ is the number of triangles in $skcx_k$ and $mcyc_k(\mathcal{V}_i)$ is a maximal centroidal cycle in frame i of the video. Two frames are similar if,

$$\delta_\Phi^2(\mathcal{V}_i, \mathcal{V}_j) = 0 \iff \|\eta_\phi(\mathcal{V}_i) - \eta_\phi(\mathcal{V}_j)\|_2 \leq th, \tag{4.3}$$

where th is a suitable threshold empirically determined.

Since it is possible for an image to have multiple maximal vortices, we compute the value of ϕ for each vortiex and then compare the value for all the possible combinations. If any of the multiple vortices in an image are similar to any in the other image, then these frames are said to be similar. Let us plot $\delta_\Phi^2(V_i, V_j)$ for $i, j = 1, \cdots, |\mathcal{V}|$, where $|\mathcal{V}|$ is the number of frames in the video. As δ_Φ^2 is a binary relation we only mark the locations for which $\delta_\Phi^2(\mathcal{V}_i, \mathcal{V}_j) = 0$. Moreover, due to the symmetry of Euclidean distance($\|\cdot\|_2$}) we can ignore the lower half below the diagonal.

Moreover, as each frame is similar to itself, the diagonal is always marked. This leads to what we call a *descriptive frame recurrence diagram* and mark it as $\mathcal{R}_\Phi(\mathcal{V}, th)$ for video $\mathcal{V}$ and threshold th. We present a formal definition of the descriptive frame recurrence diagram.

Definition 7. *Let $\mathcal{V}$ be a video η_ϕ be a feature vector as defined in Eq. 4.2 . Then,*

$$\mathbb{R}_\Phi(\mathcal{V}, th) = \{(i, j) : \|\eta_\phi(\mathcal{V}_i) - \eta_\phi(\mathcal{V}_j)\|_2 \leq th, j \geq i \, and \, i, j = 1, \cdots, |\mathcal{V}|\},$$

where $th \in \mathbb{R}^+$ and $|\mathcal{V}|$ is the number of frames in the video $\mathcal{V}$. The set $\mathcal{R}_\Phi(\mathcal{V}, th)$ is the descriptive frame recurrence diagram.

We combine all the tools presented above in a framework for video processing, illustrated in Fig. 4. Calculation of $skcx_k$ and $mcyc_k$ for all the frames of a video is the starting step. Once, we have the $mcyc_k$ we calculate $\eta_\phi(\mathcal{V}_i)$ as defined in Eq. 4.2. For this step we use wavelength,λ defined in Eq. 4.1, as the probe function ϕ. The last step is to calculate the recurrance diagram $\mathcal{R}_\Phi(\mathcal{V}, th)$ as defined in Def. 7. Thus, for every frame $\mathcal{V}_i$ a recurrance diagram tells values of $j > i$, such that $\delta^2_\Phi(\mathcal{V}_i, \mathcal{V}_j) = 0$.

5. Application of Descriptive Frame Recurrence Diagrams

This section introduces an application of descriptive frame recurrence diagrams in terms of the occurrence of similar frames in a video. In this section, we will present a pair of frames detected as similar using each of the probe functions defined in Defs. 5,6. Then we finish the section with a pair frames that are not similar for both probe functions.

We start the discussion with the vertex-based probe function defined in Def. 5. In this probe as discussed in Sec. 4, we only use the value of probe function at the vertices of cycles(centroids of triangles in spoke complexes). The feature that we are interested in is the wavelength as calculated using the Eq. 4.1. The results are displayed in the Fig. 5.

We calculate the descriptive frame recurrence diagram for the video $\mathcal{V}$. It is represented as $\mathcal{R}_\Phi(\mathcal{V}, th)$ and is shown in Fig. 5.1. For this study we set $th = 5$. It can be seen that apart from the diagonal only four other points show up in $\mathcal{R}_\Phi(\mathcal{V}, th)$. Since every frame is similar to itself we only look at the points apart from the diagonal. We select a pair of frames((15, 87)) marked with a red circle in Fig. 5.1. We display frame 15 in Fig. 5.2 and frame 87 in Fig. 5.3.

Next we display the $skcx_k$ and corresponding $mcyc_k$, $k = 1, \cdots, 4$, for both the frames. Figs. 5.4, 5.5 display $skcx_1$ and corresponding $mcyc_1$ for frame 15 and 87 respectivly. It can be seen that both the $skcx_1$ and $mcyc_1$ are identical for both the frames. $skcx_2$ and $mcyc_2$ are also identical for both the frames as shown in Figs. 5.6,5.7. The $skcx_3$ and corresponding $mcyc_3$ for frame 15 is in Fig. 5.8, and $skcx_3$ and $mcyc_3$ for frame 87 is illustrated in Fig. 5.9. The $skcx_3$ and $mcyc_3$ are slightly different. Figs. 5.10,5.11 show the $skcx_4$ and $mcyc_4$ for both the frames. It is evident from the figures that spoke complex at level 4 and the corresponding maximal centroidal cycles differ slightly for the frames.

Now that we have seen the $mcyc_k$ for both the frames, let us comapre the values of ϕ_{vrt} for different cycles across the two frames. For ease of comparison we take the average value of ϕ_{vrt} calculated at all the vertices in an $mcyc_k$ for a particular k. We plot these values as a bar graph

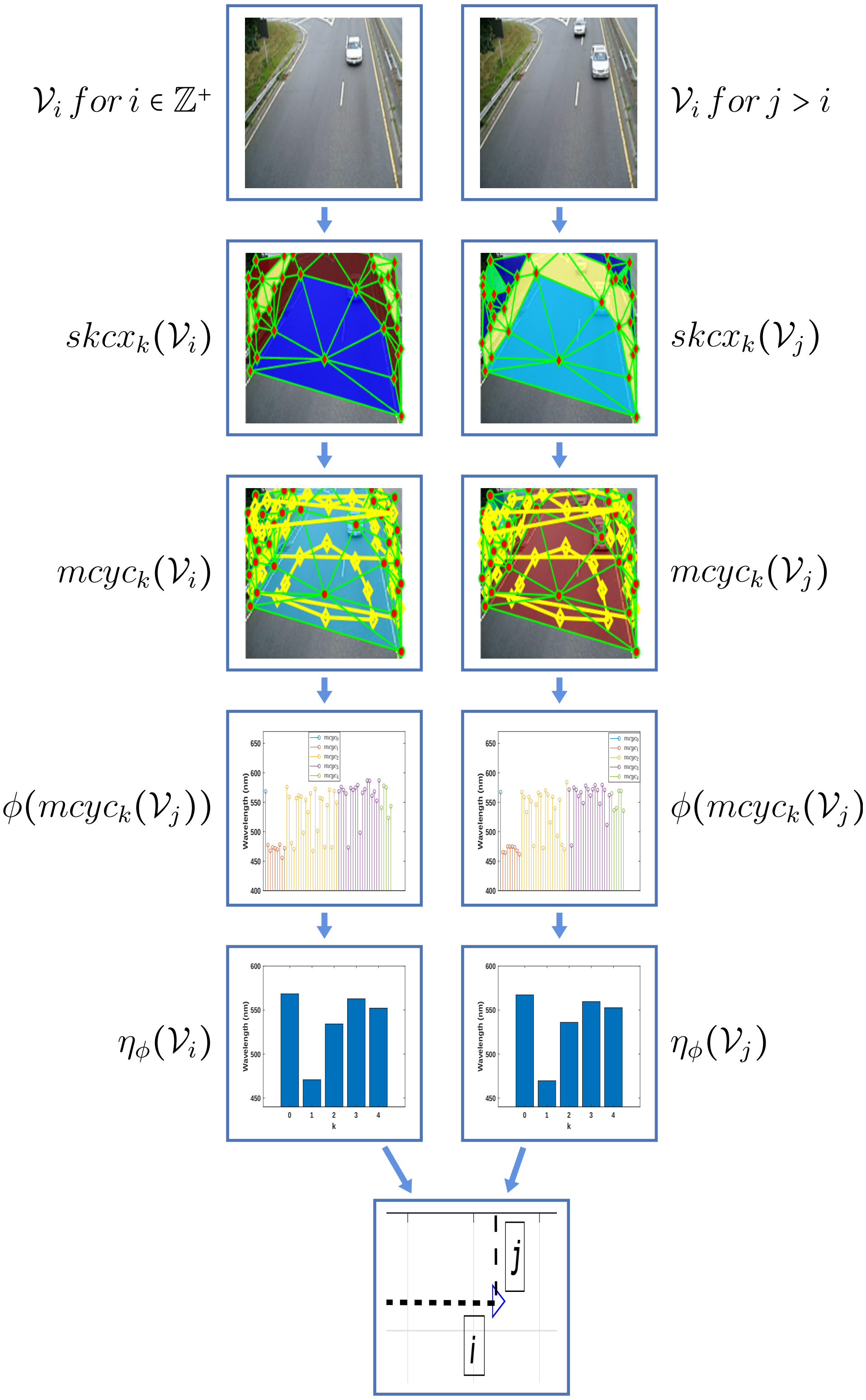
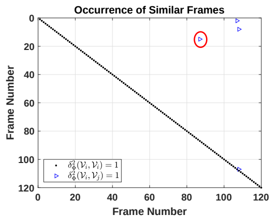
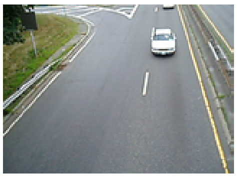


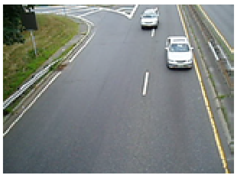
Figure 4: This flow diagram illustrates the methodology for constructing a descriptive frame recurrance diagram, $\mathcal{R}_\Phi(\mathcal{V}, th)$. Repeating thsi process for all valid pairs of (i, j), such that $i = 1, \cdots, |\mathcal{V}|$ and $j > i$. Here, $\mathcal{V}$ represents the number of frames in the video.



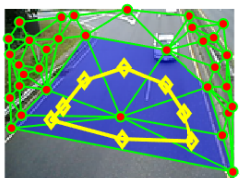
5.1: Descriptive Frame Recurrance



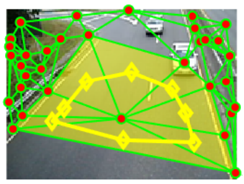
5.2: Frame 1



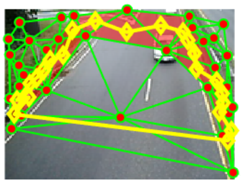
5.3: Frame 2



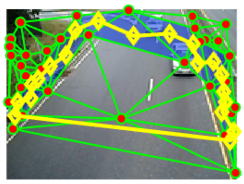
5.4: $skcx_1$ and $mcyc_1$ in Frame 1



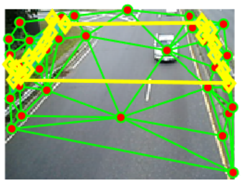
5.5: $skcx_1$ and $mcyc_1$ in Frame 2



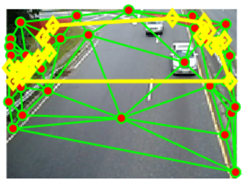
5.6: $skcx_2$ and $mcyc_2$ in Frame 1



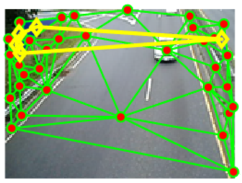
5.7: $skcx_2$ and $mcyc_2$ in Frame 2



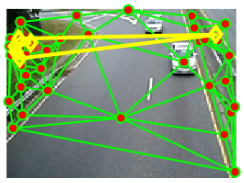
5.8: $skcx_3$ and $mcyc_3$ in Frame 1



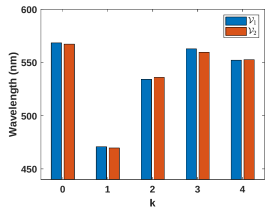
5.9: $skcx_3$ and $mcyc_3$ in Frame 2



5.10: $skcx_4$ and $mcyc_4$ in Frame 1



5.11: $skcx_4$ and $mcyc_4$ in Frame 2



5.12: Mean Wavelengths for both images

Figure 5: This figure illustrates the application of the framework developed in sec. 4. In this illustration we use the values of wavelengts at centroids only. Fig. 5.1, represents the descriptive frame recurrance diagram. The red circle annotates the frame pair being analyzed. Fig. 5.2,5.3 are the frames that have been detected as descriptivly similar. Figs. 5.4,5.6,5.8,5.10 represent $skcx_k$ and corresponding $mcyc_k, k = 1, \cdots, 4$ for frame in Fig. 5.2. Figs. 5.5,5.7,5.9,5.11 represent $skcx_k$ and corresponding $mcyc_k, k = 1, \cdots, 4$ for frame in Fig. 5.3. Figs. 5.12 represents the average wavelengts calculated at each spoke level k. $skcx_0$ and $mcyc_0$ are the nucleus. The frames are similar in terms of these values as can be seen in this figure.

comparing the frames in Fig. 5.12. It can bee seen that values almost identical for the different $mcyc_k$ across the frames. We conclude that $\delta^2_\Phi(\mathcal{V}_{15}, \mathcal{V}_{87}) = 0$. Most noticable difference occurs for $k = 3$. It must be noted that both the images shown in Fig. 5.2,5.3 have multiple vortices.

In the Fig. 5 we only show cycles across the frames that were detected as being descriptivly similar(δ^2_Φ). It is in teresting to note that frame 15 and 87 are almost identical apart from the fact that later has two cars. The position of cars, their shape and color are almost identical. The fact that location, shape and the description of these vortices are similar is due to the similarity in images.

Next, consider the case of ϕ_{avg}, region based probe defined in 6. The result for this method are displayed in Fig. 6. The structure of the figure is similar to one adopted for Fig. 5. We start by showing the descriptive frame recurrence diagram $\mathcal{R}_\Phi(\mathcal{V}, th)$ in Fig. 6.1. The value of $th = 5$ is the same for this study also. It can be noted that apart from the diagonal there are eight other points in the diagram, representing a pair of frames that is descriptivly similar. The number of off-diagonal points was only four for the case(Fig. 5) where neighborhood was not used. This can be easily explained as the ϕ_{vrt} only takes into account the value of feature at a single point(the centroid of triangles).

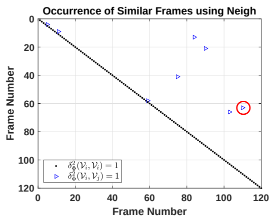
The value at a single pixel in an image can vary due a number of reasons e.g. quantization errors, noise, motion artifacts and sudden changes in illumination at the scene. Thus there is a chance that similar frames can be detected as disimilar when using ϕ_{vrt}. When we use the neighborhood, noise and illumination effects cancel due to averaging over a region. This can also lead to two different frames being classified as similar if the changes are small enough to be destroyed in averaging.

Consider next a marked as a red circle in Fig. 6.1, namely, the frame 63 shown in Fig. 6.2 and the frame 110 in Fig. 7.3. There are some similarities and differences in the frames. Frame 63 has three cars while the frame 110 has only one car. The car in frame 110 is similar to one of the cars in frame 63 but not identical and it is in the different lane.

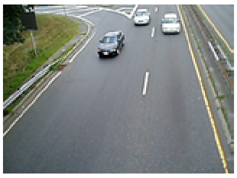
This pair of frames have similar vortices due to the fact that we are looking at wavelengths in a region as opposed to at a point. The cars are black, the same color as the road, which results in their being detected as similar in terms of wavelength averaged over the spoke triangles. Let us look at the $skcx_k$ and the corresponding $mcyc_k$ for both the frames. The Figs. 6.4,6.6,6.8,6.10,6.12 illustrate $skcx_k$ and $mcyc_k$ for $k = 1, \cdots, 5$, in frame 63(Fig.). Figs. 6.5,6.7,6.9,6.11,6.13 represent $skcx_k$ and $mcyc_k$ for frame 110. We can observe that all the spokes vary slightly interms of structure.

This structural vairation is due to the slight difference in the triangulation of frames. The reason that spoke lie in almost the same area accounts for the similarity. The black cars occurs in different spoke levels but their color is almost similar to color of road thus the difference is not registered when we average over the spoke. We ahve seen in Eq. 4.1 that the wavelength is a fucntion of hue or the color. This fact is further established when we look at the average wavelengths for each of the $mcyc_k$ and compare them across the two frames in Fig. 6.14.

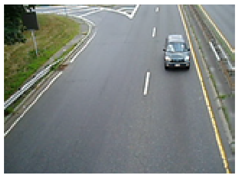
There are slight difference in the values for each value of k and the overall difference is smaller than the threshold. Hence $\delta^2_\Phi(\mathcal{V}_{63}, \mathcal{V}_{110}) = 0$. This is an example of the flexibility yielded by the neighborhood based probe function(ϕ_{avg},Def. 6) yields as compared to the vertex based probe(ϕ_{vrt},Def. 5). It can be seen that frames detected as similar by ϕ_{vrt} are almost identical in all



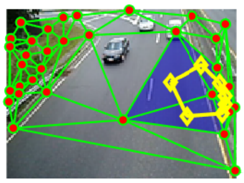
6.1: Descriptive Frame Recurrance



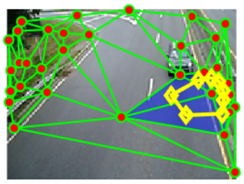
6.2: Frame 1



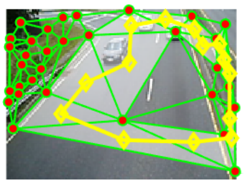
6.3: Frame 2



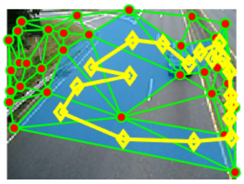
6.4: $skcx_1$ and $mcyc_1$ in Frame 1



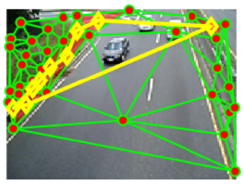
6.5: $skcx_1$ and $mcyc_1$ in Frame 2



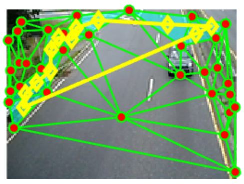
6.6: $skcx_2$ and $mcyc_2$ in Frame 1



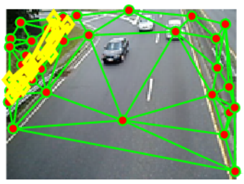
6.7: $skcx_2$ and $mcyc_2$ in Frame 2



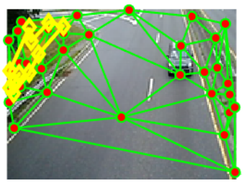
6.8: $skcx_3$ and $mcyc_3$ in Frame 1



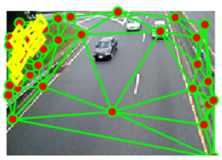
6.9: $skcx_3$ and $mcyc_3$ in Frame 2



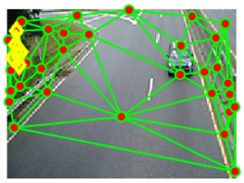
6.10: $skcx_4$ and $mcyc_4$ in Frame 1



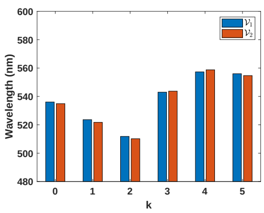
6.11: $skcx_4$ and $mcyc_4$ in Frame 2



6.12: $skcx_5$ and $mcyc_5$ in Frame 1

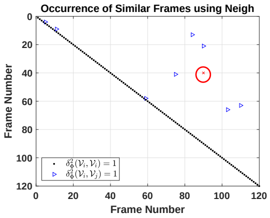


6.13: $skcx_5$ and $mcyc_5$ in Frame 2

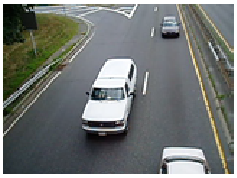


6.14: Mean Wavelengths for both images

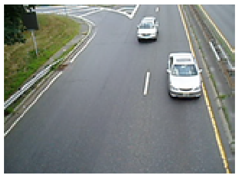
Figure 6: This figure illustrates the application of framework developed in sec. 4. In this illustration we use the values of wavelengts averaged over the spoke triangles(neighborhoods). Fig. 6.1, represents the descriptive frame recurrance diagram. The red circle annotates the frame pair being analyzed. Fig. 6.2,6.3 are the frames that have been detected as descriptivly similar. Figs. 6.4,6.6,6.8,6.10,6.12 represent $skcx_k$ and corresponding $mcyc_k, k = 1, \cdots, 5$ for frame in Fig. 6.2. Figs. 6.5,6.7,6.9,6.11,6.13 represent $skcx_k$ and corresponding $mcyc_k, k = 1, \cdots, 5$ for frame in Fig. 6.3. Figs. 6.14 represents the average wavelengts calculated at each spoke level k. $skcx_0$ and $mcyc_0$ are the nucleus. The frames are similar in terms of these values as can be seen in this figure.



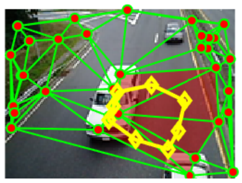
7.1: Descriptive Frame Recurrance



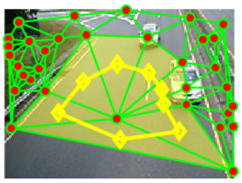
7.2: Frame 1



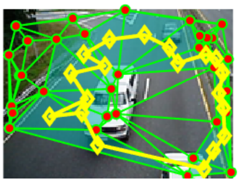
7.3: Frame 2



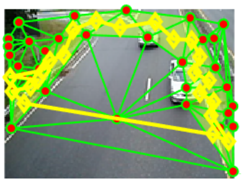
7.4: $skcx_1$ and $mcyc_1$ in Frame 1



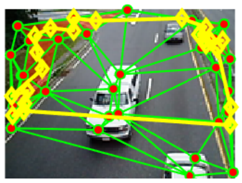
7.5: $skcx_1$ and $mcyc_1$ in Frame 2



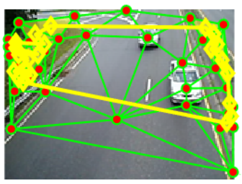
7.6: $skcx_2$ and $mcyc_2$ in Frame 1



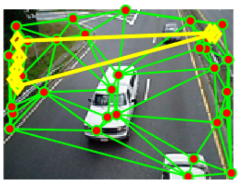
7.7: $skcx_2$ and $mcyc_2$ in Frame 2



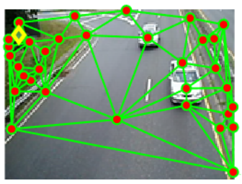
7.8: $skcx_3$ and $mcyc_3$ in Frame 1



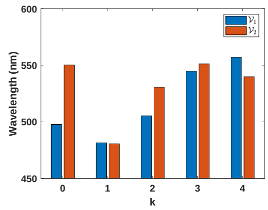
7.9: $skcx_3$ and $mcyc_3$ in Frame 2



7.10: $skcx_4$ and $mcyc_4$ in Frame 1



7.11: $skcx_4$ and $mcyc_4$ in Frame 2



7.12: Mean Wavelengths for both images

Figure 7: This figure illustrates the application of framework developed in sec. 4. In this illustration we display two frames which are dissimilar as none of the vortices are descriptivly similar. Fig. 7.1, represents the descriptive frame recurrance diagram. The red circle and cross annotates the frame pair being analyzed. Fig. 7.2,7.3 are the frames that have been detected as descriptivly not similar. Figs. 7.4,7.6,7.8,7.10 represent $skcx_k$ and corresponding $mcyc_k, k = 1, \cdots, 4$ for frame in Fig. 7.2. Figs. 7.5,7.7,7.9,7.11 represent $skcx_k$ and corresponding $mcyc_k, k = 1, \cdots, 4$ for frame in Fig. 7.3. Figs. 7.12 represents the average wavelengts calculated at each spoke level k. $skcx_0$ and $mcyc_0$ are the nucleus. The frames are not similar due to marked difference in these values as can be seen in this figure.

respects as shown in Fig. 5. The frames detected as similar by ϕ_{avg} have matching aspects such as containing the car of same color and similar background(as the camera position is fixed) but are also different in terms of the location and number of cars.

In addition to resulting from the flexibility due to ϕ_{avg}, it is also necessary to point out its extents. Let us look at a pair of frames that have been marked as dissimilar by both ϕ_{avg} and ϕ_{vrt}. The pair of frames is $(40, 90)$ and the results are displayed in Fig. 7. The pair of frames has been marked by a red cross and circle in $\mathcal{R}_\Phi(\mathcal{V}, th)$ as shown in Fig. 7.1. The threshold value is the same as previous experiments, *i.e.*, $th = 5$.

Consider, first Figs. 7.2, 7.3 to illustrate frame 40 and 90 respectively. It can be observed that the frames are very different in terms of the number and location of cars. This was also the case with Fig. 6, but the frames were detected as similar. In this case there were also cars with similar appearance (the white jeep and white sedan), but the size difference and the location of jeep forces a lot of keypoints to the center of frame 40. In contrast to this, the keypoints are distributed near the edges with a single point in the middle. Delaunay triangulation thus forces the nucleus to that point and hence the MNC also to the centre. This significant change in the location of the keypoints results in a significant change in structure of the triangulation for both frames.

Now let us move on to the $skcx_k$ and the $mcyc_k$ for both the frames. Figs. 7.4, 7.6, 7.8, 7.10 represent the $skcx_k$ and $mcyc_k$ for $k = 1, \cdots, 4$ in frame 40(Fig. 7.2). Figs. 7.5, 7.7, 7.9, 7.11 represent the $skcx_k$ and $mcyc_k$ for $k = 1, \cdots, 4$ in frame 90(Fig. 7.3). We can see that all the $skcx_k$ and the corresponding $mcyc_k$ have a different structure. Moreover, it is important to note which features of the image lie in the different spokes for each of the frames. Let us first look at the comparison of average wavelengths for the differnet $mcyc_k$. This is presented in Fig. 7.12 for both the frames. It is evident from the bar graph that the major difference lies in $mcyc_0, mcyc_2$ and $mcyc_4$.

Let us now examine the reasons for this difference. As has been established in the paper that when using the neighborhood based probe(ϕ_{avg}) we are not only looking at the vertices in $mcyc_k$ but we are looking at the triangles corresponding to the vertices. Thus, ineffect we are looking at the corresponding $skcx_k$. It must also be noted that $skcx_0$ is the nucleus thus a single point. If we look at Fig. 7.4(frame 40) we can see that the nucleus lies on the bonnet of white jeep, while the nucleus for frame 90, as shown in Fig. 7.4, lies on the black road. As wavelength is a function of the hue(color) as depicted by Eq. 4.1, this explains the difference in average wavelength for $mcyc_0$. If we look at the $skcx_2$ for frame 40 as shown in Fig. 7.6, it can be seen that majority of the areas are black road and a small portion of white car. While for frame 90, shown in Fig. 7.7, most of the regions in $skcx_2$ are the white cars and the green grass. This results in the difference in average wavelength for $skcx_2$ as observed in Fig. 7.12.

With respect to the difference in $skcx_4$, it can be seen that in frame 40, as shown in Fig. 7.10, contains the dark green tree, light green grass and the gray separator in the highway(to the right side of frame). For frame 90 the $skcx_4$ only contains the dark green tree as shown in Fig. 7.11. This leads to the different average wavelengt values depicted in Fig. 7.12. When the differences are added up as per the euclidean distance as in Def. 7, this leads to $\delta^2_\Phi(\mathcal{V}_{40}, \mathcal{V}_{90}) = 1$.

Each of these examples depicted in Figs. 5, 6 and 7 illustrate the utility of considering the descriptive similarity of maximal centroidal vortices across the frames. We have seen that the ϕ_{vrt}(vertex based probe function) imposes a more strict notion of similarity than the ϕ_{avg}(region based probe function). Moreover, we saw that there are limits to the flexibility of ϕ_{avg}, since too much change in either the description of regions or structure of the underlying triangulation can result in the frames being marked as dissimilar.

6. Conclusions

In this paper, we have introduced a proximity based framework for the study of videos. We start by developing the notion of maximal centroidal cycles and establishing their relation to the spoke complexes. Further, we have defined the notion of iso-proximal complex and cycle, explaining how they encode the proximity structure of a triangulated space. We establish that spoke complexes and maximal centroidal cycles are iso-proximal.

After having developed the structure to encode the spatial proximity in a video frame, we use the notion of vertex and region based probe functions to track the features along these cycles. This yields a framework that combines the spatial and descriptive proximity to analyze a video frame. We calculate the proposed features on each frame and then find out which frames across the video are descriptively similar. This results in a descriptive frame recurrence diagram. We analyze in great detail how the vertex and region based probes differ in terms of detecting similar frames.

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On Normal Fuzzy Submultigroups of a Fuzzy Multigroup

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Abstract

In this paper, we propose the notion of normal fuzzy submultigroups of a fuzzy multigroup. Some properties of normal fuzzy submultigroups of a fuzzy multigroup are explored and some related results are obtained. It is shown that a fuzzy submultigroup of a fuzzy multigroup is normal if and only if its alpha-cut is a normal subgroup of a given group. The concepts of commutator and normalizer in fuzzy multigroup setting are introduced and some results are deduced.

Keywords: Fuzzy comultiset, Fuzzy multiset, Fuzzy multigroup, Fuzzy submultigroup, Normal fuzzy submultigroup.
2010 MSC: 03E72, 08A72, 20N25.

1. Introduction

The concept of fuzzy sets proposed by (Zadeh, 1965) is a mathematical tool for representing vague concepts. The theory of fuzzy sets has grown stupendously over the years giving birth to fuzzy groups proposed in (Rosenfeld, 1971). Several works have been done on fuzzy groups and fuzzy normal subgroups (see Ajmal & Jahan, 2012; Malik *et al.*, 1992; Mashour *et al.*, 1990; Mordeson *et al.*, 1996; Mukherjee & Bhattacharya, 1984; Seselja & Tepavcevic, 1997; Wu, 1981).

Motivated by the work in (Zadeh, 1965), the idea of fuzzy multisets was conceived in (Yager, 1986) as the generalization of fuzzy sets in multisets framework. For some details on fuzzy multisets (see Ejegwa, 2014; Miyamoto, 1996; Syropoulos, 2012). Recently, in (Shinoj *et al.*, 2015), the concept of fuzzy multigroups was introduced as an application of fuzzy multisets to group theory, and some properties of fuzzy multigroups were presented. In fact, fuzzy multigroup is a generalization of fuzzy groups. (Baby *et al.*, 2015) continued the algebraic study of fuzzy multisets by proposing the idea of abelian fuzzy multigroups.

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The work in (Ejegwa, 2018*c*), which was built on (Shinoj *et al.*, 2015), introduced the concept of fuzzy multigroupoids and presented the idea of fuzzy submultigroups with a number of results. More properties of abelian fuzzy multigroups were explicated in (Ejegwa, 2018*b*), in the same vein, the notions of centre and centralizer in fuzzy multigroup setting where established with some relevant results. In (Ejegwa, 2018*a*), the notion of homomorphism in the context of fuzzy multigroups was defined and some homomorphic properties of fuzzy multigroups in terms of homomorphic images and homomorphic preimages, respectively, were presented. Since the notions of fuzzy multigroups, fuzzy submultigroups and abelian fuzzy multigroups have been established in literature, then it is germane to consider when a fuzzy submultigroup is said to be normal. Hence the motivation for this present research. In fact, this study is an application of fuzzy multisets to group theoretical notions like normal subgroups.

In this paper, we propose the notion of normal fuzzy submultigroups of a fuzzy multigroup and discuss some of its properties. The concepts of commutator and normalizer in fuzzy multigroup setting are also introduced, and some related results are deduced. By organization, the paper is thus presented: Section 2 provides some preliminaries on fuzzy multisets, fuzzy multigroups and fuzzy submultigroups. In Section 3, we propose the idea of normal fuzzy submultigroups of a fuzzy multigroup and discuss some of its properties. Also, the concepts of commutator and normalizer in fuzzy multigroup setting are also introduced, and some related results are obtained. Finally, Section 4 concludes the paper and provides direction for future studies.

2. Preliminaries

In this section, we review some existing definitions and results which shall be used in the sequel.

Definition 2.1. (Yager, 1986) Assume X is a set of elements. Then a fuzzy bag/multiset A drwan from X can be characterized by a count membership function CM_A such that

$$CM_A : X \to Q,$$

where Q is the set of all crisp bags or multisets from the unit interval $I = [0, 1]$.

From (Syropoulos, 2012), a fuzzy multiset can also be characterized by a high-order function. In particular, a fuzzy multiset A can be characterized by a function

$$CM_A : X \to N^I \text{ or } CM_A : X \to [0, 1] \to N,$$

where $I = [0, 1]$ and $N = \mathbb{N} \cup \{0\}$.

By (Miyamoto & Mizutani, 2004), it implies that $CM_A(x)$ for $x \in X$ is given as

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ...\},$$

where $\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ... \in [0, 1]$ such that $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x) \geq ...$, whereas in a finite case, we write

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x)\},$$

for $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x)$.

A fuzzy multiset A can be represented in the form

$$A = \{\langle \frac{CM_A(x)}{x} \rangle \mid x \in X\} \text{ or } A = \{\langle x, CM_A(x) \rangle \mid x \in X\}.$$

In a simple term, a fuzzy multiset A of X is characterized by the count membership function $CM_A(x)$ for $x \in X$, that takes the value of a multiset of a unit interval $I = [0, 1]$ (see Biswas, 1999; Mizutani *et al.*, 2008).

We denote the set of all fuzzy multisets by $FMS(X)$.

Example 2.1. Let $X = \{a, b, c\}$ be a set. Then a fuzzy multiset of X is given as

$$A = \{\langle \frac{0.5, 0.4, 0.3}{a} \rangle, \langle \frac{0.6, 0.4, 0.4}{b} \rangle, \langle \frac{0.7, 0.4, 0.2}{c} \rangle\}.$$

Definition 2.2. (see Miyamoto, 1996) Let $A, B \in FMS(X)$. Then A is called a fuzzy submultiset of B written as $A \subseteq B$ if $CM_A(x) \leq CM_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Definition 2.3. (see Syropoulos, 2012) Let $A, B \in FMS(X)$. Then the intersection and union of A and B, denoted by $A \cap B$ and $A \cup B$, respectively, are defined by the rules that for any object $x \in X$,

(i) $CM_{A \cap B}(x) = CM_A(x) \wedge CM_B(x)$,
(ii) $CM_{A \cup B}(x) = CM_A(x) \vee CM_B(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum, respectively.

Definition 2.4. (see Miyamoto, 1996) Let $A, B \in FMS(X)$. Then A and B are comparable to each other if and only if $A \subseteq B$ or $B \subseteq A$, and $A = B \Leftrightarrow CM_A(x) = CM_B(x) \forall x \in X$.

Definition 2.5. A fuzzy multiset B of a set X is said to have sup-property if for any subset $W \subset X$ $\exists\, w_0 \in W$ such that

$$CM_B(w_0) = \bigvee_{w \in W} \{CM_B(w)\}.$$

Definition 2.6. (Shinoj *et al.*, 2015) Let X be a group. A fuzzy multiset A of X is said to be a fuzzy multigroup of X if it satisfies the following two conditions:

(i) $CM_A(xy) \geq CM_A(x) \wedge CM_A(y) \forall x, y \in X$,
(ii) $CM_A(x^{-1}) \geq CM_A(x) \forall x \in X$.

It follows immediately that,

$$CM_A(x^{-1}) = CM_A(x), \forall x \in X$$

since

$$CM_A(x) = CM_A((x^{-1})^{-1}) \geq CM_A(x^{-1}).$$

Also,

$$CM_A(x^n) \geq CM_A(x) \forall x \in X,\ n \in \mathbb{N}$$

since

$$\begin{aligned} CM_A(x^n) = CM_A(x^{n-1}x) &\geq CM_A(x^{n-1}) \wedge CM_A(x) \\ &\geq CM_A(x) \wedge \ldots \wedge CM_A(x) \\ &= CM_A(x). \end{aligned}$$

It can be easily verified that if A is a fuzzy multigroup of X, then

$$CM_A(e) = \bigvee_{x \in X} CM_A(x) \; \forall x \in X,$$

that is, $CM_A(e)$ is the tip of A. The set of all fuzzy multigroups of X is denoted by $FMG(X)$.

Example 2.2. Let $X = \{e, a, b, c\}$ be a Klein 4-group such that

$$ab = c, ac = b, bc = a, a^2 = b^2 = c^2 = e.$$

Again, let

$$A = \{\langle \frac{1, 0.9}{e} \rangle, \langle \frac{0.7, 0.5}{a} \rangle, \langle \frac{0.8, 0.6}{b} \rangle, \langle \frac{0.7, 0.5}{c} \rangle\}$$

be a fuzzy multiset of X. We investigate whether $A \in MG(X)$ using Definition 2.6.

$$CM_A(ea) = CM_A(a) = 0.7, 0.5 \geq CM_A(e) \wedge CM_A(a) = 0.7, 0.5$$

$$CM_A(eb) = CM_A(b) = 0.8, 0.6 \geq CM_A(e) \wedge CM_A(b) = 0.8, 0.6$$

$$CM_A(ec) = CM_A(c) = 0.7, 0.5 \geq CM_A(e) \wedge CM_A(c) = 0.7, 0.5$$

$$CM_A(ab) = CM_A(c) = 0.7, 0.5 \geq CM_A(a) \wedge CM_A(b) = 0.7, 0.5$$

$$CM_A(ac) = CM_A(b) = 0.8, 0.6 \geq CM_A(a) \wedge CM_A(c) = 0.7, 0.5$$

$$CM_A(bc) = CM_A(a) = 0.7, 0.5 \geq CM_A(b) \wedge CM_A(c) = 0.7, 0.5$$

$$CM_A(aa) = CM_A(e) = 1, 0.9 \geq CM_A(a) \wedge CM_A(a) = 0.7, 0.5$$

$$CM_A(bb) = CM_A(e) = 1, 0.9 \geq CM_A(b) \wedge CM_A(b) = 0.8, 0.6$$

$$CM_A(cc) = CM_A(e) = 1, 0.9 \geq CM_A(c) \wedge CM_A(c) = 0.7, 0.5$$

$$CM_A(ee) = CM_A(e) = 1, 0.9 \geq CM_A(e) \wedge CM_A(e) = 1, 0.9$$

$$CM_A(a^{-1}) = CM_A(a) = 0.7, 0.5, CM_A(b^{-1}) = CM_A(b) = 0.8, 0.6$$

$$CM_A(c^{-1}) = CM_A(c) = 0.7, 0.5, CM_A(e^{-1}) = CM_A(e) = 1, 0.9$$

Because all the axioms in Definition 2.6 are satisfied $\forall x, y \in X$, it follows that A is a fuzzy multigroup of X.

Clearly, a fuzzy multigroup is a fuzzy group that admits repetition of membership values. That is, a fuzzy multigroup collapses into a fuzzy group whenever repetition of membership values is ignored.

Remark. We notice the following from Definition 2.6:

(i) every fuzzy multigroup is a fuzzy multiset but the converse is not always true.
(ii) a fuzzy multiset A of a group X is a fuzzy multigroup if $\forall x, y \in X$,

$$CM_A(xy^{-1}) \geq CM_A(x) \wedge CM(y)$$

holds.

Definition 2.7. (Shinoj *et al.*, 2015) Let A be a fuzzy multigroup of a group X. Then A^{-1} is defined by $CM_{A^{-1}}(x) = CM_A(x^{-1})\ \forall x \in X$.

Thus, we notice that $A \in FMG(X) \Leftrightarrow A^{-1} \in FMG(X)$.

Definition 2.8. (Ejegwa, 2018c) Let $A, B \in FMG(X)$. Then the product $A \circ B$ of A and B is defined to be a fuzzy multiset of X as follows:

$$CM_{A\circ B}(x) = \begin{cases} \bigvee_{x=yz}(CM_A(y) \wedge CM_B(z)), & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ 0, & \text{otherwise.} \end{cases}$$

This definition is adapted from (Shinoj *et al.*, 2015).

Definition 2.9. (Ejegwa, 2018c) Let $A \in FMG(X)$. A fuzzy submultiset B of A is called a fuzzy submultigroup of A denoted by $B \sqsubseteq A$ if B is a fuzzy multigroup. A fuzzy submultigroup B of A is a proper fuzzy submultigroup denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

Definition 2.10. (Baby *et al.*, 2015) Let $A \in FMG(X)$. Then A is said to be abelian (commutative) if for all $x, y \in X$, $CM_A(xy) = CM_A(yx)$.

Whenever A is a fuzzy multigroup of an abelian group X, it implies that A is abelian.

Definition 2.11. (see Ejegwa, 2018c; Shinoj *et al.*, 2015) Let $A \in FMG(X)$. Then the sets A_* and A^* are defined as

(i) $A_* = \{x \in X \mid CM_A(x) > 0\}$ and
(ii) $A^* = \{x \in X \mid CM_A(x) = CM_A(e)\}$, where e is the identity element of X.

Proposition 2.1. (see Ejegwa, 2018c; Shinoj *et al.*, 2015) *Let $A \in FMG(X)$. Then A_* and A^* are subgroups of X.*

Definition 2.12. Let $A \in FMG(X)$. Then the sets $A_{[\alpha]}$ and $A_{(\alpha)}$ defined as

(i) $A_{[\alpha]} = \{x \in X \mid CM_A(x) \geq \alpha\}$ and
(ii) $A_{(\alpha)} = \{x \in X \mid CM_A(x) > \alpha\}$

are called strong upper alpha-cut and weak upper alpha-cut of A, where $\alpha \in [0, 1]$.

Definition 2.13. Let $A \in FMG(X)$. Then the sets $A^{[\alpha]}$ and $A^{(\alpha)}$ defined as

(i) $A^{[\alpha]} = \{x \in X \mid CM_A(x) \leq \alpha\}$ and

(ii) $A^{(\alpha)} = \{x \in X \mid CM_A(x) < \alpha\}$

are called strong lower alpha-cut and weak lower alpha-cut of A, where $\alpha \in [0, 1]$.

Theorem 2.1. *Let $A \in FMG(X)$. Then $A_{[\alpha]}$ is a subgroup of X for all $\alpha \leq CM_A(e)$ and $A^{[\alpha]}$ is a subgroup of X for all $\alpha \geq CM_A(e)$, where e is the identity element of X and $\alpha \in [0, 1]$.*

Proof. Let $x, y \in A_{[\alpha]}$, then $CM_A(x) \geq \alpha$ and $CM_A(y) \geq \alpha$. Because $A \in FMG(X)$, we get

$$\begin{aligned} CM_A(xy^{-1}) &\geq (CM_A(x) \wedge CM_A(y)) \geq \alpha \\ &= CM_A(x) \geq \alpha \wedge CM_A(y) \geq \alpha. \end{aligned}$$

Thus, $xy^{-1} \in A_{[\alpha]}$. Hence, $A_{[\alpha]}, \alpha \in [0, 1]$ is a subgroup of X for all $\alpha \leq CM_A(e)$. The proof of the second part, that is, $A^{[\alpha]}$ is a subgroup of X $\forall$ $\alpha \geq CM_A(e)$ is similar. □

3. Main Results

In this section, some properties of normal subgroups in fuzzy multigroup setting are investigated by redefining some concepts in the light of fuzzy multigroups.

Definition 3.1. Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then A is called a normal fuzzy submultigroup of B if for all $x, y \in X$, it satisfies

$$CM_A(xyx^{-1}) \geq CM_A(y).$$

Example 3.1. Let $X = \{0, 1, 2, 3\}$ be a group of modulo 4 with respect to addition. Then a fuzzy multigroup of X is given as

$$B = \{\langle \frac{1, 0.9, 0.8}{0} \rangle, \langle \frac{0.9, 0.7, 0.5}{1} \rangle, \langle \frac{0.8, 0.7, 0.4}{2} \rangle, \langle \frac{0.9, 0.7, 0.5}{3} \rangle\},$$

and

$$A = \{\langle \frac{1, 0.8, 0.7}{0} \rangle, \langle \frac{0.8, 0.6, 0.4}{1} \rangle, \langle \frac{0.7, 0.6, 0.4}{2} \rangle, \langle \frac{0.8, 0.6, 0.4}{3} \rangle\}$$

is a fuzzy submultigroup of B. It follows that A is a normal fuzzy submultigroup of B since

$$CM_A(1 + 2 + 1^{-1}) = CM_A(1 + 2 + 3) = 0.7, 0.6, 0.4 \geq CM_A(2)$$

$$CM_A(2 + 1 + 2^{-1}) = CM_A(2 + 1 + 2) = 0.8, 0.6, 0.4 \geq CM_A(1)$$

$$CM_A(3 + 2 + 3^{-1}) = CM_A(3 + 2 + 1) = 0.7, 0.6, 0.4 \geq CM_A(2)$$

$$CM_A(2 + 3 + 2^{-1}) = CM_A(2 + 3 + 2) = 0.8, 0.6, 0.4 \geq CM_A(3)$$

$$CM_A(1 + 3 + 1^{-1}) = CM_A(1 + 3 + 3) = 0.8, 0.6, 0.4 \geq CM_A(3)$$

$$CM_A(3 + 1 + 3^{-1}) = CM_A(3 + 1 + 1) = 0.8, 0.6, 0.4 \geq CM_A(1).$$

Definition 3.2. Let $A \in FMG(X)$ and $x, y \in X$. Then x and y are called conjugate elements in A if for some $z \in X$,

$$CM_A(x) = CM_A(zyz^{-1}).$$

Two fuzzy multigroups A and B of X are conjugate to each other if for all $x, y \in X$,

$$CM_A(y) = CM_B(xyx^{-1}) \text{ or } CM_A(y) = CM_{B^x}(y)$$

and

$$CM_B(x) = CM_A(yxy^{-1}) \text{ or } CM_B(x) = CM_{A^y}(x).$$

Remark. Let A be a fuzzy submultigroup of $B \in FMG(X)$. From Definitions 2.6 and 2.7, A is normal if and only if A^{-1} is normal.

Proposition 3.1. *If $B \in FMG(X)$ and A is a normal fuzzy submultigroup of B. Then A_* and A^* are normal subgroups of X. Also, A_* is a normal subgroup of B_* and A^* is a normal subgroup of B^*.*

Proof. We know that A_* and A^* are subgroups of X by Proposition 2.1. Now, we proof that A_* and A^* are normal subgroups of X.

Let $x, y \in A_*$. By the definition of A_*, it follows that $CM_A(x) > 0$ and $CM_A(y) > 0$. That is,

$$CM_A(xyx^{-1}) \geq CM_A(y) > 0.$$

So, $xyx^{-1} \in A_* \Rightarrow A_*$ is a normal subgroup of X.

Similarly, assume $x, y \in A^*$. By the definition of A^*, it follows that

$$CM_A(x) = CM_A(e) = CM_A(y).$$

That is,

$$CM_A(xyx^{-1}) \geq CM_A(y) = CM_A(e) \geq CM_A(xyx^{-1}).$$

Thus, $CM_A(xyx^{-1}) = CM_A(e)\ \forall x, y \in X$. Hence, $xyx^{-1} \in A^*$ and the result follows.

Recall that, A is a normal fuzzy submultigroup of B, and A_* and A^* are normal subgroups of X. Synthesizing these, it implies that A_* is a normal subgroup of B_* and A^* is a normal subgroup of B^*. □

Proposition 3.2. *Let A be a normal fuzzy submultigroup of $B \in FMG(X)$. Then $A_{[\alpha]}$ is a normal subgroup of X for all $\alpha \leq CM_A(e)$ and $A^{[\alpha]}$ is a normal subgroup of X for all $\alpha \geq CM_A(e)$, where e is the identity element of X and $\alpha \in [0, 1]$. Consequently, $A_{[\alpha]}$ is a normal subgroup of $B_{[\alpha]}$ and $A^{[\alpha]}$ is a normal subgroup of $B^{[\alpha]}$.*

Proof. It implies from Theorem 2.1 that, $A_{[\alpha]}$ is a subgroup of X for all $\alpha \leq CM_A(e)$ and $A^{[\alpha]}$ is a subgroup of X for all $\alpha \geq CM_A(e)$, where $\alpha \in [0, 1]$. Now, we proof that $A_{[\alpha]}$ and $A^{[\alpha]}$ are normal subgroups of X.

Let $x, y \in A_{[\alpha]}$. By the definition of $A_{[\alpha]}$, we get

$$CM_A(x) \geq \alpha \text{ and } CM_A(y) \geq \alpha.$$

That is,

$$CM_A(xyx^{-1}) = CM_A(y) \geq \alpha.$$

Thus, $xyx^{-1} \in A_{[\alpha]}$, so $A_{[\alpha]}$ is a normal subgroup of X. Similarly, it follows that $A^{[\alpha]}$ is a normal subgroup of X.

But we know that, A is a normal fuzzy submultigroup of B, and $A_{[\alpha]}$ and $A^{[\alpha]}$ are normal subgroups of X. Synthesizing these, it happens that $A_{[\alpha]}$ is a normal subgroup of $B_{[\alpha]}$ and $A^{[\alpha]}$ is a normal subgroup of $B^{[\alpha]}$. □

Theorem 3.1. *For a fuzzy submultigroup A of $B \in FMG(X)$, the following statements are equivalent:*

(i) A is a normal submultigroup of B.

(ii) $A_{[\alpha]}$ (for $\alpha \in [0,1]$ and $\alpha \leq CM_A(e)$, where e is he identity element of X) is a normal subgroup of X. It also holds for$A^{[\alpha]}$, where $\alpha \in [0,1]$ and $\alpha \geq CM_A(e)$.

Proof. (i)⇒(ii). Let $x \in X$ and $y \in A_{[\alpha]}$. By the hypothesis, we have

$$CM_A(xyx^{-1}) = CM_A(y) \geq \alpha.$$

It follows that $y = xyx^{-1} \in A_{[\alpha]}$ and hence $A_{[\alpha]}$ is a normal subgroup of X.

(ii)⇒(i). Let $x, g \in X$. Take $\alpha = CM_A(x)$ and $\beta = CM_B(g)$, so that $x \in A_{[\alpha]}$ and $g \in B_{[\beta]}$.

Case 1: $\alpha \geq \beta$. This implies that $\alpha_0 \geq CM_A(x) \geq \beta = CM_B(g)$ for $\alpha \in [0, \alpha_0]$. Thus $\beta \in Im(B)$ and $\beta \leq \alpha_0$. By the hypothesis, $A_{[\beta]}$ is a normal subgroup of $B_{[\beta]}$. Also, $x \in A_{[\beta]}$ and $g \in B_{[\beta]}$. Hence $gxg^{-1} \in A_{[\beta]}$. So,

$$CM_A(gxg^{-1}) \geq \beta = CM_B(g) = CM_A(x) \wedge CM_B(g).$$

Case 2: $\beta \geq \alpha$. This implies that

$$CM_B(g) \geq \alpha = CM_A(x).$$

Thus $\alpha \in Im(A)$ and $x \in A_{[\alpha]}$, $g \in B_{[\alpha]}$. By the hypothesis, $A_{[\alpha]}$ is a normal subgroup of $B_{[\alpha]}$. Consequently, $gxg^{-1} \in A_{[\alpha]}$. So,

$$CM_A(gxg^{-1}) \geq \alpha = CM_A(x) = CM_A(x) \wedge CM_B(g).$$

Hence (i) holds. □

Proposition 3.3. *Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then the following statements are equivalent.*

(i) A is a normal fuzzy submultigroup of B.

(ii) $CM_A(xyx^{-1}) = CM_A(y)\ \forall x, y \in X$.

(iii) $CM_A(xy) = CM_A(yx)\ \forall x, y \in X$.

Proof. (i)⇒(ii). Suppose A is a normal fuzzy submultigroup of B. From Definition 3.1, it implies that $CM_A(xyx^{-1}) = CM_A(y)\ \forall x, y \in X$.

(ii)⇒(iii). Suppose that $CM_A(xyx^{-1}) = CM_A(y)$. Then, it implies that

$$CM_A(xy) = CM_A(yx)\ \forall x, y \in X.$$

(iii)⇒(i). Assume that $CM_A(xy) = CM_A(yx)\ \forall x, y \in X$. It follows that A is a normal fuzzy submultigroup of B since $A \subseteq B$. □

Remark. Every normal fuzzy submultigroup of a fuzzy multigroup is abelian.

Proposition 3.4. *If A is a fuzzy submultigroup of $B \in FMG(X)$ such that $CM_A(x) = CM_A(y)\ \forall x, y \in X$. Then the following assertions are equivalent.*

(i) A is a normal fuzzy submultigroup of B.
(ii) $CM_A(yx) = CM_A(xy) \wedge CM_B(y)\ \forall x, y \in X$.

Proof. (i)⇒(ii). Since A is a normal fuzzy submultigroup of B and $CM_A(x) = CM_A(y)$, it follows from Definition 3.1 and Proposition 3.3 that,

$$CM_A(yx) = CM_A(y(xy)y^{-1}) = CM_A(xy) \wedge CM_B(y)\ \forall x, y \in X.$$

(ii)⇒(i). Suppose $CM_A(yx) = CM_A(xy) \wedge CM_B(y)$. We infer that

$$CM_A(xy) = CM_A(yx) \wedge CM_B(y).$$

Then it implies that, $CM_A(xy) = CM_A(yx)$. Hence, the proof is completed by Proposition 3.3. □

Proposition 3.5. *Let A be a fuzzy submultigroup of $G \in FMG(X)$ and B be a fuzzy submultiset of G. If A and B are conjugate, then B is a fuzzy submultigroup of G.*

Proof. Since A and B are conjugate, then by Definition 3.2 it implies that $A = B$. And this completes the proof for the fact that, A is a fuzzy submultigroup of $G \in FMG(X)$. □

Proposition 3.6. *Let $A, B, C \in FMG(X)$ such that A and B are normal fuzzy submultigroups of C. If $A \subseteq B$, then $A \cap B$ and $A \cup B$ are normal fuzzy submultigroups of C.*

Proof. Since A and B are normal fuzzy submultigroups of C such that $A \subseteq B$, it follows that $A \cap B = A$ and $A \cup B = B$. Thus, $A \cap B$ and $A \cup B$ are normal fuzzy submultigroups of C. □

Theorem 3.2. *Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then A is a normal fuzzy submultigroup of B if and only if $x \in X$ is constant on the conjugacy classes of A.*

Proof. Suppose that A is a normal fuzzy submultigroup of B. Then

$$CM_A(y^{-1}xy) = CM_A(xyy^{-1}) = CM_A(x)\ \forall x, y \in X.$$

This implies that, $x \in X$ is constant on the conjugacy classes of A.

Conversely, let $x \in X$ be constant on each conjugacy classes of A. Then

$$CM_A(xy) = CM_A(xyxx^{-1}) = CM_A(x(yx)x^{-1}) = CM_A(yx)\ \forall x, y \in X.$$

Hence, A is a normal fuzzy submultigroup of B. □

We now give an alternative formulation of the notion of normal fuzzy submultigroup in terms of commutator of a group. First, we recall that if X is a group and $x, y \in X$, then the element $x^{-1}y^{-1}xy$ is usually depicted by $[x, y]$ and is called the commutator of x and y.

Theorem 3.3. *Let $A, B \in FMG(X)$ such that $A \subseteq B$. Then A is a normal fuzzy submultigroup of B if and only if*

(i) $CM_A([x, y]) \geq CM_A(x)\ \forall x, y \in X$.

(ii) $CM_A([x, y]) = CM_A(e)\ \forall x, y \in X$, *where e is the identity of X.*

Proof. (i) Suppose A is a normal fuzzy submultigroup of B. Let $x, y \in X$, then

$$\begin{aligned} CM_A(x^{-1}y^{-1}xy) &\geq CM_A(x^{-1}) \wedge CM_A(y^{-1}xy) \\ &= CM_A(x) \wedge CM_A(x) = CM_A(x). \end{aligned}$$

Conversely, assume that A satisfies the inequality. Then for all $x, y \in X$, we have

$$\begin{aligned} CM_A(x^{-1}yx) &= CM_A(yy^{-1}x^{-1}yx) \\ &\geq CM_A(y) \wedge CM_A([y, x]) = CM_A(y). \end{aligned}$$

Thus, $CM_A(x^{-1}yx) \geq CM_A(y) \forall x, y \in X$. Hence, A is a normal fuzzy submultigroup of B.

(ii) Let $x, y \in X$. Suppose A is a normal fuzzy submultigroup of B. We know that A is a normal fuzzy submultigroup of B
$\Leftrightarrow CM_A(xy) = CM_A(yx)\ \forall x, y \in X$
$\Leftrightarrow CM_A(x^{-1}y^{-1}x) = CM_A(y^{-1})\ \forall x, y \in X$
$\Leftrightarrow CM_A(x^{-1}y^{-1}xyy^{-1}) = CM_A(y^{-1})\ \forall x, y \in X$
$\Leftrightarrow CM_A([x, y]y^{-1}) = CM_A(y^{-1})\ \forall x, y \in X$.
Consequently, $CM_A([x, y]) = CM_A(y^{-1}y) = CM_A(e)\ \forall x, y \in X$.

Conversely, assume $CM_A([x, y]) = CM_A(e)\ \forall x, y \in X$. Then

$$CM_A(x^{-1}y^{-1}xy) = CM_A(e) \Rightarrow CM_A((yx)^{-1}xy) = CM_A(e).$$

That is, $CM_A(xy) = CM_A(yx)\ \forall x, y \in X$. Thus, A is a normal fuzzy submultigroup of B. □

Theorem 3.4. *Let A be a normal fuzzy submultigroup of $G \in FMG(X)$. Then $\bigcap_{x\in X} A^x$ is normal and is the largest normal fuzzy submultigroup of G that is contained in A.*

Proof. Suppose $A^x \in FMG(X)\ \forall x \in X$. Then for all $y \in X$, we observe that $A^x = A^{xy}\ \forall x, y \in X$ since

$$CM_{A^x}(z) = CM_A(xzx^{-1}) = CM_A(z)$$

and

$$CM_{A^{xy}}(z) = CM_A((xy)z(xy)^{-1}) = CM_A(z).$$

That is, $A^x = A$ whenever A is normal. Thus,

$$\begin{aligned} \bigwedge_{x\in X} CM_{A^x}(yzy^{-1}) &= \bigwedge_{x\in X} CM_A(xyzy^{-1}x^{-1}) \\ &= \bigwedge_{x\in X} CM_A((xy)z(xy)^{-1}) \\ &= \bigwedge_{x\in X} CM_{A^{xy}}(z) \\ &= \bigwedge_{x\in X} CM_{A^x}(z)\ \forall y, z \in X. \end{aligned}$$

Hence, $\bigcap_{x\in X} A^x$ is a normal fuzzy submultigroup of G. Now, let B be a normal fuzzy submultigroup of G such that $B \subseteq A$. Then $B = B^x \subseteq A^x\ \forall x \in X$. Thus, $B \subseteq \bigcap_{x\in X} A^x$. Therefore, $\bigcap_{x\in X} A^x$ is the largest normal fuzzy submultigroup of G that is contained in A. □

Definition 3.3. Let A be a submultiset of $B \in FMG(X)$. Then the normalizer of A in B is the set given by

$$N(A) = \{g \in X \mid CM_A(gy) = CM_A(yg)\ \forall y \in X\}.$$

Theorem 3.5. *Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then $N(A)$ is a subgroup of X.*

Proof. Let $g, h \in N(A)$. Then

$$CM_{A^{gh}}(x) = CM_{(A^h)^g}(x) \quad = \quad CM_{A^h}(x) = CM_A(x) \forall x \in X$$

since $CM_{A^g}(x) = CM_A(g^{-1}xg) = CM_A(x)$. Hence, $gh \in N(A)$. Again, let $g \in N(A)$. We show that $g^{-1} \in N(A)$. For all $y \in X$, $CM_A(gy) = CM_A(yg)$ and so $CM_A((gy)^{-1}) = CM_A((yg)^{-1})$. Thus, for all $y \in X$,

$$CM_A(y^{-1}g^{-1}) = CM_A(g^{-1}y^{-1})$$

and so $CM_A(yg^{-1}) = CM_A(g^{-1}y)$, since $CM_A(y) = CM_A(y^{-1})$. Thus, $g^{-1} \in N(A)$. Hence, $N(A)$ is a subgroup of X. □

Theorem 3.6. *Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then A is a normal fuzzy submultigroup of B if and only if $N(A) = X$.*

Proof. Let A be a normal fuzzy submultigroup of B and $g \in X$. Then $\forall x \in X$, we have

$$\begin{aligned} CM_{A^g}(x) &= CM_A(g^{-1}xg) = CM_A((g^{-1}x)g) \\ &= CM_A(g(g^{-1}x)) = CM_A(x). \end{aligned}$$

Thus, $CM_{A^g}(x) = CM_A(x)$ and so $g \in N(A)$. Therefore, $N(A) = X$.

Conversely, suppose $N(A) = X$. Let $x, y \in X$. To prove that A is a normal fuzzy submultigroup of B, it is sufficient we show that $CM_A(xy) = CM_A(yx)$. Now

$$\begin{aligned} CM_A(xy) &= CM_A(xyxx^{-1}) = CM_A(x(yx)x^{-1}) \\ &= CM_{A^{x^{-1}}}(yx) = CM_A(yx), \end{aligned}$$

where the last equality follows since $N(A) = X$ and so, $x^{-1} \in N(A)$. Hence, $CM_{A^{x^{-1}}}(y) = CM_A(y)$ (that is, $A^{x^{-1}} = A = A^x$). Therefore, A is a normal fuzzy submultigroup of B. □

Remark. Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then $S = N(A) = T$, if

$$S = \{x \in X \mid CM_A(xy(yx)^{-1}) = CM_A(e)\ \forall y \in X\}$$

and

$$T = \{x \in X \mid CM_A(xyx^{-1}) = CM_A(y)\ \forall y \in X\}.$$

Theorem 3.7. *Let A, B and C be fuzzy multigroups of an abelian group X such that $A \subseteq B \subseteq C$. Then*

(i) $N(A) \cap N(B) \subseteq N(A \cap B)$.

(ii) $N(A) \cap N(B) \subseteq N(A \circ B)$.

Proof. (i) Let $y \in N(A)$ and $y \in N(B) \Rightarrow y \in N(A) \cap N(B)$. For any $x, y \in X$, we get $CM_{A\cap B}(xy) = CM_{A\cap B}(yx) \Rightarrow CM_{A\cap B}(xyx^{-1}) = CM_{A\cap B}(y)$. Now,

$$\begin{aligned} CM_{A\cap B}(xyx^{-1}) &= CM_A(xyx^{-1}) \wedge CM_B(xyx^{-1}) \\ &= CM_A(yxx^{-1}) \wedge CM_B(yxx^{-1}) \\ &= CM_A(y) \wedge CM_B(y) \\ &= CM_{A\cap B}(y). \end{aligned}$$

Thus, $y \in N(A \cap B)$. Hence, $N(A) \cap N(B) \subseteq N(A \cap B)$.

(ii) Let $y \in N(A) \cap N(B)$, that is $y \in N(A)$ and $y \in N(B)$. Then for all $x \in X$,

$$\begin{aligned} CM_{A\circ B}(y) &= \bigvee_{y=ab} (CM_A(a) \wedge CM_B(b)), \forall a, b \in X \\ &= \bigvee_{y=ab} (CM_A(x^{-1}ax) \wedge CM_B(x^{-1}bx)), \forall a, b \in X \\ &\leq \bigvee_{x^{-1}yx=cd} (CM_A(c) \wedge CM_B(d)), \forall c, d \in X \\ &= CM_{A\circ B}(x^{-1}yx) \end{aligned}$$

$\Rightarrow CM_{A\circ B}(y) \leq CM_{A\circ B}(x^{-1}yx)$. The inequality holds since

$$y = ab \Rightarrow x^{-1}abx = cd \Rightarrow ab = xcdx^{-1} = (xcx^{-1})(xdx^{-1})$$

and since $a = xcx^{-1}$ and $b = xdx^{-1}$ imply $x^{-1}ax = c$ and $x^{-1}bx = d$. Again,

$$CM_{A\circ B}(x^{-1}yx) \leq CM_{A\circ B}(x(x^{-1}yx)x^{-1}) = CM_{A\circ B}(y).$$

So, $CM_{A\circ B}(y) \geq CM_{A\circ B}(x^{-1}yx)$. Thus,

$$CM_{A\circ B}(y) = CM_{A\circ B}(x^{-1}yx).$$

Hence, $y \in N(A \circ B)$. Therefore, $N(A) \cap N(B) \subseteq N(A \circ B)$. □

Corollary 3.1. *Let $A, B, C \in FMG(X)$ such that $A \subseteq B \subseteq C$ and $CM_A(e) = CM_B(e)$. Then $N(A) \cap N(B) = N(A \cap B)$.*

Proof. Recall that

$$\begin{aligned} N(A) &= \{x \in X \mid CM_A(xy) = CM_A(yx)\ \forall y \in X\} \\ &= \{x \in X \mid CM_A(xyx^{-1}y^{-1}) = CM_A(e)\ \forall y \in X\}. \end{aligned}$$

Let $y \in N(A \cap B)$. Then from the definition of $N(A)$, for all $x \in X$ we get

$$\begin{aligned} CM_{A\cap B}(xyx^{-1}y^{-1}) &= CM_A(xyx^{-1}y^{-1}) \wedge CM_B(xyx^{-1}y^{-1}) \\ &= CM_A(e) \wedge CM_B(e), \end{aligned}$$

implies $y \in N(A)$ and $y \in N(B)$. Thus, $y \in N(A) \cap N(B)$ since

$$CM_A(xyx^{-1}y^{-1}) = CM_A(e) \Rightarrow CM_A(xy) = CM_A(yx)$$

and similarly in the case of B because $CM_A(e) = CM_B(e)$. Hence, it follows that $N(A) \cap N(B) = N(A \cap B)$. □

Remark. If A and B are fuzzy submultigroups of $C \in FMG(X)$ such that $A \subseteq B$. Then $N(A) \subseteq N(B)$.

Definition 3.4. Let A be a fuzzy submultigroup of $G \in FMG(X)$. Then the fuzzy submultiset yA of G for $y \in X$ defined by

$$CM_{yA}(x) = CM_A(y^{-1}x)\ \forall x \in X$$

is called the left fuzzy comultiset of A. Similarly, the fuzzy submultiset Ay of G for $y \in X$ defined by

$$CM_{Ay}(x) = CM_A(xy^{-1})\ \forall x \in X$$

is called the right fuzzy comultiset of A.

Proposition 3.7. *Let A be a normal fuzzy submultigroup of $B \in FMG(X)$. Then $CM_{xA}(xz) = CM_{xA}(zx) = CM_A(z)\ \forall x, z \in X$.*

Proof. Let $x, z \in X$. Suppose A is a normal fuzzy submultigroup of B, then by Proposition 3.3 and Definition 3.4, we get

$$CM_{xA}(xz) = CM_{xA}(zx) = CM_A(x^{-1}zx) = CM_A(z).$$

Hence,

$$CM_{xA}(xz) = CM_{xA}(zx) = CM_A(z)\ \forall z \in X.$$

□

Theorem 3.8. *Let $A, B \in FMG(X)$ such that $A \subseteq B$. Then A is a normal fuzzy submultigroup of B if and only if for all $x \in X$, $Ax = xA$.*

Proof. Suppose A is a normal fuzzy submultigroup of B. Then for all $x \in X$, we have

$$\begin{aligned} CM_{Ax}(y) &= CM_A(yx^{-1}) = CM_A(x^{-1}y) \\ &= CM_{xA}(y)\ \forall y \in X. \end{aligned}$$

Thus, $Ax = xA$.

Conversely, let $Ax = xA$ for all $x \in X$. We get,

$$\begin{aligned} CM_A(xy) &= CM_{x^{-1}A}(y) = CM_{Ax^{-1}}(y) \\ &= CM_A(yx)\ \forall y \in X. \end{aligned}$$

Hence, A is a normal fuzzy submultigroup of B by Proposition 3.3. □

Theorem 3.9. *Let X be a finite group and A be a fuzzy submultigroup of $B \in FMG(X)$. Define*

$$\begin{aligned} H &= \{g \in X \mid CM_A(g) = CM_A(e)\}, \\ K &= \{x \in X \mid CM_{Ax}(y) = CM_{Ae}(y)\}, \end{aligned}$$

where e denotes the identity element of X. Then H and K are subgroups of X. Again, $H = K$.

Proof. Let $g, h \in H$. Then

$$\begin{aligned} CM_A(gh) &\geq CM_A(g) \wedge CM_A(h) \\ &= CM_A(e) \wedge CM_A(e) \\ &= CM_A(e) \end{aligned}$$

$\Rightarrow CM_A(gh) \geq CM_A(e)$.

But, $CM_A(gh) \leq CM_A(e)$ from Definition 2.6. Thus, $CM_A(gh) = CM_A(e)$, implying that $gh \in H$. Since X is finite, it follows that H is a subgroup of X.

Now, we show that $H = K$. Let $k \in K$. Then for $y \in X$ we get

$$CM_{Ak}(y) = CM_{Ae}(y) \Rightarrow CM_A(yk^{-1}) = CM_A(y).$$

Choosing $y = e$, we obtain

$$CM_A(k^{-1}) = CM_A(e) \Rightarrow k^{-1} \in H,$$

and so, $k \in H$ since H is a subgroup of X. Thus, $K \subseteq H$.

Again, let $h \in H$. Then $CM_A(h) = CM_A(e)$. Also,

$$CM_{Ah}(y) = CM_A(yh^{-1})\ \forall y \in X$$

and

$$CM_{Ae}(y) = CM_A(y)\ \forall y \in X.$$

Thus, to show that $h \in K$, it suffices to prove that

$$CM_A(yh^{-1}) = CM_A(y)\ \forall y \in X.$$

Now,

$$\begin{aligned} CM_A(yh^{-1}) &\geq CM_A(y) \wedge CM_A(h^{-1}) \\ &= CM_A(y) \wedge CM_A(h) \\ &= CM_A(y) \wedge CM_A(e) \\ &= CM_A(y). \end{aligned}$$

Again,

$$\begin{aligned} CM_A(y) &= CM_A(yh^{-1}h) \\ &\geq CM_A(yh^{-1}) \wedge CM_A(h) \\ &= CM_A(yh^{-1}) \wedge CM_A(e) \\ &= CM_A(yh^{-1}) \end{aligned}$$

$\Rightarrow CM_A(yh^{-1}) = CM_A(y)$, thus $H \subseteq K$. Hence, $H = K$. Therefore, K is a subgroup of X. □

Corollary 3.2. *With the same notation as in Theorem 3.9, H is a normal subgroup of X if A is a normal fuzzy submultigroup of B.*

Proof. Let $y \in X$ and $x \in H$. Then

$$\begin{aligned} CM_A(yxy^{-1}) &= CM_A(yy^{-1}x) \text{ since } A \text{ is normal in } B \\ &= CM_A(x) = CM_A(e). \end{aligned}$$

Thus, $yxy^{-1} \in H$. Hence, H is normal in X. □

Definition 3.5. Let A and B be fuzzy submultigroups of $C \in FMG(X)$. Then the commutator of A and B is the fuzzy multiset (A, B) of X defined as follows:

$$CM_{(A,B)}(x) = \begin{cases} \bigvee_{x=[a,b]}\{CM_A(a) \wedge CM_B(b)\}, & \text{if } x \text{ is a commutator in } X \\ 0, & \text{otherwise.} \end{cases}$$

That is,

$$CM_{(A,B)}(x) = \bigvee_{x=aba^{-1}b^{-1}} \{CM_A(a) \wedge CM_B(b)\}.$$

Since the supremum of an empty set is zero, $CM_{(A,B)}(x) = 0$ if x is not a commutator.

Definition 3.6. Let A and B be fuzzy submultigroups of $C \in FMG(X)$. Then the commutator fuzzy multigroup of A and B is the fuzzy multigroup generated by the commutator (A, B). It is denoted by $[A, B]$.

Definition 3.7. Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then the fuzzy submultigroup of B generated by A is the least fuzzy submultigroup of B containing A. It is denoted by $< A >$. That is

$$< A >= \bigcap \{A_i \in FMG(X) | CM_A(x) \leq CM_{A_i}(x)\}.$$

With the aid of Definitions 3.5 and 3.6, we obtain the result that follows.

Theorem 3.10. *Let A and B be normal fuzzy submultigroups of* $C \in FMG(X)$. *Then* $[A, B] \subseteq A \cap B$.

Proof. Let $x \in X$. Now if x is not a commutator, then $CM_{(A,B)}(x) = 0$ and therefore there is nothing to prove. Suppose that $x = aba^{-1}b^{-1}$ for some $a, b \in X$. Then

$$\begin{aligned} CM_{A\cap B}(x) &= CM_A(x) \wedge CM_B(x) \\ &= CM_A(aba^{-1}b^{-1}) \wedge CM_B(aba^{-1}b^{-1}) \\ &\geq (CM_A(a) \wedge CM_A(ba^{-1}b^{-1})) \wedge (CM_B(aba^{-1}) \wedge CM_B(b^{-1})) \\ &\geq (CM_A(a) \wedge CM_C(b)) \wedge (CM_B(b) \wedge CM_C(a)) \\ &= CM_A(a) \wedge CM_B(b). \end{aligned}$$

This implies that

$$\begin{aligned} CM_{A\cap B}(x) &\geq \bigvee_{x=aba^{-1}b^{-1}} CM_A(a) \wedge CM_B(b) \\ &= CM_{(A,B)}(x). \end{aligned}$$

Consequently, $CM_{A\cap B}(x) \geq CM_{(A,B)}(x)$. Thus $[A, B] \subseteq A \cap B$. □

4. Conclusion

We have introduced and also studied the concept of normal fuzzy submultigroups of a fuzzy multigroup and explored some of its properties. Also, the ideas of commutator and normalizer in fuzzy multigroup setting were proposed and some related results were established. However, more properties of normal fuzzy submultigroups could still be exploited.

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A Common Fixed Point Theorem in Cone Metric Spaces over Banach Algebras

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Abstract

In this paper, a common fixed point theorem for four mappings in cone metric spaces over Banach algebras is proved without assuming the normality of underlying cone. The results of this paper unify, generalize and extend some known results in cone metric spaces over Banach algebras. An example is presented which shows the significance of the result proved herein.

Keywords: Cone metric space, coincidence point, common fixed point.
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1. Introduction

The study of K-metric and K-normed spaces were introduced in the mid-20th century (Aliprantis & Tourky, 2007; Kantorovich, 1957; Vandergraft, 1967; Zabreĭko, 1997). In these papers, the set of real numbers was replaced by an ordered Banach space, as the codomain for a metric. In 2007, such spaces are reintroduced by (Huang & Zhang, 2007) under the name of cone metric spaces. (Huang & Zhang, 2007) defined convergent and Cauchy sequences in cone metric spaces in terms of interior points of underlying cone. Some basic versions of the fixed point theorems in cone metric spaces can be found in (Huang & Zhang, 2007). (Abbas & G.Jungck, 2008) proved some common fixed point results in these spaces. (Radenović, 2009) obtained a coincidence point theorem for two mappings in this new setting, which satisfy a new type of contractive condition. The result of (Radenović, 2009) was extended by (Rangamma & Prudhvi, 2012) for three mappings which satisfy a generalized contractive condition without exploiting the notion of continuity.

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In the papers (Radenović, 2009) and (Rangamma & Prudhvi, 2012) the contractive conditions were generalized by using the norm function. Notice that, the norm function is defined from the Banach space into the set of real numbers, hence, the results of (Huang & Zhang, 2007) can not be obtained by the results of (Rangamma & Prudhvi, 2012). Inspired by this fact, (Malhotra *et al.*, 2012) used a more competent function ϕ instead of the norm function $\|\cdot\|$. The benefit of using the function ϕ was that, the new results generalize and unify the results of (Rangamma & Prudhvi, 2012) as well as (Huang & Zhang, 2007) and (Vetro, 2007). (Malhotra *et al.*, 2012) defined a function ϕ from a normal cone into another normal cone, and so, their results cannot be applied if the cone is non-normal.

Some recent studies (see, (Çakallı *et al.*, 2012; Du, 2010; Feng & Mao, 2010; Kadelburg *et al.*, 2011)) show that the fixed point results proved in cone metric space are direct consequences of their usual metric versions. To overcome this drawback, recently, (Liu & Xu, 2013) improved the concept of cone metric spaces by defining the cone metric with values in a Banach algebra, instead of a Banach space, so that, the contractive conditions can involve the vector constants. The fixed point results thus obtained cannot be derived by their usual metric versions which was shown by an example in (Liu & Xu, 2013).

Inspired by the results of (Liu & Xu, 2013), in this paper, we prove some coincident and common fixed point results for four mappings in cone metric spaces over Banach algebras with solid cone which are not necessarily normal. We improve the definition of function ϕ used by (Malhotra *et al.*, 2012), by removing the normality condition from the domain and codomain cones of ϕ, as well as, we use a vector constant, instead the scalar in the contractive condition involving the function ϕ. Our result generalizes and unifies the results of (Liu & Xu, 2013), (Radenović, 2009) and (Rangamma & Prudhvi, 2012) and several other results, in cone metric spaces over Banach algebras.

2. Preliminaries

We first state some known definitions and facts which will be used throughout the paper.

Let E be a real Banach algebra with a unit e_E and a zero element 0_E. A nonempty closed subset P of E is called a cone if the following conditions hold:

(1) $\{0_E, e_E\} \subset P$;

(2) if $\alpha, \beta \in [0, \infty)$, then $\alpha P + \beta P \subseteq P$;

(3) $P^2 = PP \subset P$;

(4) $P \cap (-P) = \{0_E\}$.

A cone P is called a solid cone if P° is nonempty, where P° stands for the interior of P.

We can always define a partial ordering $\preceq_P$ with respect to P by $x \preceq_P y$ if and only if $y - x \in P$. We shall write $x \ll_P y$ to indicate that $y - x \in P^\circ$. We shall also write $\|\cdot\|$ as the norm on E. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0_E \preceq_P x \preceq_P y$ implies $\|x\| \leq K\|y\|$.

Throughout the paper, we consider the real Banach algebras.

Definition 2.1. (Liu & Xu, 2013) Let X be a nonempty set and E be a Banach algebra. A mapping $d\colon X \times X \to E$ is called a cone metric if it satisfies:

(i) $0_E \preceq_P d(x,y)$, for all $x, y \in X$, $d(x,y) = 0_E$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii) $d(x,y) \preceq_P d(x,z) + d(z,y)$ for all $x, y \in X$.

In this case, the pair (X, d) is called a cone metric space over Banach algebra E. If the cone P is normal then (X, d) is called a normal cone metric space.

Definition 2.2. (Dordević *et al.*, 2011) A sequence $\{u_n\}$ in a P is said to be a c-sequence in P if for each $c \gg_P 0_E$ (i.e., $0_E \ll_P c$), there exists $N \in \mathbb{N}$ such that $u_n \ll_P c$ for all $n > N$.

Definition 2.3. (Huang *et al.*, 2017) Let (X, d) be a cone metric space over Banach algebra E and $\{x_n\}$ be a sequence in X. We say that

(i) $\{x_n\}$ converges to $x \in X$ if $\{d(x_n, x)\}$ is a c-sequence and in this case we write $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is a Cauchy sequence if $\{d(x_n, x_m)\}$ is a c-sequence for n, m, i.e., for each $c \gg_P 0_E$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll_P c$ for all $n, m > N$.

(iii) (X, d) is complete if every Cauchy sequence in X is convergent.

It is obvious that the limit of a convergent sequence in a cone metric space (X, d) over Banach algebra E is unique.

Lemma 2.1. *(Janković et al., 2011) Let E be a Banach algebra and $u, v, w \in E$. Then*
(1) $u \ll_P w$ if $u \preceq_P v \ll_P w$ or $u \ll_P v \preceq_P w$;
(2) $u = 0_E$ if $0_E \preceq_P u \ll_P c$ for each $c \gg_P 0_E$.

The following results are well known and will be used in the sequel.

Lemma 2.2. *Let E be a Banach algebra and $u \in E$. Then the spectral radius of u is equal to $\rho(u) = \lim_{n\to\infty} \|u^n\|^{\frac{1}{n}}$.*

Lemma 2.3. *Let E be a Banach algebra and $k \in E$. If $\rho(k) < \lambda$, for some $\lambda > 0$ then $\lambda e_E - k$ is invertible in E, moreover, $(\lambda e_E - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}}$.*

Lemma 2.4. *(Huang et al., 2016) Let P be a cone in a Banach algebra E, $\{u_n\}$ and $\{v_n\}$ be two c-sequences in P, and $\alpha, \beta \in P$ be vectors, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence in P.*

Lemma 2.5. *(Huang & Radenović, 2015) Let P be a cone and $k \in P$ with $\rho(k) < 1$. Then $\{k^n\}$ is a c-sequence in P.*

Let X be a nonempty set and f, g be two self-maps on X and $x, z \in X$. Then x is called coincidence point of pair (f, g) if $fx = gx$, and z is called point of coincidence of pair (f, g) if $fx = gx = z$. The pair (f, g) is called weakly compatible if f and g commutes at their coincidence point, i.e. $fgx = gfx$, whenever $fx = gx$ for some $x \in X$.

For results on weakly compatible mappings in cone metric spaces, see (Janković *et al.*, 2010; Jungck *et al.*, 2009).

Now we can state our main results.

3. Main Results

Let E, B be two real Banach algebras, P and C be solid cones in E and B respectively. Let "$\preceq_P$" and "$\preceq_C$" be the partial orderings induced by P and C in E and B respectively, 0_E and 0_B are the zero vectors of E and B respectively; and e_E and e_B are the units of E and B respectively.

The following definition of function ϕ is an improved version of the definition used by (Malhotra *et al.*, 2012) (see, also, (Khan *et al.*, 2015)). Let $\phi : P \to C$ be a function satisfying:

(i) if $a, b \in P$ with $a \preceq_P b$ then $\phi[a] \preceq_C k\phi[b]$, for some positive real k;
(ii) $\phi[a+b] \preceq_C \phi[a] + \phi[b]$ for all $a, b \in P$;
(iii) the sequence $\{\phi[a_n]\}$ is c-sequence in C if and only if the sequence $\{a_n\}$ is a c-sequence in P.

We denote the set of all such functions by $\Phi(P, C)$, i.e., $\phi \in \Phi(P, C)$ if ϕ satisfies all above properties. It is clear that $\phi[a] = 0_B$ if and only if $a = 0_E$.

Let (X, d) be a cone metric space with solid cone P and $\phi \in \Phi(P, C)$. Then $d(x, y) \preceq_P d(x, z) + d(z, y)$ for all $x, y, z \in X$, therefore

$$\phi[d(x, y)] \preceq_C k\phi[d(x, z)] + k\phi[d(z, y)]. \tag{3.1}$$

Example 3.1. Let $E = L[0, 1]$ be the real Banach algebra of integrable functions $f(x)$ such that $\int_0^1 |f(x)|dx < \infty$ with norm $\|f\| = \int_0^1 |f(x)|dx$, the point-wise multiplication and the unit 1. Let $P = \{f \in E\colon f(t) \geq 0, t \in [0, 1]\}$ be the solid cone in E. Let $B = C^1_{\mathbb{R}}[0, 1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, point-wise multiplication and unit 1. Let $C = \{f \in B\colon f(t) \geq 0, t \in [0, 1]\}$ be the solid cone in B. Define $\phi\colon P \to B$ by $\phi[f] = \int_0^t f(x)dx$, $t \in [0, 1]$ for all $f \in P$. Then $\phi \in \Phi(P, C)$ with $k = 1$.

Example 3.2. Let E be a Banach algebra with solid cone P. Define $\phi\colon P \to P$ by $\phi[a] = a$, for all $a \in P$. Then $\phi \in \Phi(P, C)$ with $E = B, P = C$ and $k = 1$.

Example 3.3. Let $E = \mathbb{R}^2, P = \{(a, b) : a, b \in \mathbb{R}$ and $a, b \geq 0\}$ and $B = \mathbb{R}^3, C = \{(a, b, c) : a, b, c \in \mathbb{R}$ and $a, b, c \geq 0\}$, with coordinatewise multiplication and units $(1, 1)$ and $(1, 1, 1)$ respectively. Then P and C are solid cones. Define $\phi\colon P \to C$ by $\phi[(x, y)] = (x, y, ax + by)$, where a, b are positive constants, then $\phi \in \Phi(P, C)$ with $k = 1$.

Example 3.4. Let E be any real Banach algebra with normal cone P and normal constant K. Define $\phi : P \to [0, \infty)$ by $\phi[a] = \|a\|$, for all $a \in P$. Then $\phi \in \Phi(P, C)$ with $B = \mathbb{R}, C = [0, \infty)$ and $k = K$.

Example 3.5. Let E be the real vector space defined by

$$E = \{ax + b\colon a, b \in \mathbb{R}, x \in [1/2, 1]\}$$

with supremum norm and $P = \{ax + b \in E\colon a \leq 0, b \geq 0\}$. Then E is a real Banach algebra with point-wise multiplication and unit $e_E = 1$ and P is a normal cone with normal constant $K > 2$ (see (Rezapour & Hamlbarani, 2008)). Let $B = \mathbb{R}^2$ be with Euclidean norm, coordinate-wise multiplication, unit $e_B = (1, 1)$ and $C = \{(a, b)\colon a \geq 0, b \geq 0\}$. Then C is a normal cone with normal constant $K = 1$. Define $\phi\colon P \to C$ by

$$\phi\,[ax + b] = (-a, b) \text{ for all } ax + b \in P.$$

Then, $\phi \in \Phi(P, C)$ with $k = 1$.

The following theorem is an improved version of the main results of (Radenović, 2009), (Rangamma & Prudhvi, 2012) and (Liu & Xu, 2013), and unifies and generalizes these results.

Theorem 3.1. *Let (X, d) be a complete cone metric space over Banach algebra E and P be a solid cone. Suppose that f, g, h, l be four self-maps of X, $f(X) \subset l(X), g(X) \subset h(X)$ and the following condition is satisfied: there exist $\phi \in \Phi(P, C)$ and $\alpha \in C$ such that $\rho(\alpha) < 1$ and*

$$\phi[d(fx, gy)] \preceq_C \alpha\phi[d(hx, ly)] \text{ for all } x, y \in X. \tag{3.2}$$

If $h(X)$, $l(x)$ are closed subsets of X, and the pairs (f, h), (g, l) are weakly compatible, then the mappings f, g, h and l have a unique common fixed point.

Proof. Suppose, x_0 be any arbitrary point of X. Since $f(X) \subset l(X)$, there exists $x_1 \in X$ such that $fx_0 = lx_1$. Again, as $g(X) \subset h(X)$, there exists $x_2 \in X$ such that $gx_1 = hx_2$. Continuing in this manner, we obtain a sequence $\{z_n\}$ such that

$$z_{2n} = fx_{2n} = lx_{2n+1},$$
$$z_{2n+1} = gx_{2n+1} = hx_{2n+2} \text{ for all } n \geq 0.$$

We shall prove that $\{z_n\}$ is a Cauchy sequence in X.

Note that, if there exists $n \in \mathbb{N}$ such that $z_n = z_{n+1}$, e.g., suppose, $z_{2n_0} = z_{2n_0+1}$, then it follows from (3.2) that

$$\begin{aligned} \phi[d(z_{2n_0+2}, z_{2n_0+1})] &= \phi[d(fx_{2n_0+2}, gx_{2n_0+1})] \\ &\preceq_C \alpha\phi[d(hx_{2n_0+2}, lx_{2n_0+1})] \\ &= \alpha\phi[d(z_{2n_0+1}, z_{2n_0})]. \end{aligned}$$

As, $z_{2n_0} = z_{2n_0+1}$ the above inequality yields

$$\phi[d(z_{2n_0+2}, z_{2n_0+1})] = 0_B.$$

As, $\phi \in \Phi(P, C)$ therefore the above equality implies that $d(z_{2n_0+2}, z_{2n_0+1}) = 0_E$, i.e., $z_{2n_0+2} = z_{2n_0+1}$. Similarly, we obtain that

$$z_{2n_0} = z_{2n_0+1} = z_{2n_0+2} = z_{2n_0+3} = \cdots.$$

Therefore, $\{z_n\}$ is a Cauchy sequence.

Now, suppose that z_n and z_{n+1} are distinct for all $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$ we obtain from (3.2) that

$$\begin{aligned} \phi[d(z_{2n}, z_{2n+1})] &= \phi[d(fx_{2n}, gx_{2n+1})] \\ &\preceq_C \alpha\phi[d(hx_{2n}, lx_{2n+1})] \\ &= \alpha\phi[d(z_{2n-1}, z_{2n})]. \end{aligned}$$

Writing $d_n = \phi[d(z_n, z_{n+1})]$, we obtain

$$d_{2n} \preceq_C \alpha d_{2n-1} \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

Again, for each $n \in \mathbb{N}$ we obtain from (3.2) that

$$\begin{aligned} \phi[d(z_{2n+2}, z_{2n+1})] &= \phi[d(fx_{2n+2}, gx_{2n+1})] \\ &\preceq_C \alpha\phi[d(hx_{2n+2}, lx_{2n+1})] \\ &= \alpha\phi[d(z_{2n+1}, z_{2n})]. \end{aligned}$$

It follows from the above inequality that

$$d_{2n+1} \preceq_C \alpha d_{2n} \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$d_n \preceq_C \alpha d_{n-1} \text{ for all } n \in \mathbb{N}. \tag{3.5}$$

Repeated use of (3.5) that

$$d_n \preceq_C \alpha d_{n-1} \preceq_C \alpha^2 d_{n-2} \preceq_C \cdots \preceq_C \alpha^n d_0 \text{ for all } n \in \mathbb{N}. \tag{3.6}$$

Let $n, m \in \mathbb{N}$ and $m > n$, then by (3.1) and (3.6) we obtain

$$\begin{aligned} \phi[d(z_n, z_m)] &\preceq_C k\phi[d(z_n, z_{n+1})] + k\phi[d(z_{n+1}, z_{n+2})] + \cdots + k\phi[d(z_{m-1}, z_m)] \\ &= k[d_n + d_{n+1} + \cdots + d_{m-1}] \\ &\preceq_C k[\alpha^n d_0 + \alpha^{n+1} d_0 + \cdots + \alpha^{m-1} d_0] \\ &= k\alpha^n [e_B + \alpha + \alpha^2 + \cdots + \alpha^{m-n-1}] d_0 \\ &\preceq_C k\alpha^n \left[\sum_{i=0}^{\infty} \alpha^i\right] d_0 \\ &= k\alpha^n (e_B - \alpha)^{-1} d_0. \end{aligned}$$

Since $\rho(\alpha) < 1$, therefore, by Lemma 2.4 and Lemma 2.5, the sequence $\{k\alpha^n (e_B - \alpha)^{-1} d_0\}$ is a c-sequence in C. Hence, by Lemma 2.1 and the fact that $\phi \in \Phi(P, C)$, we have, the sequence $\{z_n\} = \{hx_{n+1}\}$ is a Cauchy sequence.

Since X is complete, there exists $w \in X$ such that $z_n \to w$ as $n \to \infty$. Since, $z_{2n} = lx_{2n+1} \in l(X)$, $z_{2n+1} = hx_{2n+2} \in h(X)$ for all $n \in \mathbb{N}$ and $l(X)$, $h(X)$ are closed subsets of X, there exist $u, v \in X$ such that

$$w = lu = hv.$$

Therefore, from by (3.1) and (3.2) we obtain

$$\begin{aligned} \phi[d(fv, w)] &\preceq_C k\phi[d(fv, gx_{2n+1})] + k\phi[d(gx_{2n+1}, w)] \\ &\preceq_C k\alpha\phi[d(hv, lx_{2n+1})] + k\phi[d(gx_{2n+1}, w)] \\ &= k\alpha\phi[d(w, z_{2n})] + k\phi[d(z_{2n+1}, w)]. \end{aligned}$$

Since, $z_n \to w$ as $n \to \infty$, the sequences $\{d(w, z_{2n})\}$ and $\{d(z_{2n+1}, z)\}$ are c-sequences in P. As, $\phi \in \Phi(P, C)$, the sequences $\phi[\{d(w, z_{2n})\}]$ and $\phi[\{d(z_{2n+1}, z)\}]$ are c-sequences in C. By Lemma 2.1, Lemma 2.4 and the above inequality, the sequence $\{\phi[d(fv, w)]\}$ is a c-sequence in C. This shows that the sequence $\{d(fv, w)\}$ is a c-sequence in P, and so, we must have $d(fv, w) = 0_E$, i.e., $fv = w$. Therefore

$$\begin{aligned} \phi[d(w, gu)] &= \phi[d(fv, gu)] \\ &\preceq_C \alpha\phi[f(hv, lu)] \\ &= \alpha\phi[f(w, w)] \\ &= \theta_B. \end{aligned}$$

Hence, $d(w, gu) = 0_E$, i.e., $w = gu$. Thus

$$w = gu = lu = hv = fv. \tag{3.7}$$

As, the pairs (f, h), (g, l) are weakly compatible it follows from (3.7) that $fw = fhv = hfv = hw$ and $gw = glu = lgu = lw$. Hence

$$gw = lw = hw = fw. \tag{3.8}$$

Using (3.2), (3.7) and (3.8) we obtain

$$\begin{aligned} \phi[d(w, gw)] &= \phi[d(fv, gw)] \\ &\preceq_C \alpha\phi[d(hv, lw)] \\ &= \alpha\phi[d(w, gw)]. \end{aligned}$$

Thus, $\phi[d(w, gw)] \preceq_C \alpha\phi[d(w, gw)]$. Successive use of this inequality yields

$$\phi[d(w, gw)] \preceq_C \alpha\phi[d(w, gw)] \preceq_C \alpha^2\phi[d(w, gw)] \preceq_C \cdots \preceq_C \alpha^n\phi[d(w, gw)].$$

As, $\rho(\alpha) < 1$, the sequence $\{\alpha^n\}$ is a c-sequence in C, and by Lemma 2.1, Lemma 2.4 we obtain that the sequence $\{\phi[d(w, gw)]\}$ is a c-sequence in C. By definition, $d(w, gw)$ is a c-sequence in P, and so, $d(w, gw) = 0_E$, i.e., $gw = w$. Hence, we obtain from (3.8) that

$$w = gw = lw = hw = fw. \tag{3.9}$$

Thus, w is a common fixed point of the mappings f, g, h and l.

For uniqueness of fixed point, suppose w' is a common fixed point of the mappings f, g, h and l and w and w' are distinct. Then, we have

$$w' = gw' = lw' = hw' = fw'.$$

Using (3.2) we obtain

$$\begin{aligned} \phi[d(w, w')] &= \phi[d(fw, gw')] \\ &\preceq_C \alpha\phi[d(hw, lw')] \\ &= \alpha\phi[d(w, w')]. \end{aligned}$$

Again, since $\rho(\alpha) < 1$, the above inequality yields $\phi[d(w, w')] = 0_B$. This shows that $d(w, w') = 0_E$, i.e., $w = w'$. This contradiction proves the uniqueness of common fixed point. □

The following corollary is a generalized version of Theorems 2.1 of (Rangamma & Prudhvi, 2012) and Theorem 2.1 of (Radenović, 2009).

Corollary 3.1. *Let (X, d) be a complete cone metric space over a Banach algebra E and P be solid cone. Suppose that f, g, h are self-maps of X, $f(X) \cup g(X) \subset h(X)$ and the following condition is satisfied: there exists a number $a \in [0, 1)$ such that*

$$\|d(fx, gy)\| \leq a\|d(hx, hy)\| \text{ for all } x, y \in X.$$

If $h(X)$ is a closed subset of X, and the pairs (f, h), (g, h) are weakly compatible, then the mappings f, g and h have a unique common fixed point.

Proof. Take $B = \mathbb{R}, C = [0, \infty)$, $lx = hx$ for all $x \in X$, and $\phi[a] = \|a\|$ for all $a \in P$, in Theorem 3.1. Then $\phi \in \Phi(P, C)$ with $k = K$ =normal constant of P and the result follows from Theorem 3.1. □

The following corollary is an improved and generalized version of (Huang & Zhang, 2007) and (Liu & Xu, 2013).

Corollary 3.2. *Let (X, d) be a complete cone metric space over a Banach algebra E and P be solid cone. Suppose that f, g, h are self-maps of X, $f(X) \cup g(X) \subset h(X)$ and the following condition is satisfied: there exists $\alpha \in C$ such that $\rho(\alpha) < 1$ and*

$$d(fx, gy) \preceq \alpha d(hx, hy) \text{ for all } x, y \in X.$$

If $h(X)$ is a closed subset of X, and the pairs (f, h), (g, h) are weakly compatible, then the mappings f, g and h have a unique common fixed point.

Proof. Take $E = B, P = C$ and define $\phi : P \to P$ by $\phi[a] = a$, for all $a \in P$, in Theorem 3.1. Then $\phi \in \Phi(P, C)$ with $k = 1$ and the result follows from Theorem 3.1. □

Example 3.6. Let $E = \mathbb{R}^2$ be the Banach algebra with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$ and the multiplication defined by $(x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$ for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. The unit of E is $e_E = (1, 0)$. Let $P = \{(x_1, x_2) \colon x_1, x_2 \geq 0\}$. Then, P is a solid cone. Define $\phi \colon P \to P$ by $\phi[a] = a$, for all $a \in P$. Then $\phi \in \Phi(P, C)$ with $E = B, P = C$ and $k = 1$.

Let $X = \mathbb{R}^2$ and let $d \colon X \times X \to \mathbb{R}$ be defined by

$$d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1|, |x_2 - y_2|).$$

Then, (X, d) is a complete cone metric space over Banach algebra E. Define the mappings $f, g, h \colon X \to X$ by

$$f(x_1, x_2) = g(x_1, x_2) = (\ln(1 + |x_1|), \arctan(2 + |x_2|) + 2ax_1), \; g(x_1, x_2) = h(x_1, x_2) = (2x_1, x_2)$$

for all $(x_1, x_2) \in X$, where $a > 0$. Then, it is easy to see that the condition (3.2) is satisfied with $\alpha = \left(\frac{1}{2}, a\right)$. Obviously, $h(X)$ is a closed subset of X and $f(X) \subset h(X)$. It is easy to see that f and h are weakly compatible. All the conditions of Theorem 3.1 are satisfied, hence f and h has a unique common fixed point.

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Introduction to Non-Diophantine Number Theory

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Abstract

In the 19^{th} century, non-Euclidean geometries were discovered and studied. In the 20th century, non-Diophantine arithmetics were discovered and studied. Construction of non-Diophantine arithmetics is based on more general mathematical structures, which are called abstract prearithmetics, as well as on the projectivity relation between abstract prearithmetics. In a similar way, as set theory gives a foundation for mathematics, the theory of abstract prearithmetics provides foundations for the theory of the Diophantine and non-Diophantine arithmetics. In this paper, we use abstract prearithmetics for developing fundamentals of non-Diophantine number theory, which can be also called non-Diophantine higher arithmetic as the conventional number theory is called higher arithmetic. In particular, we prove the Fundamental Theorem of Arithmetic for a wide range of abstract prearithmetics.

Keywords: number, arithmetic, number theory, divisibility, prime number, factorization, addition, difference, multiplication

1. Introduction

The arithmetic $\mathbb{N}$ of all natural numbers is one of the most basic objects in mathematics. People in general and mathematicians in particular believe that the laws of this arithmetic are universal and unique. The equality $2 + 2 = 4$ is treated as an eternal absolute truth. However, the best thinkers started questioning universality of $\mathbb{N}$ long ago suggesting various examples when the rules of this arithmetic, which is called the Diophantine arithmetic, are not true (cf., for example, ((Helmholtz, 1887), (Kline, 1982), (Kline, 1985), (Davis, 1972) (Davis & Hersh, 1998), (Burgin, 1997), (Burgin, 2001), (Gardner, 2005), (Cleveland, 2008)).

Here we give only three of such examples although it is possible to find much more.

1. One raindrop added to another raindrop does not make two raindrops (Helmholtz, 1887). Mathematically, it is described by the equality $1 + 1 = 1$.
2. If one puts a lion and a rabbit in a cage, one will not find two animals in the cage later on (cf. (Kline, 1985)). In terms of numbers, it will mean $1 + 1 = 1$.

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3. When a cup of milk is added to a cup of popcorn then only one cup of mixture will result because the cup of popcorn will very nearly absorb a whole cup of milk without spillage (Davis & Hersh, 1998). So, in this case we also have 1 + 1 = 1.

Besides, recently the expression 1 + 1 = 3 has become a popular metaphor for synergy in a variety of areas: in business and industry (cf., for example, (Beechler, 2013), (Gottlieb, 2013), (Grant & Johnston, 2013), (Marks & Mirvis, 2010)), in economics and finance (cf., for example, (Burgin & Meissner, 2017)), in psychology and sociology (cf., for example, (Brodsky *et al.*, 2004), (Bussmann, 2013), (Enge, 2017), (Frame & Meredith, 2008), (Klees, 2006), (Mane, 1952), (Trott, 2015)), library studies (cf., for example, (Marie, 2007)), biochemistry and bioinformatics (cf., for example, (Kroiss *et al.*, 2009)), computer science (cf., for example, (Derboven, 2011), (Glyn, 2017), (Lea, 2016)), physics (cf., for example, (Lang, 2014)), medicine (cf., for example, (Lawrence, 2011), (Phillips, 2016), (Trabacca *et al.*, 2012)) and pedagogy (cf., for example, (Nieuwmeijer, 2013)).

All these situations indicated existence of other non-Diophantine arithmetics in which it would be possible to describe all these situations in a rigorous mathematical way. Thus, the first class of non-Diophantine arithmetics was discovered and explored in 1975 while the first publication appeared in 1977 (Burgin, 1977). Later other classes were introduced (Burgin, 2010). Recently non-Diophantine arithmetics found applications in physics (Czachor, 2016), (Czachor, 2017*a*), (Czachor & Posiewnik, 2016) and psychology (Czachor, 2017*b*). Following the classical understanding of arithmetic, here non-Diophantine arithmetics are considered as arithmetics of natural numbers.

Construction of non-Diophantine arithmetics is based on more general mathematical structures, which are called abstract prearithmetics, as well as on projectivity between abstract prearithmetics. (Burgin, 2010). In a similar way, as set theory is a foundation for mathematics, the theory of abstract prearithmetics provides foundations for the theory of non-Diophantine arithmetics and the Diophantine arithmetic. In addition, the theory of abstract prearithmetics includes theories of various conventional mathematical structures, such as rings, fields, ordered rings, ordered fields, lattices and Boolean algebras, as its subtheories. This allows using constructions from the theory of abstract prearithmetics for its subtheories of conventional mathematical structures. Abstract prearithmetics also provide a unified algebraic context for some traditional mathematical constructions, such as logarithmic scales, modular arithmetics and computer arithmetics, which are used in many applications in mathematics, science and technology.

In essence, an abstract prearithmetic is a universal algebra (algebraic system) with two operations and a partial order. Operations are called addition and multiplication but in a general case, there are no restrictions on these operations. Some of abstract prearithmetics are numerical, that is, their elements are numbers, e.g., natural numbers or real numbers. A numerical prearithmetic that satisfies additional conditions, in particular, containing all natural numbers and no other elements is called an arithmetic of natural numbers. A numerical prearithmetic that satisfies additional conditions, in particular, contains all whole numbers and no other elements is called an arithmetic of whole numbers. Everybody knows the conventional Diophantine arithmetic $\mathbb{N}$. However, there are also non-Diophantine arithmetics of natural numbers introduced and studied in (Burgin, 1977), (Burgin, 1997), (Burgin, 2001), (Burgin, 2007), (Burgin, 2010).

In this paper, we use abstract prearithmetics for developing fundamentals of non-Diophantine number theory, which can be also called non-Diophantine higher arithmetic as the conventional number theory is called higher arithmetic (cf., for example, (Broadbent, 1971), (Davenport, 1999), (Hayes, 2009). Number theory has three basic goals:

- Exploration of properties of and relations between natural numbers
- Classification of natural numbers and formation of important and interesting classes of natural numbers
- Exploration of properties of and relations between classes of natural numbers.

In this paper, we pursue these goals in the context of abstract prearithmetics. We pay the main attention to the problems of divisibility and primality. In particular, we prove the Fundamental Theorem of Arithmetic for a wide range of abstract prearithmetics.

In what follows, we use the following notation:

- N is the set of all natural numbers,
- $\mathbb{N}$ is the conventional (Diophantine) arithmetic of all natural numbers,
- W is the set of all whole numbers
- $\mathbb{W}$ is the conventional (Diophantine) arithmetic of all natural numbers,
- R is the set of all real numbers,
- $\mathbb{R}$ is the conventional (Diophantine) arithmetic of all real numbers.

2. Abstract prearithmetics

An *abstract prearithmetic* is a set (often a set of numbers) A with a partial order $\leq$ and two binary operations + (addition) and · (multiplication), which are defined for all its elements. It is denoted by $\mathbb{A} = (A; +, \cdot, \leq)$. The set A is called *the set of the elements or numbers* or *the carrier* of the prearithmetic $\mathbb{A}$. As always, if $x \leq y$ and $x \neq y$, then we denote this relation by $x < y$. Operation + is called *addition* and operation · is called *multiplication* in the abstract prearithmetic $\mathbb{A}$. Note that an abstract prearithmetic can have more than two operations and more than one order relations.

Example 2.1. Naturally, the conventional Diophantine arithmetic $\mathbb{N}$ of all natural numbers, the conventional arithmetic $\mathbb{R}$ of all real numbers and the conventional Diophantine arithmetic $\mathbb{W}$ of all whole numbers are an abstract prearithmetics.

Example 2.2. Another example of abstract prearithmetics is a *modular arithmetic*, which is sometimes known as *residue arithmetic* or *clock arithmetic* (Kurosh, 1963)). It is studied in mathematics and used in physics and computing. In modular arithmetic, operations of addition and multiplication are defined but in contrast to the conventional arithmetic, its numbers form a cycle upon reaching a certain value, which is called *the modulus*. The rigorous approach to the theory of modular arithmetic was worked out by Carl Friedrich Gauss.

All these examples show that conventional arithmetics are abstract prearithmetics. However, there are many unusual abstract prearithmetics.

Example 2.3. Let us consider the set N of all natural numbers with the standard order $\leq$ and introduce the following operations:

$$a \oplus b = a \cdot b$$

$$a \otimes b = a^b$$

Then the system $\mathbb{A} = (\mathbb{N}; \oplus, \otimes, \leq)$ is an abstract prearithmetic with addition $\oplus$ and multiplication $\otimes$.

Example 2.4. Let us consider the set $\mathbb{R}^{++}$ of all positive real numbers is with the standard order $\leq$ and introduce the following operations:

$$a \boxplus b = a + b$$

$$a \divideontimes b = a \div b$$

Then the system $\mathbb{B} = (\mathbb{R}^{++}; \boxplus, \divideontimes, \leq)$ is an abstract prearithmetic with addition $\boxplus$ and multiplication $\divideontimes$.

Example 2.5. Many algebraic structures studied in algebra are abstract prearithmetics with a trivial order, i.e., any ring, lattice, Boolean algebra, linear algebra, field, Ω-group, Ω-ring, Ω-algebra (Kurosh, 1963), (Baranovich & Burgin, 1975), topological ring, topological field, normed ring, normed algebra, normed field, and in essence, any universal algebra with two operations is an abstract prearithmetic with a trivial order. The same structures with nontrivial order are also abstract prearithmetics. Examples are given by ordered rings, ordered linear algebras and ordered fields. Besides, it is possible to treat universal algebras with one operation as abstract prearithmetics with a trivial order and trivial multiplication.

Elements 0 and 1 have very special properties in the conventional Diophantine arithmetic. We explore these properties in the general setting of abstract prearithmetics.

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$.

Definition 2.1. a) An element z, which is usually denoted by 0 or 0_A, is called an *additive zero* of $\mathbb{A}$ if $a + z = z + a = a$ for any element a from A.

b) An element z, which is usually denoted by 0 or 0_{mA}, is called a *multiplicative zero* of $\mathbb{A}$ if $a \cdot z = z \cdot a = z$ for any element a from A.

c) An element b, which is usually denoted by 1 or 1_A, is called a *multiplicative one* of $\mathbb{A}$ if $a \cdot b = b \cdot a = a$ for any element z from A.

Lemma 2.1. *An additive zero is unique.*

Indeed, suppose an abstract prearithmetic $\mathbb{A}$ has two additive zeros 0_1 and 0_2. Then we have

$$0_1 = 0_1 + 0_2 = 0_2$$

Lemma 2.2. *A multiplicative zero is unique.*

Indeed, suppose an abstract prearithmetic $\mathbb{A}$ has two multiplicative zeros 0_1 and 0_2 . Then we have

$$0_1 = 0_1 \cdot 0_2 = 0_2$$

Lemma 2.3. *A multiplicative* 1 *is unique.*

Indeed, suppose an abstract prearithmetic $\mathbb{A}$ has two additive zeros 1_1 and 1_2 . Then we have

$$1_1 = 1_1 \cdot 1_2 = 1_2$$

The number 0 in the conventional Diophantine arithmetic is both additive and multiplicative zero while the number 1 is the multiplicative one. However, in a general case of abstract prearithmetics, additive and multiplicative zeros do not coincide as the following examples demonstrate.

Example 2.6. Let us define an abstract prearithmetic $\mathbb{A} = (N; \oplus, \otimes, \leq)$ where N is the set of all natural numbers by the following rules:

$$m \oplus n = m + n$$

$$m \otimes n = m \cdot n + 3$$

where $m + n$ are arbitrary natural numbers, while + is conventional addition and · is conventional multiplication of natural numbers.

We can see that 0 is the additive zero but not the multiplicative zero in $\mathbb{A}$.

Example 2.7. Let us define an abstract prearithmetic $\mathbb{A} = (Z; \oplus, \otimes, \leq)$ where Z is the set of all integer numbers by the following rules:

$$m \oplus n = m + n + 2$$

$$m \otimes n = m \cdot n$$

where $m + n$ are arbitrary integer numbers, while + is conventional addition and · is conventional multiplication of integer numbers.

We can see that 0 is the multiplicative zero but not the multiplicative zero in $\mathbb{A}$. At the same time, -2 is the additive zero in $\mathbb{A}$.

However, in some cases, additive and multiplicative zeros coincide.

Proposition 1. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ has the additive zero* 0*, contains an opposite element $-x$ for each element x, multiplication is distributive with respect to addition and preserves opposite elements, i.e., $z \cdot (-x) = -(z \cdot x)$ for any elements z and x from A, then* 0 *is also the multiplicative zero.*

Proof. Let us take an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ that satisfies all initial conditions of the Proposition. We remind that if x is an element from A, then an element y is called opposite to x and denoted by x when $x + y = 0$. Taking arbitrary elements x and z from A, we have

$$0 \cdot x = (z + (-z)) \cdot x = z \cdot x + (-z) \cdot x = z \cdot x + (-(z \cdot x)) = 0$$

The identity $x \cdot 0 = 0$ is proved in a similar way. Consequently, 0 is also the multiplicative zero. □

Proposition is proved.

As a corollary, we obtain a well-known result from the theory of rings (cf. (Kurosh, 1963)).

Corollary 2.1. *In a ring, additive and multiplicative zeros coincide.*

An important property of the Diophantine arithmetic $\mathbb{N}$ is existence of the successor Sx and the predecessor Px for any number x from $\mathbb{N}$. The successor Sx is defined by the following conditions

$$x < S\,x \text{ and if } x \leq z \leq S\,x, \text{ then } z \text{ is equal either to } x \text{ or to } S\,x$$

The predecessor Px is defined by the following conditions

$$Px < x \text{ and if } Px \leq z \leq x, \text{ then } z \text{ is equal either to } x \text{ or to} Px$$

In what follows we assume that all considered abstract prearithmetics have this property, i.e., any element x has the successor Sx and the predecessor Px.

It is possible to extend addition and multiplication in abstract prearithmetics to n-ary addition and multiplication by the following formulas using induction on n.

$$\sum_{i=1}^{1} a_i = a_1$$

$$\sum_{i=1}^{2} a_i = a_1 + a_2$$

If $\sum_{i=1}^{n-1} a_i$ is defined, then

$$\sum_{i=1}^{n} a_i = (\sum_{i=1}^{n-1} a_i) + a_n$$

In the same way, we have

$$\prod_{i=1}^{1} a_i = a_1$$

$$\prod_{i=1}^{2} a_i = a_1 \cdot a_2$$

If $\prod_{i=1}^{n-1} a_i$ is defined, then

$$\prod_{i=1}^{n} a_i = (\prod_{i=1}^{n-1} a_i) \cdot a_n$$

Note that while in the conventional Diophantine arithmetic, addition and multiplication are commutative and associative, for arbitrary abstract prearithmetic, this is not always true and it is possible to define other n-ary operations.

When all a_i are equal to the same element, say a, we use the following notation

$$\sum_{i=1}^{n} a_i = n[a]$$

$$\prod_{i=1}^{n} a_i = [a]^n$$

Definitions imply the following result.

Lemma 2.4. *For any natural number n and any element a, we have* $(n+1)[a] = n[a] + a$ *and* $[a]^{n+1} = [a]^n \cdot a$*. When addition + is associative, it is possible to remove parentheses and we have*

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

$$n[a] = na.$$

and when multiplication · is associative, it is also possible to remove parentheses and we have

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot \ldots \cdot a_n$$

$$[a]^n = a^n$$

The Diophantine (conventional) arithmetic of natural numbers has the, so-called, Archimedean property, which is named after the great ancient Greek mathematician Archimedes of Syracuse and is important for proofs of many results in arithmetic and number theory. For instance, the Archimedean property, which is often called the Archimedean axiom, is important for proving that the set of all natural numbers and the set of all prime numbers are infinite. This property (axiom) is also very important for axiomatics in geometry (cf. (Veronese, 1889), (Hilbert, 1899)).

The Archimedean property (axiom) states that if we take any two natural numbers m and n, in spite that n may be enormously larger than m, it is always possible to add m enough times to itself, i.e., to take a sum $m + m + \ldots + m$, so that the result will be larger than n.

In contrast to the Diophantine arithmetic $\mathbb{N}$, the Archimedean property is invalid in many Diophantine arithmetics, such as $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{C}$, and many non-Diophantine arithmetics. Other examples of non-Archimedean arithmetics are: the arithmetic of cardinal numbers (cf., for example, (Fraenkel *et al.*, 1973)), the nonstandard arithmetic of hyperreal numbers (Robinson, 1966) , and the arithmetic of real hypernumbers (Burgin, 2012). However, many non-Diophantine arithmetics have the Archimedean property and we study it because it is important for number theory. In abstract prearithmetics, there are three principal structures - one is relational and two are operational, that is why it is natural to consider four Archimedean properties, which do not coincide in the general case.

Definition 2.2. a) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ satisfies the *Successively Archimedean Property,* or is a *successively Archimedean prearithmetic,* if the inequality $a < b$ for $a, \ b \in A$ implies existence of a natural number n such that $S^n a$ is larger than or equal to b.

b) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ satisfies the *Additively Archimedean Property*, or is *additively Archimedean prearithmetic,* if the inequality $a < b$ for $a, \ b \in A$ implies existence of a natural number n such that

$$b \leq n[a]. \tag{2.1}$$

c) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ satisfies the *Multiplicatively Archimedean Property*, or is *multiplicatively Archimedean prearithmetic,* if for any elements a and b from A, the inequality $a < b$ implies existence of a natural number n such that

$$b \leq [a]^n. \tag{2.2}$$

d) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the additive 0 satisfies the *left Binary Archimedean Property* for addition, or is a *binary for addition Archimedean prearithmetic* from the left, if for any elements a and b from A, the inequality $0 < a < b$ implies that there is an element q less than b such that

$$b \leq q + a \tag{2.3}$$

and satisfies the *right Binary Archimedean Property* for addition, or is a *binary for addition Archimedean prearithmetic* from the right, if for any elements a and b from A, the inequality $0 < a < b$ implies that there is an element q less than b such that

$$b \leq a + q \tag{2.4}$$

When addition + is commutative, then the right Binary Archimedean Property for addition coincides with the left Binary Archimedean Property for addition. When an abstract prearithmetic has both the right and left Binary Archimedean Properties for addition, then it has the *Binary Archimedean Property for addition.*

Example 2.8. The conventional arithmetic $2\mathbb{N}$ of all even numbers and conventional arithmetic $3\mathbb{N}$ of all natural numbers divisible by 3 have all four properties - the Successively Archimedean Property, Binary Archimedean Property for addition, Additively Archimedean Property and Multiplicatively Archimedean Property.

However, in general, these properties are independent because there are prearithmetics and arithmetics, which have only one part of the Archimedean Properties. For instance, the Diophantine arithmetic $\mathbb{N}$ does not have the Multiplicatively Archimedean Property but has other Archimedean Properties. That is why when the Archimedean Property is defined for multiplicative groups or semigroups, its validity is assumed for all elements but the unit element e (Fuchs, 1963).

Example 2.9. The conventional Diophantine arithmetic $\mathbb{W}$ of all whole numbers has the Successively Archimedean Property and Binary Archimedean Property for addition but does not have the Multiplicatively Archimedean Property and Additively Archimedean Property because these properties do not hold for the number 0.

That is why when the Archimedean Property is defined for additive groups or semigroups, its validity is assumed for all elements but the zero 0 (Fuchs, 1963).

Example 2.10. The conventional arithmetic $\mathbb{R}_1$ of all larger than 1 real numbers does not have the Successively Archimedean Property because real numbers do not have successors but has the Binary Archimedean Property for addition, Multiplicatively Archimedean Property, and Additively Archimedean Property.

There are also prearithmetics and arithmetics, which do not have any of the Archimedean Properties. Examples are: the arithmetic ***Ord*** of all ordinal numbers, the arithmetic $\mathbb{NW}$ of all nonstandard whole numbers (Robinson, 1966) and the arithmetic NH of all whole hypernumbers (Burgin, 2012).

Lemma 2.5. *An additively Archimedean prearithmetic cannot have the additive zero* 0.

Indeed, if $0 < b$, then any element $n[0]$ is equal to 0 and still less than b.

Lemma 2.6. *A multiplicatively Archimedean prearithmetic cannot have the multiplicative one* 1 *or the multiplicative zero* 0.

Indeed, if $1 < b$, then any element $[1]^n$ is equal to 1 and still less than b. In a similar way, if $0 < b$, then any element $[0]^n$ is equal to 0 and is still less than b.

Proposition 2. *Any additively Archimedean prearithmetic is a binary for addition Archimedean prearithmetic from the left.*

Proof. Let us consider an additively Archimedean abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ and its elements $a < b$. By Definition 2.2, A satisfies condition (2.3), i.e., for some natural number n, we have $b \leq n[a] = (\ldots(((a + a) + a) + a)\ldots) + a$. As $a < b$ and $1 \leq n$, we can take the least n such that $b \leq n[a]$. It means that $q = (n-1)[a] < b$ and $b \leq (n-1)[a] + a = q + a$. □

Proposition is proved.

Corollary 2.2. *Any additively Archimedean prearithmetic with associative addition is a binary for addition Archimedean prearithmetic from the left and from the right.*

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with a discrete order $\leq$ and in which addition $+$ preserves the order $\leq$.

Lemma 2.7. *If $Sb \leq b + a$ for any elements a and b from A, then for any natural number n, we have $S^n a \leq (n + 1)[a]$.*

Proof. We use induction on n to show that $S^n a \leq (n + 1)[a]$. For $n = 1$, taking a as b, we have

$$Sa \leq a + a = 2[a]$$

For $n = 2$, taking Sa as b, we have

$$S^2 a = S(Sa) \leq Sa + a \leq (a + a) + a = 3[a]$$

as addition $+$ preserves the order $\leq$. Let us assume that our statement is true for $n - 1$, i.e.,

$$Sn - 1a \leq n[a]$$

Then we have

$$S^n a = S(S^{n-1} a) \leq S^{n-1} a + a \leq n[a] + a = (n + 1)[a]$$

as addition $+$ preserves the order $\leq$. The principle (axiom) of the mathematical induction gives us the necessary result. □

Lemma is proved.

Proposition 3. *If in a successively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, we have $Sb \leq b + a$ for any elements a and b from A and addition $+$ preserves the order $\leq$, then $\mathbb{A}$ is an additively Archimedean prearithmetic.*

Proof. Let us consider elements a and b from A such that $a < b$. As $\mathbb{A}$ is a successively Archimedean prearithmetic, there is a natural number n such that $S^n a$ is larger than or equal to b, i.e., $b \leq S^n a$. Then by Lemma 2.7, we have

$$b \leq S^n a \leq (n + 1)[a]$$

It means that $\mathbb{A}$ is an additively Archimedean prearithmetic. □

Proposition is proved.

Proposition 4. *If in a successively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the additive zero 0, addition $+$ preserves the order $\leq$ and 0 is the least element in $\mathbb{A}$, then $\mathbb{A}_P = (A \setminus \{0\}; +, \cdot, \leq)$ is an additively Archimedean prearithmetic.*

Proof. As $0 < a$ for any element a from $\mathbb{A}_P$, we have $b = b + 0 \leq b + a$. By Definition 2.2, we have

$$b < Sb \leq b + a$$

Thus, by Proposition 2, $\mathbb{A}_P$ is an additively Archimedean prearithmetic. □

Proposition is proved.

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with a discrete order $\leq$ and multiplication $\cdot$ preserves the order $\leq$.

Lemma 2.8. *If $Sb \leq b \cdot a$ for any elements a and b from A, then for any natural number n, we have $S^n a \leq [a]^{(n+1)}$.*

Proof. We use induction on n to show that $S^n a \leq [a]^{(n+1)}$. For $n = 1$, taking a as b, we have

$$Sa \leq a \cdot a = [a]^2$$

For $n = 2$, taking Sa as b, we have

$$S^2 a = S(Sa) \leq Sa \cdot a \leq (a \cdot a) \cdot a = [a]^3$$

as multiplication $\cdot$ preserves the order $\leq$. Let us assume that our statement is true for $n - 1$, i.e.,

$$S^{n-1} a \leq [a]^n$$

Then we have

$$S^n a = S(S^{n-1} a) \leq S^{n-1} a \cdot a \leq [a]^n \cdot a = [a]^{(n+1)}$$

as multiplication $\cdot$ preserves the order $\leq$. The principle (axiom) of the mathematical induction gives us the necessary result. □

Lemma is proved.

Proposition 5. *If in a successively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with a discrete order $\leq$, we have $Sb \leq b \cdot a$ for any elements a and b from A and multiplication + preserves the order $\leq$, then $\mathbb{A}$ is a multiplicatively Archimedean prearithmetic.*

Proof. Let us consider elements a and b from A such that $a < b$. As $\mathbb{A}$ is a successively Archimedean prearithmetic, there is a natural number n such that $S^n a$ is larger than or equal to b, i.e., $b \leq S^n a$. Then by Lemma 2.8, we have

$$b \leq S^n a \leq [a]^{(n+1)}$$

It means that $\mathbb{A}$ is a multiplicatively Archimedean prearithmetic. □

Proposition is proved.

Corollary 2.3. *If in a successively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with a discrete order $\leq$, we have $Sb \leq b \cdot a$ for any elements a and b from A and multiplication $\cdot$ preserves the order $\leq$, then $\mathbb{A}$ is a binary Archimedean prearithmetic.*

Proposition 6. *If in a successively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the multiplicative 1, multiplication $\cdot$ preserves the order $\leq$ and 1 is the smallest element in $\mathbb{A}$, then $\mathbb{A}_C = (A \backslash \{1\}; +, \cdot, \leq)$ is a multiplicatively Archimedean prearithmetic.*

Proof. As $1 < a$ for any element a from $\mathbb{A}_C$, we have $b = b \cdot 1 \leq b \cdot a$. By Definition 2.2, we have

$$b < Sa \leq b \cdot a$$

Thus, by Proposition 5, $\mathbb{A}_C$ is an multiplicatively Archimedean prearithmetic. □

Proposition is proved.

Let us study relations between multiplication and addition.

Lemma 2.9. *If $b + a \leq b \cdot a$ for any elements a and b from A, then for any natural number n, we have $n[a] \leq [a]^n$.*

Proof. We use induction on n to prove the lemma. For $n = 2$, taking a as b, we have

$$2[a] = a + a \leq a \cdot a = [a]^2.$$

Let us assume that our statement is true for $n - 1$, i.e.,

$$n[a] \leq [a]^n$$

Then we have

$$(n+1)[a] = n[a] + a \leq [a]^n \cdot a = [a]^{(n+1)}$$

The principle (axiom) of the mathematical induction gives us the necessary result. □

Lemma is proved.

Proposition 7. *If in an additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ we have $b + a \leq b \cdot a$ for any elements a and b from A, then $\mathbb{A}$ is a multiplicatively Archimedean prearithmetic.*

Proof. Let us consider elements a and b from A such that $a < b$. As $\mathbb{A}$ is an additively Archimedean prearithmetic, there is a natural number n such that $n[a]$ is larger than or equal to b, i.e., $b \leq n[a]$. Then by Lemma 2.9, we have

$$b \leq n[a] \leq [a]^n$$

□

Proposition is proved.

Corollary 2.4. *If in a additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ we have $b + a \leq b \cdot a$ for any elements a and b from A, then $\mathbb{A}$ is a binary Archimedean prearithmetic.*

Lemma 2.10. *In a additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ where addition preserves order, we have $a < a + a$ for any element a from A, which is not maximal.*

Proof. Let us assume that $a + a \leq a$ for some element a from A. Then

$$(a + a) + a \leq a + a \leq a$$

By induction we can prove that $n[a] \leq a$ for any natural number n.
As the element a is not maximal, $a < b$ for some element b from A. At the same time, we have

$$n[a] \leq a < b$$

This means that $\mathbb{A}$ is not an additively Archimedean prearithmetic. Thus, in an additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ where addition preserves order, we have $a < a + a$ for any element a from A, which is not maximal. □

Lemma is proved.

Corollary 2.5. *In a totally ordered additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ where addition preserves strict order, the inequality $m < n$ implies the inequality $m[a] < n[a]$ for any element a from A, which is not maximal.*

When addition is associative and commutative, we have a stronger result.

Lemma 2.11. *In a totally ordered additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ where addition strictly preserves order and is associative and commutative, we have $a < a + b$ for any elements a and b from A, which are not maximal.*

Proof. If $a \leq b$, then the statement of the lemma follows from Lemma 2.10 because

$$a < a + a \leq a + b$$

as addition strictly preserves order and order relation is transitive.
Let us consider the case when $b < a$ and assume $a + b \leq a$. Because $\mathbb{A}$ is additively Archimedean prearithmetic, we have $a \leq n[b]$ for some natural number n. As addition strictly preserves order and order relation is transitive, we have

$$b + b < a + b \leq a$$

Adding b to both sides of the inequality $b + b \leq a + b$, we obtain

$$b + b + b < a + b + b \leq a + b \leq a$$

Continuing this process, we obtain

$$n[b] < a + b \leq a$$

This contradicts the equality $a \leq n[b]$ and by the Principle of excluded middle completes the proof. □

Lemma 2.12. *If $b \cdot a \leq b + a$ for any elements a and b from A, then for any natural number n, we have $[a]^n \leq n[a]$.*

Proof is similar to the proof of Lemma 2.1.16.

Proposition 8. *If in a multiplicatively Archimedean prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$, *we have* $b \cdot a \leq b + a$ *for any elements a and b from A, then* $\mathbb{A}$ *is an additively Archimedean prearithmetic.*

Proof is similar to the proof of Proposition 7.

Lemma 2.13. *In a totally ordered multiplicatively Archimedean prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$ *where multiplication preserves order, we have* $a < a \cdot a$ *for any element a from A, which is not maximal.*

Proof is similar to the proof of Lemma 2.10.

Corollary 2.6. *In a totally ordered multiplicatively Archimedean prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$ *where multiplication preserves strict order, the inequality* $m < n$ *implies the inequality* $[a]^m < [a]^n$ *for any element a from A, which is not maximal.*

When addition is associative and commutative, we have a stronger result.

Lemma 2.14. *In a totally ordered multiplicatively Archimedean prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$ *where multiplication strictly preserves order and is associative and commutative, we have* $a < a \cdot b$ *for any elements a and b from A, which are not maximal.*

Proof is similar to the proof of Lemma 2.11.
We also introduce and study exact Archimedean properties.

Definition 2.3. a) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *Exactly Successively Archimedean* if for any elements a and b from A, the inequality $a < b$ implies that there is natural number n such that $S^n a$ is equal to b.

b) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *Exactly Additively Archimedean* if there is an element d from A, which is called the *additive generator* of $\mathbb{A}$, such that for any elements a and b from A, the inequality $a < b$ implies that there is a natural number n such that

$$a + n[d] = b \tag{2.5}$$

c) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *Exactly Multiplicatively Archimedean* if there is an element d from A, which is called the *multiplicative generator* of $\mathbb{A}$, such that for any elements a and b from A, the inequality $a < b$ implies that there is a natural number n such that

$$a \cdot [d]^n = b \tag{2.6}$$

d) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the multiplicative 1 satisfies the *left Exactly Binary Archimedean Property* for addition, or is *left exactly additive prearithmetic* from the left, if for any elements a and b from A, the inequality $1 < a < b$ implies that there is an element q less than b such that

$$b = q + a \tag{2.7}$$

and satisfies the *right Exactly Binary Archimedean Property*, or is *exactly additive prearithmetic from the right*, if for any elements a and b from A, the inequality $1 < a < b$ implies that there is an element q less than b such that

$$b = a + q \tag{2.8}$$

The Diophantine arithmetic $\mathbb{N}$ is Exactly Successively Archimedean and Exactly Additively Archimedean because 1 is the additive generator of $\mathbb{N}$. The conventional arithmetic $\mathbb{W}$ of all whole numbers is also Exactly Successively Archimedean and Exactly Additively Archimedean because 1 is the additive generator of $\mathbb{W}$.

Remark. Exact Archimedean properties are intrinsically related to the concept of the natural order in partially ordered groupoids, groups and semigroups. We remind that if $H$is a partially ordered groupoid (semigroup, group), then its order is natural if $a < b$ implies $ax = ya = b$ for some elements a and b from H (Fuchs, 1963). Thus, the order in an Exact Archimedean partially ordered groupoid (semigroup, group) is natural.

Lemma 2.15. *An abstract prearithmetic with the linear order is Exactly Successively Archimedean if and only it is Successively Archimedean.*

Proof. Necessity. Any Exactly Successively Archimedean abstract prearithmetic is Successively Archimedean because for any element a in a partially ordered set, we have $a \leq a$, i.e., $S^n a = b$ implies $b \leq S^n a$.

Sufficiency. Let us assume that an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is Successively Archimedean and for some elements a and b from A, we have $a < b$. Then by definition, there is the least natural number n such that $b \leq S^n a$. As the order $\leq$ is linear, it means that we have the inequalities

$$S^{n-1} a \leq b \leq S^n a$$

Because $S^n a = S(S^{n-1} a)$ and by definition, there are no elements that larger than $S^{n-1} a$ and smaller than $S(S^{n-1} a)$, we have either $b = S^{n-1} a$ or $b = S^n a$. Consequently, the abstract prearithmetic $\mathbb{A}$ is Exactly Successively Archimedean. □

Lemma is proved.

Remark. For Exactly Additively Archimedean prearithmetics and Exactly Multiplicatively Archimedean prearithmetics a similar statement is not always true.

Lemma 2.16. *In an Exactly Additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with linear order and associative commutative addition, which strictly preserves order, either* 0 *or the additive generator d of $\mathbb{A}$, which is not maximal, is the least element.*

Proof. At first, we show that any element $b = n[d]$ if $d \leq b$. As $\mathbb{A}$ is Exactly Additively Archimedean, in this case, $b = d + n[d]$. As addition is commutative, $b = n[d] + d = (n + 1)[d]$.

At the same time, if there is an element b with $d < b$. Then assuming $d + d \leq d$, we obtain

$$(d + d) + d \leq d + d \leq d$$

By induction we can prove that $n[d] \leq d < b$ for any natural number n and the equality $b = n[d]$ becomes impossible. Consequently, we have

$$d < d + d = 2[d] < \ldots < n[d] < (n + 1)[d] < \ldots$$

as addition strictly preserves order and order relation is transitive.

Now let us suppose there is an element a that is less than d. As $\mathbb{A}$ is Exactly Additively Archimedean and d is the additive generator of $\mathbb{A}$, in this case, $d = a + n[d]$. As we demonstrated, we have $d < n[d]$. Then by the same token, if $d < a + d$, then $d < a + n[d]$ for any natural number n. If $d > a + d$, then $d > a + n[d]$ for any natural number n. Thus, we come to conclusion that $d = a + d$.

Applying mathematical induction, we see that $a + n[d] = n[d]$ for any natural number n. Thus, for any element b larger than d, we have $b = a + b$, i.e., a is the additive zero for all elements $b \geq d$.

Let us suppose there is an element c that is less than a. Then $a = c + n[d]$ and $d = c + m[d]$ because $c < a < d$ and d is the additive generator of $\mathbb{A}$.

At the same time, $c + d < c + n[d] < d$. Thus,

$$d > c + d > c + d + d = 2[d] > \ldots > m[d]$$

This contradict the equality $d = c + m[d]$ demonstrating that a is the least element and the additive zero in $\mathbb{A}$. □

Lemma is proved.

The Exactly Additively Archimedean Property allows representing successors $S\,a$ using addition as it is done in the conventional Diophantine arithmetic $\mathbb{N}$ of natural numbers where $S\,n = n+1$. The same is true for many abstract prearithmetics.

Proposition 9. *If $\mathbb{A} = (A; +, \cdot, \leq)$ is an Exactly Additively Archimedean prearithmetic with linear order, the successor function S, the additive generator d and strictly monotone associative addition, then $Sa = a + d$ for any element a from A.*

Proof. Let us take an Exactly Additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the successor function S, an the additive generator d and monotone addition. As by definition $a < Sa$ and the prearithmetic $\mathbb{A}$ is Exactly Additively Archimedean, we have

$$Sa = a + n[d] \tag{2.9}$$

As the order in A is linear, we have three options: $a > a + d, a = a + d$, or $a < a + d$.

If we have the first option, i.e., $a + d < a$, then

$$a + 2[d] = a + d + d < a + d < a,$$

because addition is strictly monotone and associative. By induction, for any natural number n, we have

$$a + n[d] < a$$

This contradicts equality (2.9) and shows that the first option is impossible.

If we have the second option, i.e., $a + d = a$, then

$$a + 2[d] = a + d + d = a + d = a,$$

because addition is associative. By induction, for any natural number n, we have

$$a + n[d] = a$$

This contradicts equality (2.9) and shows that the second option is impossible.

If we have the third option, i.e., $a + d > a$, then

$$a + 2[d] = a + d + d > a + d > a,$$

because addition is strictly monotone and associative.

By the definition of the successor Sa, if $a \leq z \leq Sa$, then z is equal either to a or to Sa. Because $a + 2[d] > a + d > a$, equality (2.9) implies $n = 1$ in (2.5) and $Sa = a + d$. □

Proposition is proved.

Applying Proposition 9 several times, we obtain the following result.

Proposition 10. *If $\mathbb{A} = (A; +, \cdot, \leq)$ is an Exactly Additively Archimedean prearithmetic (ESAPA) with linear order, the successor function S, an additive generator d and strictly monotone associative addition, then for any element a from A, $S^n a = a + n[a]$.*

Proof is left as an exercise.

Proposition 11. *Any Exactly Additively Archimedean prearithmetic is an exactly additive prearithmetic from the right.*

Proof. Let us consider a additively Archimedean abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ and its elements $a < b$. By Definition 2.3, it satisfies condition (2.5), i.e., for some natural number n, we have $b = a + n[d] = a + (\ldots(((d + d) + d) + d)\ldots) + d)$ where d is an additive generator. It means that we can take $q = n[d]$ and $b = a + n[d] = a + q$. □

Proposition is proved.

Corollary 2.7. *Any exactly additively Archimedean prearithmetic with commutative addition is an exactly additively Archimedean prearithmetic from the left and from the right.*

3. Elements of non-Diophantine number theory

Obtained properties of abstract prearithmetics allow building non-Diophantine number theory, which is also called non-Diophantine higher arithmetic. Here we develop only the fundamentals of this theory starting with such properties as subtractability and divisibility.

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$.

Definition 3.1. a) An element b from $\mathbb{A}$ is subtractable *from the right (from the left)* by an element a from $\mathbb{A}$ if $b = d + a$ (correspondingly, $b = a + d$) for some element d from $\mathbb{A}$, which is called the *difference from the right* (correspondingly, *from the left*) of b and a. We call a and d *additive factors* of b and denote *subtractability from the right* by $b\lceil a$ (*from the left* by $a\rceil b$) and the *difference from the right* (correspondingly, *from the left*) by $d = b \rightharpoondown a$ (correspondingly, by $d = b \leftharpoondown a$).

b) An element a from $\mathbb{A}$ is *subtractable* by an element b if it is subtractable by b from the right and from the left with the same difference, i.e., $b = d + a = a + d$. We denote *subtractability* by $b(a$ and the difference of b and a by $d = b - a$.

For instance, in the conventional Diophantine arithmetic $\mathbb{N}$, any number is subtractable from the right and from the left by any smaller number because $n = 1 + (n - 1) = (n - 1) + 1$ for any natural number $n > 1$. However, this is not true for many abstract prearithmetics and non-Diophantine arithmetics.

Example 3.1. Let us consider the set N of all natural numbers with the standard order $\leq$ and introduce the following operations:

$$a \oplus b = a \cdot b$$

$$a \otimes b = a^b$$

Then the system $\mathbb{A} = (N; \oplus, \otimes, \leq)$ is an abstract prearithmetic. Taking numbers 5_A and 3_A from this prearithmetic, we see that there is no number n_A in $\mathbb{A}$ such that $3_A \oplus b = 5_A$. It means that 5_A is not subtractable by 3_A . Moreover, we can see that in this prearithmetic, subtractability means divisibility.

This example shows that subtractability is an additive counterpart of divisibility.

Lemma 3.1. *If addition + is commutative in an abstract prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$*, then for any elements a and b from* $\mathbb{A}$*, a is subtractable by b if and only if it is subtractable by b from the right or from the left.*

Proof is left as an exercise.

Proposition 12. *If addition + is associative in an abstract prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$*, then for any elements a, b and c from* $\mathbb{A}$*, we have:*

(1) *$a\lceil b$ and $b\lceil c$ imply $a\lceil c$.*

(2) *$b\rceil a$ and $c\rceil b$ imply $c\rceil a$.*

(3) *$a(b$ and $b(c$ imply $a(c$.*

Proof. (1) If $a\lceil b$, then $a = d + b$ for some element d from $\mathbb{A}$. If $b\lceil c$, then $b = e + c$ for some element e from $\mathbb{A}$. Consequently,

$$a = d + b = d + e + c = (d + e) + c$$

It means that $a\lceil c$. Statements (2) and (3) are proved in a similar way. □

Proposition is proved.

Proposition 13. *If addition + is associative in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, then for any elements a, b and c from $\mathbb{A}$, we have:*

(1) *$a\lceil b$ imply $a + c\lceil b + c$.*

(2) *$b\rceil a$ imply $c + b\rceil c + a$.*

(3) *$a(b$ imply $a + c(b + c$ when addition + is also commutative.*

Proof. (1) If $a\lceil b$, then $a = d + b$ for some element d from $\mathbb{A}$. Consequently,

$$a + c = d + b + c = d + (b + c)$$

It means that $a + c\lceil b + c$.
(2) If $b\rceil a$, then $a = b + d$ for some element $\mathbb{A}$. Consequently,

$$c + a = c + b + d = (c + b) + d$$

It means that $c + b\rceil c + a$. By Lemma 3.1, the statement (3) follows from statements (1) and (2) when addition + is commutative. □

Proposition is proved.

Proposition 14. *If addition + is associative in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, then for any elements a, b and c from $\mathbb{A}$, we have:*

(1) *$a\lceil b$ imply $c + a\lceil b$.*

(2) *$b\rceil a$ imply $b\rceil a + c$.*

(3) *$a(b$ imply $a + c(b$ when addition + is also commutative.*

Proof. (1) If $a\lceil b$, then $a = d + b$ for some element d from $\mathbb{A}$. Consequently,

$$c + a = c + d + b = (c + d) + b$$

It means that $c + a\lceil b$. (2) If $b\rceil a$, then $a = b + d$ for some element d from $\mathbb{A}$. Consequently,

$$a + c = b + d + c = (c + b) + d + c$$

It means that $a + c\rceil b$. By Lemma 3.1, the statement (3) follows from statements (1) and (2) when addition + is commutative. □

Proposition is proved.

Proposition 15. *In an Exactly Additively Archimedean abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, for any elements a, b and c from $\mathbb{A}$, if $a < b$, then b is subtractable from the left by a.*

Proof is left as an exercise.

An important property of numbers is the cancelation law. For instance, if $n + 5 = m + 5$, then $n = m$ and if $n + 5 > m + 5$, then $n > m$ for any whole numbers n and m. Here we consider nine forms of the cancelation law and study them for abstract prearithmetics.

Definition 3.2. a) $\mathbb{A}$ is an abstract *prearithmetic with ordered additive cancelation* if $a + c \leq b + c$ implies $a \leq b$.

b) $\mathbb{A}$ is an abstract *prearithmetic with additive cancelation* if $a + c = b + c$ implies $a = b$.

c) $\mathbb{A}$ is an abstract *prearithmetic with strict additive additive cancelation from the right* if $a + c < b + c$ implies $a < b$.

d) $\mathbb{A}$ is an abstract *prearithmetic with ordered additive cancelation from the left* if $c + a \leq c + b$ implies $a \leq b$.

e) $\mathbb{A}$ is an abstract *prearithmetic with additive cancelation from the left* if $c + a = c + b$ implies $a = b$.

f) $\mathbb{A}$ is an abstract *prearithmetic with strict additive cancelation from the left* if $c + a < c + b$ implies $a < b$.

g) $\mathbb{A}$ is an abstract *prearithmetic with ordered additive cancelation* if it is an abstract prearithmetic with ordered additive cancelation from the left and from the right.

h) $\mathbb{A}$ is an abstract *prearithmetic with additive cancelation* if it is an abstract prearithmetic with additive cancelation from the left and from the right.

j) $\mathbb{A}$ is an abstract *prearithmetic with strict additive cancelation* if it is an abstract prearithmetic with strict additive cancelation from the left and from the right.

Let us consider some examples.

Example 3.2. The arithmetic $\mathbb{N}$ of all natural numbers is an abstract prearithmetic with additive cancelation, with ordered additive cancelation and with strict additive cancelation.

Example 3.3. However, the Diophantine arithmetic $\mathbb{W}$ of all whole numbers is an abstract prearithmetic with ordered additive cancelation but does not have additive cancelation or strict additive cancelation because any number multiplied by 0 is equal to 0.

Lemma 3.2. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with additive cancelation has the additive zero 0_A , then for any element a from $\mathbb{A}$, $a - a$ is defined and equal to 0_A.*

Indeed, by Definition 3.1, $a + 0_A = a$. Consequently, $a - a = 0_A$ because if $a + c = a$, then $c = 0_A$.

Proposition 16. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with additive cancelation has the additive zero 0_A, then for any element a from $\mathbb{A}$, $a - 0_A$ is defined and equal to a.*

Proof is left as an exercise.

Let us find conditions for ordered additive cancelation.

Lemma 3.3. $\mathbb{A} = (A; +, \cdot, \leq)$ *is an abstract prearithmetic with ordered additive cancelation if in* $\mathbb{A}$*, the order is linear (total) and addition preserves the strict order.*

Indeed, let us assume that $a + c \leq b + c$. If it is not true $a \leq b$, then $b < a$ because the order is total. However, as addition preserves the strict order, it would be $b + c < a + c$. As this contradicts our assumption, we conclude that $a \leq b$.

Lemma 3.4. $\mathbb{A} = (A; +, \cdot, \leq)$ *is an abstract prearithmetic with ordered additive cancelation (from the right or from the left) if it is with additive cancelation (from the right or from the left) and strict additive cancelation(from the right or from the left).*

Proof is left as an exercise.

Lemma 3.5. $\mathbb{A} = (A; +, \cdot, \leq)$ *is an abstract prearithmetic with ordered additive cancelation (from the right or from the left), then it is an abstract prearithmetic with additive cancelation (from the right or from the left).*

Indeed, $a + c = b + c$ implies $a + c \leq b + c$ and $b + c \leq a + c$. As $\mathbb{A}$ is an abstract prearithmetic with ordered additive cancelation, this implies $a \leq b$ and $b \leq a$. Consequently, $a = b$, which means that $\mathbb{A}$ is an abstract prearithmetic with additive cancelation.

Cancelation property allows strengthening of results in Lemma 3.5.

Lemma 3.6. *For any elements a and b from an abstract prearithmetic* $\mathbb{A}$ *with additive cancelation, we have:*

a) $a \lceil b$ *if and only if subtraction from the left* $a \leftharpoondown b$ *is defined.*

b) $b \rceil a$ *if and only if subtraction from the right* $a \rightharpoondown b$ *is defined.*

c) $a (b$ *if and only if full subtraction* $a - b$ *is defined.*

Proof is left as an exercise.

Lemma 3.7. *For any elements a and b from an abstract prearithmetic* $\mathbb{A}$ *with additive cancelation, we have:*

a) *If subtraction from the left* $\leftharpoondown$ *is defined in an abstract prearithmetic* $\mathbb{A}$*, then* $\mathbb{A}$ *is with additive cancelation from the left.*

b) *If subtraction from the right* $\rightharpoondown$ *is defined in an abstract prearithmetic* $\mathbb{A}$*, then* $\mathbb{A}$ *is with additive cancelation from the right.*

c) *If full subtraction* $a-$ *is defined in an abstract prearithmetic* $\mathbb{A}$*, then* $\mathbb{A}$ *is with additive cancelation.*

Proof is left as an exercise.

In some cases, additive and multiplicative zeros coincide.

Proposition 17. *If an abstract prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$ *with additive cancelation has the additive zero* 0_A *and multiplication is distributive over addition, then* 0_A *is also the multiplicative zero.*

Proof. Let us take an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ that satisfies all initial conditions. Taking arbitrary element x from $\mathbb{A}$, we have

$$0_A \cdot x = 0_A \cdot x + 0_A = (0_A + 0_A) \cdot x = 0_A \cdot x + 0_A \cdot x \tag{3.1}$$

As we have additive cancelation in $\mathbb{A}$, it is possible to cancel $0_A \cdot x$ in (3.1). This gives us $0_A \cdot x = 0_A$. The identity $x \cdot 0_A = 0_A$ is proved in a similar way. Consequently, 0_A is also the multiplicative zero. □

Proposition is proved.

Corollary 3.1. *In a semiring with additive cancelation, additive and multiplicative zeros coincide.*

An important property of numbers is divisibility. Here we study it for abstract prearithmetics.

Definition 3.3. a) An element a from $\mathbb{A}$ is *divisible* from the right (from the left) by an element b from $\mathbb{A}$ if $a = d \cdot b$ ($a = b \cdot d$) for some element d from $\mathbb{A}$. We call b and d *multiplicative factors* or *divisors* of a, the element a is called a *multiple* of b from the left (from the right), and *divisibility from the right* is denoted by $a\lfloor b$ (*from the left* by $b\rfloor a$).

b) An element a from $\mathbb{A}$ is *divisible* by an element b if it is divisible by b from the right and from the left. We denote this by $a|b$ and a is called a *multiple* of b.

Remark. If a is divisible by b, it is also denoted by $b|a$ in some publications.

Lemma 3.8. *If multiplication* $\cdot$ *is commutative in an abstract prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$ *with the multiplicative one* 1*, then in it, any element a is divisible by b if and only if it is divisible by b from the right or from the left.*

Proof is left as an exercise.

Proposition 18. *If multiplication* $\cdot$ *in an abstract prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$ *is associative, then for any elements* a, b *and* c *from* $\mathbb{A}$*, we have:*

(1) $a\lfloor b$ *and* $b\lfloor c$ *imply* $a\lfloor c$.

(2) $a\rfloor b$ *and* $b\rfloor c$ *imply* $a\rfloor c$.

(3) $a|b$ *and* $b|c$ *imply* $a|c$.

Proof. (1) If $a\lfloor b$, then $a = d \cdot b$ for some element d from $\mathbb{A}$. If $b\lfloor c$, then $b = e \cdot c$ for some element e from $\mathbb{A}$. Consequently,

$$a = d \cdot b = d \cdot e \cdot c = (d \cdot e) \cdot c$$

It means that $a\lfloor c$. Statements (2) and (3) are proved in a similar way. □

Proposition is proved.

Corollary 3.2. ((Landau *et al.*, 1999): Theorem 2). *If a, b and c are integer numbers, then $a|b$ and $b|c$ imply $a|c$.*

In other words, Proposition 18 and Corollary 3.2 mean that a divisor of a divisor of an element is a divisor of this element. It is also possible to say that a multiple of a multiple of an element is a multiple of this element.

Proposition 19. *If multiplication $\cdot$ in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is associative, then for any elements a, b and c from $\mathbb{A}$, we have:*

(1) *$a\lfloor b$ imply $a \cdot c\lfloor b \cdot c$.*

(2) *$b\rfloor a$ imply $c \cdot b\rfloor c \cdot a$.*

(3) *$a|b$ imply $a \cdot c|b \cdot c$ when multiplication $\cdot$ is also commutative.*

Proof. (1) If $a\lfloor b$, then $a = d \cdot b$ for some element d from $\mathbb{A}$. Consequently,

$$a \cdot c = d \cdot b \cdot c = d \cdot (b \cdot c)$$

It means that $a \cdot c\lfloor b \cdot c$.
(2) If $b\rfloor a$, then $a = b \cdot d$ for some element d from $\mathbb{A}$. Consequently,

$$c \cdot a = c \cdot b \cdot d = (c \cdot b) \cdot d$$

It means that $c \cdot b\rfloor c \cdot a$. By Lemma 3.8, the statement (3) follows from statements (1) and (2) when multiplication $\cdot$ is commutative. □

Proposition is proved.

Corollary 3.3. ((Landau *et al.*, 1999): Theorem 3b). *If a, b and c are integer numbers, then $a|b$ imply $ac|bc$.*

Proposition 20. *If multiplication $\cdot$ in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is associative, then for any elements a, b and c from $\mathbb{A}$, we have:*

(1) *$a\lfloor b$ imply $c \cdot a\lfloor b$.*

(2) *$b\rfloor a$ imply $b\rfloor a \cdot c$.*

(3) *$a|b$ imply $a \cdot c|b$ when multiplication $\cdot$ is also commutative.*

Proof. (1) If $a \lfloor b$, then $a = d \cdot b$ for some element d from $\mathbb{A}$. Consequently,

$$c \cdot a = c \cdot d \cdot b = (c \cdot d) \cdot b$$

It means that $c \cdot a \lfloor b$

(2) If $b \rfloor a$, then $a = b \cdot d$ for some element d from $\mathbb{A}$. Consequently,

$$a \cdot c = b \cdot d \cdot c = b \cdot (d \cdot c)$$

It means that $b \rfloor a \cdot c$. By Lemma 3.8, the statement (3) follows from statements (1) and (2) when multiplication $\cdot$ is commutative. □

Proposition is proved.

Corollary 3.4. ((Landau *et al.*, 1999): Theorem 4). *If a, b and c are integer numbers, then $a|b$ imply $ac|b$.*

In other words, Proposition 20 and Corollary 3.4 mean that a divisor of an element is also a divisor of any multiple of this element. It is also possible to say that a multiple of a multiple of an element is a multiple of this element.

Proposition 21. *For any elements a, b and c from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, we have:*

(1) *$a \lfloor c$ and $b \lfloor c$ imply $a + b \lfloor c$ when multiplication $\cdot$ is distributive from the right over addition $+$.*

(2) *$c \rfloor a$ and $c \rfloor b$ imply $c \rfloor a + b$ when multiplication $\cdot$ is distributive from the left over addition $+$.*

(3) *$a|c$ and $b|c$ imply $a + b|c$ when multiplication $\cdot$ is also commutative.*

Proof. (1) If $a \lfloor c$, then $a = d \cdot c$ for some element d from $\mathbb{A}$. If $b \lfloor c$, then $b = e \cdot c$ for some element e from $\mathbb{A}$. Consequently, by distributivity from the right, we have

$$a + b = d \cdot c + e \cdot c = (e + d) \cdot c$$

It means that $a + b \lfloor c$.

(2) If $c \rfloor a$, then $a = c \cdot d$ for some element d from $\mathbb{A}$. If $c \rfloor b$, then $b = c \cdot e$ for some element e from $\mathbb{A}$. Consequently, by distributivity from the left, we have

$$a + b = c \cdot d + c \cdot e = c \cdot (e + d)$$

It means that $c \rfloor a + b$. The statement (3) follows from statements (1) and (2) when multiplication $\cdot$ is commutative. □

Proposition is proved.

Corollary 3.5. ((Landau *et al.*, 1999): Theorem 5). *If a, b and c are integer numbers, then $a|b$ and $b|c$ imply $(a + b)|c$.*

In other words, Proposition 21 and Corollary 3.5 mean that that a common divisor of two elements is also a divisor of the sum of these elements.

Corollary 3.6. *For any elements a_i ($i = 1, 2, 3, \dots, n$) from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, we have:*

(1) *$a_i | c$ for all $i = 1, 2, 3, \dots, n$ imply $\sum_{i=1}^{n} a_i | c$ when multiplication $\cdot$ is associative and distributive from the right over addition $+$.*

(2) *$c | a_i$ for all $i = 1, 2, 3, \dots, n$ imply $c | \sum_{i=1}^{n} a_i$ when multiplication $\cdot$ is associative and distributive from the left over addition $+$.*

(3) *$a_i | c$ for all $i = 1, 2, 3, \dots, n$ imply $\sum_{i=1}^{n} a_i | c$ when multiplication $\cdot$ is also commutative.*

Proposition 22. *For any elements a, b, k, h and c from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, we have:*

(1) *$a \lfloor c$ and $b \lfloor c$ imply $(a \cdot k + b \cdot h) \lfloor c$ when multiplication $\cdot$ is associative and distributive from the right over addition $+$.*

(2) *$c \rfloor a$ and $c \rfloor b$ imply $c \rfloor (a \cdot k + b \cdot h)$ when multiplication $\cdot$ is associative and distributive from the left over addition $+$.*

(3) *$a | c$ and $b | c$ imply $(a \cdot k + b \cdot h) | c$ when multiplication $\cdot$ is also commutative.*

Proof. (1) By Proposition 20, for any elements a, b, k, h and c from $\mathbb{A}$, we have:

$$(a \cdot k) \lfloor c \text{ and } (b \cdot h) \lfloor c$$

Thus, By Proposition 20,

$$(a \cdot k + b \cdot h) \lfloor c$$

Statements (2) and (3) are proved in the same way based on Proposition 20 and 21. □

Proposition is proved.

Corollary 3.7. ((Landau *et al.*, 1999): Theorem 6). *If a, b, k, h and c are integer numbers, then $a | c$ and $b | c$ imply $(a \cdot k + b \cdot h) | c$.*

Proposition 23. *For any elements a, b and c from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, we have:*

(1) *$a \lfloor c$ and $b \lfloor c$ imply $a \rightharpoondown b \lfloor c$ when the right difference $a \rightharpoondown b$ of a and b exists and multiplication $\cdot$ is distributive from the right over the right difference.*

(2) *$c \rfloor a$ and $c \rfloor b$ imply $c \rfloor a \leftharpoondown b$ when the left difference $a \leftharpoondown b$ of a and b exists and multiplication $\cdot$ is distributive from the left over the left difference.*

(3) *$a|c$ and $b|c$ imply $a - b|c$ when the difference $a - b$ of a and b exists and multiplication $\cdot$ is commutative and distributive over the difference.*

Proof. (1) If $a \lfloor c$, then $a = d \cdot c$ for some element d from $\mathbb{A}$. If $b \lfloor c$, then $b = e \cdot c$ for some element e from $\mathbb{A}$. Consequently, by distributivity from the right, we have

$$a \rightharpoondown b = d \cdot c \rightharpoondown e \cdot c = (e \rightharpoondown d) \cdot c$$

It means that $a \rightharpoondown b \lfloor c$.
(2) If $c \rfloor a$, then $a = c \cdot d$ for some element d from $\mathbb{A}$. If $c \rfloor b$, then $b = c \cdot e$ for some element e from $\mathbb{A}$. Consequently, by distributivity from the left, we have

$$a \leftharpoondown b = c \cdot d \leftharpoondown c \cdot e = c \cdot (e \leftharpoondown d)$$

It means that $c \rfloor a \leftharpoondown b$. The statement (3) follows from statements (1) and (2) when multiplication $\cdot$ is commutative and distributive over the difference. □

Proposition is proved.

Corollary 3.8. ((Landau *et al.*, 1999): Theorem 5). *If a, b and c are integer numbers, then $a|c$ imply $b|c$ imply $(a - b)|c$.*

In other words, Proposition 23 and Corollary 3.8 mean that that a common divisor of two elements is also a divisor of the difference of these elements.

Lemma 3.9. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ has the multiplicative zero 0^m, then 0^m is divisible by any element from $\mathbb{A}$.*

Indeed, we have $0^m = a \cdot 0^m = 0^m \cdot a$ for any element a from $\mathbb{A}$.

Existence of the additive zero impacts subtractability.

Lemma 3.10. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ has the additive zero 0, then in it, any element is subtractable by itself and by 0.*

Indeed, we have $a = a + 0 = 0 + a$ for any element a from $\mathbb{A}$.

Lemma 3.11. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the multiplicative one 1_A, then in it, any element is divisible by itself and by 1_A.*

Indeed, we have $a = a \cdot 1_A = 1_A \cdot a$ for any element a from $\mathbb{A}$.

Number theory begins with classification of numbers and studying their properties. An important class of numbers in the Diophantine arithmetic $\mathbb{N}$ consists of prime numbers, which are extensively studied in number theory (cf., for example, (Davenport, 1999), (Landau *et al.*, 1999)).

In abstract prearithmetics in general and in non-Diophantine arithmetics in particular, there are two classes of prime numbers - additively prime numbers and multiplicatively prime numbers. There are also additively composite numbers and multiplicatively composite numbers. They are counterparts of the well-known concepts of prime and composite numbers in the Diophantine arithmetic $\mathbb{N}$. Here we define these classes in abstract prearithmetics.

Definition 3.4. a) An element p from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the additive zero 0_A is *additively prime* in $\mathbb{A}$ if there are no elements $a, b \neq 0_A$ in $\mathbb{A}$ such that $p = a + b$.

b) An element p from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the multiplicative one 1_A is *multiplicatively prime* in $\mathbb{A}$ if $p \neq 0_A$, $p \neq 1_A$ and there are no elements $a, b \neq 1_A$ in $\mathbb{A}$ such that $p = a \cdot b$.

c) An element a from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *additively composite* in $\mathbb{A}$ if it is not additively prime.

d) An element a from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *multiplicatively composite* in $\mathbb{A}$ if it is not multiplicatively prime.

Example 3.4. In the Diophantine arithmetic $\mathbb{N}$, there is only one additively prime number 1 and infinitely many multiplicatively prime numbers. That is why in the conventional (Diophantine) number theory, additively prime numbers are not even introduced but multiplicatively prime numbers, which are simply called prime numbers, are studied with great interest by many mathematicians.

Remark. It is interesting that the great Greek philosopher Aristotle defined additively prime numbers and found only two additively prime numbers 2 and 3 because at that time, Greek mathematicians did not consider 1 as a number (Aristotle, 1984).

Note that there are many abstract prearithmetics that do not have additively prime numbers, i.e., all numbers are composite.

Example 3.5. In any modular arithmetic $\mathbb{Z}_n$, there are no additively prime numbers because any number in $\mathbb{Z}_n$ is a sum of two non-zero numbers.

This is a particular case of the following result.

Lemma 3.12. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is a group with respect to addition, then it does not have additively prime elements.*

Indeed, any element a in $\mathbb{A}$ has the opposite element $-a$ and if $a \neq b$ in $\mathbb{A}$, then $b = (b+(-a))+a$ where $b + (-a) \neq 0_A$ i.e., b is an additively composite element.

At the same time, there are many abstract prearithmetics that have infinitely many additively prime numbers as the following example demonstrates.

Example 3.6. Let us consider an abstract prearithmetic $\mathbb{A} = (N; \oplus, \circ, \leq)$ which contains the set all natural numbers N and in which operations are defined by the following formula

$$n \oplus m = (n + m)^2$$

$$n \circ m = (n \cdot m)^2$$

where + is the standard addition and $\cdot$ is the standard multiplication of natural numbers.

Then all natural numbers that are not squares in the Diophantine arithmetic $\mathbb{N}$ will be additively prime numbers in this prearithmetic $\mathbb{A}$.

Proposition 24. *There is continuum of abstract prearithmetics that have infinitely many additively prime numbers.*

Proof. Abstract prearithmetics are different when they have different multiplication. Because multiplication in abstract prearithmetics is defined independently from addition, we can take the abstract prearithmetic A_f, in which addition is defined as in the abstract prearithmetic $\mathbb{A}$ from Example 3.6 while multiplication is defined by an arbitrary function f from $N \times N$ into N. By construction, all natural numbers that are not squares in $\mathbb{N}$ will be additively prime numbers in this prearithmetic A_f. As there is continuum of such functions f (Fraenkel *et al.*, 1973), there is also continuum of abstract prearithmetics that have infinitely many additively prime numbers. □

Proposition is proved.

However, the following result shows that for an arbitrary abstract prearithmetic, the situation can be essentially different.

Theorem 3.1. *There are infinite abstract prearithmetics, in which for any natural number $n > 1$, there are exactly n additively prime elements.*

Proof. Let us consider the set W of all whole numbers, take a natural number n and define the following functions

$$g(m) = \begin{cases} 0 & \text{when } m = 0 \\ n + m & \text{when } m > 0 \end{cases}$$

and

$$h(q) = \begin{cases} q & \text{when } 0 \leq q < n + 1 \\ q - n & \text{when } q > n \end{cases}$$

We can build an abstract prearithmetic $\mathbb{A}_n = (W; \oplus, \otimes, \leq)$ with addition defined for whole numbers m and n larger than 0 by the following formula

$$m \oplus k = h(g(m) + g(k)) = (m + n) + (k + n) - n = m + k + n$$

Besides,

$$0 \oplus m = m \oplus 0 = h(g(m) + 0) = (m + n) - n = m$$

for all $m > 0$.

Then the least additively composite number is $n + 2 = 1 \oplus 1$. At the same time, any larger number $r = 2 + n + k$ is also additively composite because $2 \oplus k = h(g(2) + g(k)) = 2 + k + n = r$. Consequently, there are exactly $n + 1$ additively prime elements in $\mathbb{A}_n$ and it is possible to build such a prearithmetic $\mathbb{A}_n$ for all $n = 1, 2, 3, \ldots$ □

Theorem is proved.

Considering multiplicatively prime numbers, we see that there are also many abstract prearithmetics that do not have multiplicatively prime numbers.

Example 3.7. In the modular arithmetic $\mathbb{Z}_p$ where p is a prime number, there are no multiplicatively prime numbers because any non-zero number in $\mathbb{Z}_p$ is a product of two non-zero numbers.

This is a particular case of the following result.

Lemma 3.13. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is a group with respect to multiplication, then it does not have multiplicatively prime elements because any non-zero element a in $\mathbb{A}$ has the inverse element a^{-1}.*

Indeed, any element a in $\mathbb{A}$ has the inverse element a^{-1} and if $a \neq b$ in $\mathbb{A}$, then $b = (b \cdot a^{-1}) \cdot a$ where $b \cdot a^{-1} \neq 1_A$.

At the same time, there are many abstract prearithmetics that have infinitely many multiplicatively prime numbers.

Proposition 25. *There is continuum of abstract prearithmetics that have infinitely many multiplicatively prime numbers.*

Proof. Abstract prearithmetics are different when they have different addition. Because multiplication in abstract prearithmetics is defined independently from addition, we can take the abstract prearithmetic $\mathbb{N}_f$, in which multiplication is defined as in the Diophantine arithmetic $\mathbb{N}$ while addition is defined by an arbitrary function f from $N \times N$ into N. As it is proved (cf., for example, (Davenport, 1999)) that $\mathbb{N}$ has infinitely many multiplicatively prime numbers. Thus, there are infinitely many multiplicatively prime elements in this prearithmetic N_f . As there is continuum of such functions f (Fraenkel *et al.*, 1973), there is also continuum of abstract prearithmetics that have infinitely many multiplicatively prime numbers. □

Proposition is proved.

However, the following result shows that for an arbitrary abstract prearithmetic, the situation can be essentially different.

Theorem 3.2. *There are infinite abstract prearithmetics, in which for any natural number $n > 3$, there are exactly n multiplicatively prime elements.*

Proof. Let us consider the set N of all natural numbers, take a natural number n and define the following functions

$$g(m) = \begin{cases} 1 & \text{when } m = 1 \\ 2^{n+m} & \text{when } m > 1 \end{cases}$$

and

$$h(q) = \begin{cases} q & \text{when } 1 \leq q < n + 1 \\ |\log_2 q| - n & \text{when } q > 2^n \end{cases}$$

Note that if $q = 2^m$, then $h(q) = m - n$. We can build an abstract prearithmetic $\mathbb{B}_n = (N; \oplus, \otimes, \leq)$ with multiplication defined for whole numbers m and n larger than 1 by the following formula

$$m \otimes k = h(g(m) + g(k)) = h(2^{n+m} \cdot 2^{n+k}) = h(2^{2n+m+k}) =$$

$$|\log_2 2^{2n+m+k}| - n = \log_2 2^{2n+m+k} - n = m + k + n$$

Besides,

$$1 \otimes m = m \otimes 1 = h(g(m) \cdot 1) = \log_2 2^{n+m} - n = (m + n) - n = m$$

for all $m > 0$.

Then the least composite number is $n + 4 = 2 \otimes 2$. At the same time, any larger number $r = n + 4 + k$ with $k = 1, 2, 3, \ldots$ is also composite as it is divisible by 2 because $2 \otimes (2 + k) = h(g(2) + g(k)) = 2 + k + 2 + n = r$. Consequently, there are exactly $n + 3$ additively prime elements in $\mathbb{B}_n$ and it is possible to build such a prearithmetic $\mathbb{B}_n$ for all $n = 1, 2, 3, \ldots$ □

Theorem is proved.

Let us also consider other traditional classes of natural numbers, for example, even and odd numbers. It is also natural to define even and odd elements in abstract prearithmetics.

Definition 3.5. a) An element a from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *additively even* in $\mathbb{A}$ (with respect to an element b) if there is an element c in $\mathbb{A}$ such that $a = 2 + c$ ($a = b + c$).

b) An element a from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *multiplicatively even* in $\mathbb{A}$ (with respect to an element b) if there is an element c in $\mathbb{A}$ such that $a = 2 \cdot c$ ($a = b \cdot c$).

c) An element a from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *additively odd (with respect to* an element b in $\mathbb{A}$) if it is not additively even (with respect to the element b)

d) An element a from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *multiplicatively odd (with respect to* an element b in $\mathbb{A}$) if it is not multiplicatively even (with respect to the element b)

Note that when an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ has the number 2, then multiplicatively even in $\mathbb{A}$ with respect to 2 elements are simply *even elements (numbers)* although they might be essentially different from even numbers in the Diophantine arithmetic $\mathbb{N}$. Naturally, all elements, which are not multiplicatively even in $\mathbb{A}$ with respect to 2, are *odd elements (numbers)*.

Example 3.8. In the arithmetic $2\mathbb{N}$, of all even numbers with conventional addition and multiplication, all numbers larger than 2 are additively and multiplicatively even.

However, there are many abstract prearithmetics that have only one additively (multiplicatively) even number as the following example demonstrates

Example 3.9. Let us consider an abstract prearithmetic $\mathbb{A} = (N; \oplus, \circ, \leq)$ which contains the set all natural numbers N and in which addition is defined by the following formulas

$$n \oplus 2 = 2 \oplus n = 2$$

$$n \oplus m = n + m \text{ if } n, m \neq 2$$

where + is the standard addition of natural numbers. We see that in $\mathbb{A}$, only 2 is an additively even number.

Example 3.10. Let us consider an abstract prearithmetic $\mathbb{A} = (N; \oplus, \circ, \leq)$ which contains the set all natural numbers N and in which multiplication is defined by the following formulas

$$n \circ 2 = 2 \circ n = 2$$

$$n \circ m = n \cdot m \text{ if } n, m \neq 2$$

where $\cdot$ is the standard addition of natural numbers. We see that in $\mathbb{A}$, only 2 is an multiplicatively even number.

In some abstract prearithmetics, even numbers have usual properties.

Proposition 26. *If addition in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is associative, then the sum of any additively even in $\mathbb{A}$ element with respect to an element b with any element c is additively even in $\mathbb{A}$ with respect to the element b.*

Indeed, if a is an additively even in $\mathbb{A}$ element with respect to an element b, then $a = b + d$. Consequently, as addition in $\mathbb{A}$ is associative, we have

$$a + c = (b + d) + c = b + (d + c)$$

It means that $a + c$ is additively even in $\mathbb{A}$ with respect to the element b.

Proposition 27. *If multiplication in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is distributive with respect to addition, then the sum of two multiplicatively even with respect to an element b in $\mathbb{A}$ elements is multiplicatively even with respect to the element b.*

Indeed, if a and d are multiplicatively even in $\mathbb{A}$ elements with respect to an element b, then $a = b \cdot u$ and $d = b \cdot w$. Consequently, as multiplication in $\mathbb{A}$ is distributive with respect to addition, we have

$$a + d = (b \cdot u) + (b \cdot w) = b \cdot (u + w)$$

It means that $a + d$ is multiplicatively even in $\mathbb{A}$ with respect to an element b.

Proposition 28. *If multiplication in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is associative, then the product of any multiplicatively even in $\mathbb{A}$ element with respect to an element b with any element c is additively even in $\mathbb{A}$ with respect to the element b.*

Indeed, if a is an additively even in $\mathbb{A}$ element with respect to an element b, then $a = b \cdot d$. Consequently, as multiplication in $\mathbb{A}$ is associative, we have

$$a \cdot c = (b \cdot d) \cdot c = b \cdot (d \cdot c)$$

It means that $a \cdot c$ is multiplicatively even in $\mathbb{A}$ with respect to the element b.

One of the basic results of the conventional number theory is the prime decomposition theorem, proofs of which it is possible to find in many books (cf., for example, (Landau *et al.*, 1999), (Davenport, 1999)).

Theorem 3.3. ***Prime Decomposition Theorem.*** *For any natural number larger than 1 in the conventional Diophantine arithmetic* $\mathbb{N}$, *there is a unique up to the order of factors decomposition (factoring) of this number into the product of prime numbers.*

It is also called the Fundamental Theorem of Arithmetic. According to (Davenport, 1999) Davenport, 1992), the first clear statement and proof of this theorem seem to have been given by Gauss in 1801. An equivalent form of the Fundamental Theorem of Arithmetic states that any factoring of a natural number can be extended to a unique up to the order of factors prime factorization. It is interesting that the same result is evidently true for additively prime numbers. Namely, we have the following result.

Proposition 29. *For any natural number in the conventional Diophantine arithmetic* $\mathbb{N}$, *there is a unique up to the order of factors decomposition of this number into the sum of additively prime numbers.*

Indeed, 1 is an additively prime number and any natural number is the sum of some number of $1s$.

Note that in the Diophantine arithmetics $\mathbb{N}$ and $\mathbb{W}$, there is only one additively prime number. At the same time, as it is demonstrated in Proposition 24, there are prearithmetics that have an infinite set of additively prime numbers. An equivalent form of Proposition 29 states that any decomposition (factoring) of a natural number into a sum can be extended to a unique up to the order of factors decomposition (factorization) of this number into the sum of additively prime numbers.

These results bring us to the following concepts.

Definition 3.6. a) An abstract prearithmetic $\mathbb{A}$ has the *additive factoring property* if for any of its non-zero elements, any factorization (additive decomposition) of this element into a sum can be extended to a factorization (additive decomposition) of this number into the sum of additively prime elements.

b) An abstract prearithmetic $\mathbb{A}$ has the *strong additive factoring property* if for any of its non-zero elements, any factorization (additive decomposition) of this element into a sum can be extended to a unique up to the order of factors factorization (additive decomposition) of this number into the sum of additively prime elements.

c) An abstract prearithmetic $\mathbb{A}$ has the *multiplicative factoring property* if for any of its elements but zero 0 and the multiplicative one 1, any factorization (multiplicative decomposition) of this element into a product can be extended to a factorization (multiplicative decomposition) of this number into the product of multiplicatively prime elements.

d) An abstract prearithmetic $\mathbb{A}$ has the *strong multiplicative factoring property* if for any of its elements but zero 0 and the multiplicative one 1, any factorization (multiplicative decomposition) of this element into a product can be extended to a unique up to the order of factors factorization (multiplicative decomposition) of this number into the product of multiplicatively prime elements.

For abstract prearithmetics and even for whole-number and natural number prearithmetics (cf. Section 2.7), the additive factoring property is not true in a general case as the following example demonstrates.

Example 3.11. Let us consider the arithmetic $\mathbb{R}^{++} = (\mathbb{R}^{++}; +, \cdot, \leq)$ of all positive real numbers with standard addition, multiplication and order. The prearithmetic $\mathbb{R}^{++}$ does not have additively prime numbers because any positive real number a is equal to $a/2$ plus $a/2$. Consequently, this prearithmetic does not have the additive factoring property and Proposition 29 is not true for this prearithmetic.

Lemma 3.14. *The strong additive factoring property implies the additive factoring property.*

Proof is left as an exercise.

The inverse implication is not true as the following example demonstrates.

Example 3.12. Let us consider the set $F = \{0, 1, \frac{1}{3}, \frac{1}{2}\}$ and the set P of all expressions of the form $a_1 + a_2(\frac{1}{2}) + a_3(\frac{1}{3})$ where a_1, a_2 and a_3 are natural numbers. We see that the sum of these expressions has the same form. The multiplication is defined by the following formula

$$(a_1 + a_2(\frac{1}{2}) + a_3(\frac{1}{3})) \cdot (b_1 + b_2(\frac{1}{2}) + b_3(\frac{1}{3})) = (a_1 \cdot b_1) + (a_2 \cdot b_2)(\frac{1}{2}) + (a_3 \cdot b_3)(\frac{1}{3}))$$

Now we can define the set A_2 of numbers that can be represented as expressions from P. Naturally, some of these polynomials define the same number. For instance, $1 = 2(\frac{1}{2})$ or $(\frac{1}{2}) \cdot (\frac{1}{3}) = 0$.

This gives us the abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, in which order is the same as in the arithmetic $\mathbb{R}$ of all real numbers, while addition and multiplication are defined above. In it, $\frac{1}{2}$ and $\frac{1}{3}$ are additively prime elements, while number 1 has two additive prime decompositions (factorizations)

$$1 = 2(\frac{1}{2}) = 1(\frac{1}{2}) + 1(\frac{1}{2})$$

and

$$1 = 3(\frac{1}{3}) = 1(\frac{1}{3}) + 2(\frac{1}{3})$$

Similar to the additive factoring property, for abstract prearithmetics and even for whole-number and natural number prearithmetics, the multiplicative factoring property is not true in a general case as the following example demonstrates.

Example 3.13. Let us consider the arithmetic $\mathbb{R}^{++} = (\mathbb{R}^{++}; +, \cdot, \leq)$ of all positive real numbers with standard addition, multiplication and order. The prearithmetic $\mathbb{R}^{++}$ does not have multiplicatively prime numbers because any positive real number a is equal to $a^{\frac{1}{2}}$ times $a^{\frac{1}{2}}$. Consequently, the Fundamental Theorem of Arithmetic is not true for this prearithmetic and it does not have the multiplicative factoring property.

There are also modular arithmetics, which do not have multiplicatively prime numbers.

Example 3.14. Let us consider the modular arithmetic $\mathbb{Z}_5$. It has five elements 0, 1, 2, 3 and 4. There are following multiplicative decompositions in $\mathbb{Z}_5$:

$$2 \cdot 3 = 1; \quad 3 \cdot 4 = 2; \quad 2 \cdot 4 = 3; \quad and \quad 2 \cdot 2 = 4$$

This shows that all numbers in $\mathbb{Z}_5$ are multiplicatively composite.
As a result, we can build multiplicative decompositions of an arbitrary length. For instance, we have

$$2 = 3 \cdot 4 = (2 \cdot 4) \cdot 4 = ((3 \cdot 4) \cdot 4) \cdot 4 = \ldots$$

At the same time, some modular arithmetics have multiplicatively prime numbers. For instance, 3 is a multiplicatively prime number in $\mathbb{Z}_4$.

Lemma 3.15. *The strong multiplicative factoring property implies the multiplicative factoring property.*

Proof is left as an exercise.
The inverse implication is not true as the following example demonstrates.

Example 3.15. Let us consider the set $F = \{1, 2, 2^{\frac{1}{3}}, 2^{\frac{1}{2}}\}$ and the set P of all expressions of the form $2^{a_1 + a_2(\frac{1}{2}) + a_3(\frac{1}{3})}$ where a_1, a_2 and a_3 are natural numbers. We see that the products of these expressions has the same form. Now we can define the set A_2 of numbers that can be represented as expressions from P and their arbitrary sums. Naturally, some of these polynomials define the same number. For instance, $2 = 2^{2(\frac{1}{2})} = 2^{3(\frac{1}{3})}$.

This gives us the abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, in which order is the same as in the arithmetic $\mathbb{R}$ of all real numbers, while addition and multiplication are defined above. In it, $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ are multiplicatively prime elements, while as we have demonstrated, number 2 has two multiplicative prime decompositions (factorizations).

Theorem 3.4. *An infinite abstract well-ordered additively Archimedean prearithmetic $\mathbb{A}$=(A;+,·,≤), with associative and commutative addition ,which strictly preserves the order, has the additive factoring property.*

Proof. As the abstract prearithmetic $\mathbb{A}$ is infinite and well-ordered, it does not have maximal elements. Indeed, if a is a maximal element in a well-ordered set, then there is only a finite number of elements less than a. At the same time, as well-ordering is also a total ordering (Fraenkel *et al.*, 1973), only one maximal element can exist and thus, the set has to be finite. Consequently, as the prearithmetic $\mathbb{A}$ is infinite, it does not have maximal elements.

Then by Lemma 2.11, the sum $a + b$ is larger than both its factors a and b. In other words, an additive factor of an element is less than this element. Consequently, any element a from $\mathbb{A}$ has only a finite numbers of additive factors by the properties of well-ordered sets (Fraenkel *et al.*, 1973).

Let us consider an element a from $\mathbb{A}$ that is not equal to the additive zero 0. If it is additively prime, then the statement of the theorem is valid for a. If a is composite, then for $n > 1$, there is a factoring

$$a = a_1 + a_2 + \ldots + a_n$$

.

If one of the elements a_i is not additively prime, then we will have more additive factors of a. As a has only a finite numbers of additive factors, at some step of the decomposition of the element a into additive factors, we will have only additively prime factors. □

Theorem is proved. Let us consider a natural example of an arithmetic with the additive factoring property.

Example 3.16. Let us take the arithmetic $\mathbb{N}_1$ of all whole numbers larger than 1 with conventional addition, multiplication and order. In this arithmetic, there are two additively prime numbers 2 and 3. Then it is possible to represent any even number from $\mathbb{N}_1$ as the sum of numbers all of which are equal to 2. It is also possible to represent any odd number from $\mathbb{N}_1$ as the sum of numbers some of which are equal to 2 while others are equal to 3. It will give an additive factorization of any number into the sum of additively prime numbers. However, this factorization is not unique. For instance, we have $6 = 2 + 2 + 2 = 3 + 3$.

Let us consider an infinite abstract well-ordered multiplicatively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, with associative and commutative multiplication, which strictly preserves the order.

Lemma 3.16. *The least element b of the set B of elements from A that are not equal to the multiplicative one* 1 *is multiplicatively prime.*

Indeed, as the set B is well-ordered, it has the least element b (cf.,(Fraenkel *et al.*, 1973)). If b is not prime, then $b = a \cdot d$ where by Lemma 2.14, both factors a and d are less than b and are not equal to the multiplicative one 1. As b is the least element of B, this is impossible and thus, b is multiplicatively prime.

Theorem 3.5. *An infinite abstract well-ordered multiplicatively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, with associative and commutative multiplication, which strictly preserves the order, has the multiplicative factoring property.*

Proof. At first, let us show that the prearithmetic $\mathbb{A}$ has the multiplicative factoring property. As the abstract prearithmetic $\mathbb{A}$ is infinite and well-ordered, it does not have maximal elements. Indeed, if a is a maximal element in a well-ordered set, then there is only a finite number of elements less than a. At the same time, as well-ordering is also a total ordering (cf.,(Fraenkel *et al.*, 1973)), only one maximal element can exist and thus, the set has to be finite. Consequently, as the prearithmetic $\mathbb{A}$ is infinite, it does not have maximal elements.

Then by Lemma 2.14, the product $a \cdot b$ is larger than both its factors a and b. In other words, a divisor of an element is less than this element. Consequently, any element a from A has only a finite numbers of divisors by the properties of well-ordered sets (Fraenkel *et al.*, 1973).

Let us consider an element a from $\mathbb{A}$ that is not equal to the multiplicative one 1. If we have a factoring

$$a = a_1 \cdot a_2 \cdot \ldots \cdot a_n$$

and one of the elements a_i is not multiplicatively prime, then we will have more divisors of a. As a has only a finite numbers of divisors, at some step of the decomposition of the element a into multiplicative factors, we will have only multiplicatively prime factors. □

Theorem is proved.

Multiplication in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is distributive from the left with respect to difference if for any elements a, b, c and d from $\mathbb{A}$, the equality $c \cdot a + d = c \cdot b$ implies the equality $d = c \cdot e$ where the element e is the difference of b and a. For instance, for integer numbers, it means that if $k = n - m$, $m = uw$ and $n = uv$, then

$$k = n - m = uv - uw = u(v - w)$$

Proposition 30. *The prime factorization obtained in Theorem 3.5 is unique if the abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is, in addition, countable, exactly additive with multiplicative cancellation and in which multiplication is distributive with respect to addition and difference.*

Proof. To prove uniqueness up to the order of factors of prime factoring, it is possible to use mathematical induction because the prearithmetic $\mathbb{A}$ is countable and totally ordered.

By Lemma 3.16, the least element b of the set B of elements from $\mathbb{A}$ that are not equal to the multiplicative one 1 is prime. Consequently, it has the unique prime factorization.

Let us assume that a least element in $\mathbb{A}$, which has two or more different prime factorizations

$$a = p_1 \cdot p_2 \cdot \ldots \cdot p_n = q_1 \cdot q_2 \cdot \ldots \cdot q_m \tag{3.2}$$

As multiplication is commutative and is a total order, it is possible to assume

$$p_1 \leq p_2 \leq \ldots \leq p_n$$

and

$$q_1 \leq q_2 \leq \ldots \leq q_m$$

Suppose that two elements, say p_1 and q_1, coincide. Then we have

$$p_1 \cdot p_2 \cdot \ldots \cdot p_n = p_1 \cdot q_2 \cdot \ldots \cdot q_m$$

We can cancel p_1 from both sides of this equality. As a result, we obtain different prime factorizations of a divisor of a, which is less than a by Lemma 2.14. This contradicts minimality of a and shows that any equality $p_i = q_j$ is impossible. Let us take the least elements p_1 and q_1 from both decompositions. As $p_1 \leq q_1$, it is possible to suppose that $p_1 < q_1$. As multiplication strictly preserves the order, we have

$$p_1 \cdot q_1 < q_1 \cdot q_1 \leq q_1 \cdot q_2 \leq q_1 \cdot q_2 \cdot q_3 \cdot \ldots \cdot q_m = a$$

Then

$$a = p_1 \cdot q_1 + c$$

because the prearithmetic $\mathbb{A}$ is exactly additive.

Element c is the difference of a and $p_1 \cdot q_1$. Multiplication in $\mathbb{A}$ is distributive with respect to difference and a is divisible by p_1 and q_1 . Consequently, c is also divisible by p_1 and q_1. Because c is less than a, it has unique prime factorization of the form

$$c = p_1 \cdot q_1 \cdot r_3 \cdot r_4 \cdot \ldots \cdot r_t$$

Multiplication in $\mathbb{A}$ is distributive with respect to addition. It gives us

$$a = p_1 \cdot p_2 \cdot \ldots \cdot p_n = p_1 \cdot q_1 \cdot r_3 \cdot r_4 \cdot \ldots \cdot r_t + p_1 \cdot q_1 = p_1 \cdot (q_1 \cdot r_3 \cdot r_4 \cdot \ldots \cdot r_t + q_1)$$

Cancelling p_1, we obtain

$$p_2 \cdot \ldots \cdot p_n = q_1 \cdot r_3 \cdot r_4 \cdot \ldots \cdot r_t + q_1$$

As multiplication in A is distributive with respect to addition, we have

$$p_2 \cdot \ldots \cdot p_n = q_1 \cdot (r_3 \cdot r_4 \cdot \ldots \cdot r_t + 1)$$

Thus, q_1 is a divisor of $p_2 \cdot \ldots \cdot p_n$. As q_1 is multiplicatively prime and all $p_2, \ldots, p_n$ are multiplicatively prime, q_1 has to be equal to one of the elements $p_2 \cdot \ldots \cdot p_n$ because the prime factorization $p_2 \cdot \ldots \cdot p_n$ is unique up to the order of factors. However, before we found that such an equality is impossible.

This implies that a has the unique prime factorization. By the principle of mathematical induction, this is true for any element from $\mathbb{A}$. □

Proposition is proved.

Thus, we proved the Fundamental Theorem of Arithmetic for a wide range of abstract prearithmetics because Theorem 3.5 and Proposition 30 imply the following result.

Theorem 3.6. ***Generalized Fundamental Theorem of Arithmetic.*** *An infinitely countable abstract well-ordered multiplicatively Archimedean exactly additive prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, with multiplicative cancellation, distributive with respect to addition and difference, associative and commutative multiplication, which strictly preserves the order, has the multiplicative factoring property.*

Corollary 3.9. *(Fundamental Theorem of Arithmetic). For any natural number larger than* 1 *in the conventional Diophantine arithmetic* $\mathbb{N}$, *there is a unique up to the order of factors decomposition (factoring) of this number into the product of prime numbers.*

Note that there are non-Diophantine arithmetics in which not all natural numbers have prime factorization (Burgin, 1997).

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, which is totally ordered, additively Archimedean and exactly additive and in which addition preserves the order.

Theorem 3.7. *a) If for some elements a and b from* $\mathbb{A}$, *we have* $a < b$, *then for some natural number n either*

$$b = n[a] \tag{3.3}$$

or

$$b = n[a] + r \tag{3.4}$$

where $r < a$.

b) If in addition, the abstract prearithmetic $\mathbb{A}$ is with additive cancellation from the left, then the representation (3.3) or (3.4) is unique.

Proof. a) As the abstract prearithmetic $\mathbb{A}$ is additively Archimedean, for some natural number n, we have

$$b \leq (n+1)[a] \tag{3.5}$$

It is possible to assume that n is the least number for which the inequality (3.5) is true. If in the inequality (3.5), we have equality, then b satisfies formula (3.3) and the statement (a) is proved.

If the inequality (3.5) is strict and n is the least natural number for which the inequality (3.5) is valid, then we obtain

$$n[a] \leq b \leq (n+1)[a] = n[a] + a \tag{3.6}$$

As the abstract prearithmetic $\mathbb{A}$ is exactly additive, then for some natural number k, we have $b = n[a] + r$.

By construction, $r < a$. Indeed, if this is not true, then $a \leq r$ because the order $\leq$ is total. As addition preserves the order, we have

$$n[a] + a = (n+1)[a] \leq n[a] + r$$

.

Because it is assumed $b < (n+1)[a]$, we come to a contradiction, which by the principle of excluded middle, concludes the proof of the part (a).

b) By construction the part $n[a]$ in the representation (3.4) is unique because n is the largest natural number for which $n[a] \leq b$. Now let us suppose

$$b = n[a] + r = n[a] + q$$

. Because the abstract prearithmetic $\mathbb{A}$ is with additive cancellation from the left, $r = q$. □

Theorem is proved.

When the abstract prearithmetic $\mathbb{A}$ has the additive zero 0 and is additively Archimedean for all non-zero elements, then it is possible to reduce formulas (3.3) and (3.4) to one formula. Namely, we have the following result.

Corollary 3.10. *a) If for some elements a and b from $\mathbb{A}$, we have $a < b$, then for some natural number n, we have*

$$b = n[a] + r$$

where $0 \leq r < a$.

b) If in addition, the abstract prearithmetic $\mathbb{A}$ is with additive cancellation from the left for all non-zero elements, then this representation is unique.

Theorem 3.7 also implies a well-known important result from number theory.

Corollary 3.11. ((Landau *et al.*, 1999): Theorem 7). *If for natural numbers a and b, we have $a < b$, then there is a natural number n such that*

$$b = n[a] + r$$

where $0 \leq r < a$.

Note that Theorem 3.7 implies that decompositions (3.3) and (3.4) are true not only for arithmetics of natural or whole numbers but also for the arithmetic of all positive rational numbers, arithmetic of all positive real numbers, arithmetic $2\mathbb{N}$ of all even numbers as well as for many non-Diophantine arithmetics (Burgin, 1997).

Corollary 3.12. *a) In a non-Diophantine arithmetic $\mathbb{A} = (N; +, \cdot, \leq)$, which is totally ordered, additively Archimedean and exactly additive and in which addition preserves the order, the inequality $a < b$, implies either*

$$b = n[a] \tag{3.7}$$

or

$$b = n[a] + r \tag{3.8}$$

for some natural number n and $r < a$.

b) If in addition, the non-Diophantine arithmetic $\mathbb{A}$ is with additive cancellation from the left, then the representation (3.7) or (3.8) is unique.

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, which is totally ordered, multiplicatively Archimedean and exactly multiplicative and in which multiplication preserves the order.

Theorem 3.8. *If for some elements a and b from $\mathbb{A}$, we have $a < b$, then*

$$b = [a]^n$$

or

$$b = [a]^n \cdot r$$

where $r < a$.

Proof is similar to the proof of Theorem 3.7.

This result is a multiplicative counterpart of Theorem 3.7. It is not valid for the Diophantine arithmetic $\mathbb{N}$ but there are abstract prearithmetics and arithmetics that have this property. For instance, let us consider the arithmetic $\mathbb{A}_{pow} = (A; +, \cdot, \leq)$, in which A consists of powers of some natural number m, i.e., $A = \{m^n; n = 1, 2, 3, \ldots\}$, multiplication is the same as the conventional multiplication of natural numbers and addition is trivial, i.e., the sum of any two numbers from A is equal to m. In this arithmetic, Theorem 3.8 is valid.

Corollary 3.13. *If for natural numbers a and b from* $\mathbb{A}_{pow}$*, we have* $a < b$*, then there is a natural number n such that*

$$b = a^n \cdot r$$

where $1 \leq r < a$*.*

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, which is totally ordered, additively and multiplicatively Archimedean and exactly additive and in which addition is associative and preserves the order.

Theorem 3.9. *a) For any elements a and b from* $\mathbb{A}$*, the following property is valid*

$$b = k_n[a]^n + k_{n-1}[a]^{n-1} + \ldots + k_1[a] + k_0[r] \tag{3.9}$$

where $r < a$*, the element* k_0 *is either* 1 *or the symbol* $\oslash$*, and for* $i = 1, 2, 3, \ldots, n$*, the element* k_i *is either a natural number or the symbol* $\oslash$*, which means that the corresponding element* $[a]^i$ *is absent in the left part of (3.9).*

b) If the abstract prearithmetic $\mathbb{A}$ *is with additive cancellation from the left, then the representation (3.9) is unique.*

Proof. a) To prove existence, we use mathematical induction on n.

Given two elements a and b from $\mathbb{A}$, we have either $a > b$ or $a = b$ or $a < b$ because the order in $\mathbb{A}$ is total. In the first two cases, the statement (a) is evident. Indeed, if $a > b$, we can take

$$b = k_0[r] = r$$

If $a = b$, we can take

$$b = 1[a] = a$$

In the case when $a < b$, we suppose that for all elements d from $\mathbb{A}$, such that $a \leq d < b$ the statement (a) is true and prove the equality (3.9). As the abstract prearithmetic $\mathbb{A}$ is multiplicatively Archimedean, there is a natural number n for which

$$b < [a]^{n+1} \tag{3.10}$$

because by Corollary 2.6, $[a]^n < [a]^{n+1}$ for all $n = 1, 2, 3, \ldots$. Taking the least n for which the inequality is valid, we obtain

$$[a]^n \leq b < [a]^{n+1}$$

where by our supposition, $n > 1$. If $b = [a]^n$, then the statement (a) is proved because

$$b = [a]^n = 1[a]^n,$$

If $b > [a]n$, then by Theorem 3.7, we have

$$b = k_n[a]^n + c \tag{3.11}$$

where $c < [a]^n$. Then by our supposition, formula (3.9) is true for c, i.e., we obtain the following equality

$$c = k_m[a]^m + k_{m-1}[a]^{m-1} + \ldots + k_1[a] + k_0[r] \tag{3.12}$$

with $m < n$ and $r < a$. If we substitute c in the equality (3.11) by the right side of the equality (3.12) and add the necessary number of expressions when $m < n - 1$, we obtain the equality (3.9) as addition is associative. The principle of mathematical induction implies that the statement (a) is true for all elements a and b from $\mathbb{A}$.

b) By Theorem 3.7, decomposition (3.11) is unique, while uniqueness of decomposition (3.12) is assumed according to the proof by induction. Uniqueness of decompositions (3.11) and (3.12) implies uniqueness of decomposition (3.9) for the chosen element b. Then the principle of mathematical induction allows us to conclude that the decomposition (3.9) is also unique for any element from $\mathbb{A}$, which is larger than a where a is an arbitrary element from $\mathbb{A}$.

□

Theorem is proved.

Corollary 3.14. ((Landau *et al.*, 1999): Theorem 8). *If a number a is larger than* 1, *then a natural number b can be expressed in one and only one way in the form*

$$b = k_n a^n + k_{n-1} a^{n-1} + \ldots + k_1 a + k_0$$

where $n > 0$, $k_0 > 0$ *and* $0 \leq k_j < a$ *for all* $j = 1, 2, \ldots, n$.

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, which has multiplicative 1, is totally ordered, additively Archimedean and exactly additive and in which addition preserves the order and multiplication is associative and distributive from the left with respect to difference and addition.

Theorem 3.10. *If m is the smallest common multiple of elements a and b from $\mathbb{A}$, then any common multiple u of elements a and b is divisible by m.*

Proof. As m is the smallest common multiple of elements a and b, we have $m < u$. Then by Theorem 3.7, $u = n[m]$ or $u = n[m] + r$ and $r < m$.

In the first case, $u = m \cdot n[1]$ because multiplication is distributive from the left with respect to addition and $m \cdot 1 = m$. It means that the statement of Theorem 3.7 is true.

As it was demonstrated, $n[m]$ is divisible by m and thus, by Proposition 19, it is divisible by a and b. As u is also divisible by a and b, the element r is divisible by a and b. It means that in the second case, r is a common multiple of elements a and b. However, this contradicts to the condition that m is the smallest common multiple of elements a and b. Consequently, only the first case is possible. □

Theorem is proved.

Corollary 3.15. ((Landau *et al.*, 1999): Theorem 9). *If m is the smallest common multiple of natural numbers k and h, then any common multiple n of numbers k and h is divisible by m.*

The proved results show that it is possible to develop number theory in abstract prearithmetics similar to the conventional number theory when abstract prearithmetics satisfy relevant conditions such as having an Archimedean property or associative multiplication.

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On Orthogonal Polynomials

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Abstract

The aim of this article is to present some general properties of orthogonal sequence of functions, respectively orthogonal polynomials. We obtain some connections between the classical orthogonal polynomials: Legendre, Chebyshev, Laguerre and Hermite.

Keywords: orthogonal polynomials, orthogonal basis, special functions.
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1. Introduction

The (classical) orthogonal polynomials represent one of the most important problems, extensively investigated in the literature from different points of view (see (Carlitz, 1968), (Chihara, 2011), (Dominici & Maier, 2008), (Grinshpun, 2004), (Mason & Handscomb, 2002) and references therein).

In (McCarthy *et al.*, 1993) the authors study the Dirichlet polynomials as a generalization of the Legendre polynomials and prove a formula of Rodrigues type and some results for the zeros of the generalized Legendre polynomials and an interesting approach is made in (Sun, 2014).

Relations for products of Chebyshev polynomials and the related generating functions are shown in (Cesarano, 2012) and in (Siyi, 2015) some connections between the Chebyshev polynomials, Fibonacci numbers and Lucas numbers are emphasized.

The Laguerre polynomials have been studied in mathematical physics, combinatorics and special functions (see (Koeph, 1997), (Micu & Papp, 2005)) and T. Kim, D. S. Kim, K. W. Hwang and J. J. Seo ((Kim *et al.*, 2016)) derive a family of ordinary differential equations from the generating function of the Laguerre polynomials and prove new identities for those polynomials.

Y. He and F. Yang ((He & Yang, 2018)) obtain recurrence formulas for the Hermite polynomials using generating function methods and Padé approximation techniques and in (Kim & Kim, 2013) a formula for a product of two Hermite polynomials is given. Also, recently, the differential equations associated with squared Hermite polynomials are treated in (Kim *et al.*, 2017).

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In this paper we treat the orthogonal sequence of functions and the (classical) orthogonal polynomials. Some relations for the classical orthogonal polynomials (Legendre, Chebyshev, Laguerre, Hermite) are proved.

2. The main results

Definition 2.1. We consider $\rho : (a,b) \to \mathbb{R}$ a nonnegative function on (a,b) and $\{f_n\}_{n\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a,b)$. We assume that there exist $\int_a^b f_n^2(x)\rho(x)\,dx$. The mapping $\langle\cdot,\cdot\rangle : L^2_{\mathbb{R}}(a,b) \times L^2_{\mathbb{R}}(a,b) \to \mathbb{R}$,

$$\langle f_m, f_n\rangle = \int_a^b f_m(x) f_n(x)\rho(x)\,dx$$

is *inner product* on $L^2_{\mathbb{R}}(a,b)$ and the function ρ is called *the weight function* of the inner product.

Proposition 1. *The inner product from Definition 2.1 satisfies the following properties:*

(*i*) $\forall\{f_m\}_{m\in\mathbb{N}}, \{f_n\}_{n\in\mathbb{N}}, \{f_p\}_{p\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a,b) : \langle f_m + f_n, f_p\rangle = \langle f_m, f_p\rangle + \langle f_n, f_p\rangle$;

(*ii*) $\forall\alpha \in \mathbb{R},\ \forall\{f_m\}_{m\in\mathbb{N}}, \{f_n\}_{n\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a,b) : \langle \alpha f_m, f_n\rangle = \alpha\langle f_m, f_n\rangle$;

(*iii*) $\forall\{f_m\}_{m\in\mathbb{N}}, \{f_n\}_{n\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a,b) : \langle f_m, f_n\rangle = \langle f_n, f_m\rangle$;

(*iv*) $\forall\{f_n\}_{n\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a,b) : \langle f_n, f_n\rangle \geq 0$.

Definition 2.2. The sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is called *orthogonal* if

$$\langle f_m, f_n\rangle = 0,\ \forall m \neq n.$$

Proposition 2. *Any orthogonal sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ on (a,b) is a linearly independent system.*

Theorem 2.1. *Any linearly independent sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ can be orthogonalized.*

Definition 2.3. The orthogonal polynomials $\{p_n\}_{n\in\mathbb{N}}$ on (a,b) are called *classical orthogonal polynomials* relative to the weight function ρ if the following differential equation is checked:

$$(\sigma(x)\rho(x))' = \tau(x)\rho(x),$$

where τ is a polynomial of degree 1 and σ is given by:

$$\sigma(x) = \begin{cases} (x-a)(b-x), & \text{if } a, b \in \mathbb{R} \\ x-a, & \text{if } a \in \mathbb{R}, b = \infty \\ b-x, & \text{if } a = -\infty, b \in \mathbb{R} \\ 1, & \text{if } a = -\infty, b = \infty \end{cases}$$

and

$$\lim_{x\to a} x^n\sigma(x)\rho(x) = \lim_{x\to b} x^n\sigma(x)\rho(x) = 0,\ \forall n \in \mathbb{N}.$$

Remark 1. From Definition 2.3, we obtain the expression of the function ρ:

$$\rho(x) = \begin{cases} (x-a)^{\alpha}(b-x)^{\beta}, & \alpha = \frac{\tau(a)}{b-a} - 1, \beta = -\frac{\tau(b)}{b-a} - 1 \ (a, b \in \mathbb{R}) \\ (x-a)^{\alpha} e^{x\tau'(x)}, & \alpha = \tau(a) - 1 \ (a \in \mathbb{R}, b = \infty) \\ (b-x)^{\beta} e^{-x\tau'(x)}, & \beta = -\tau(b) - 1 \ (a = -\infty, b \in \mathbb{R}) \\ e^{\int \tau(x)dx}, & a = -\infty, b = \infty \end{cases}$$

Remark 2. In particular, considering in Definition 2.3

- $(a, b) = (-1, 1)$ and $\rho(x) = 1$, $\sigma(x) = 1 - x^2$, $\tau(x) = -2x$, $\forall x \in (-1, 1)$, we obtain *the Legendre polynomials* denoted P_n;
- $(a, b) = (-1, 1)$ and $\rho(x) = \dfrac{1}{\sqrt{1-x^2}}$, $\sigma(x) = 1-x^2$, $\tau(x) = -x$, $\forall x \in (-1, 1)$, we obtain *the Chebyshev polynomials of the first kind* denoted T_n;
- $(a, b) = (-1, 1)$ and $\rho(x) = \sqrt{1-x^2}$, $\sigma(x) = 1-x^2$, $\tau(x) = -3x$, $\forall x \in (-1, 1)$, we obtain *the Chebyshev polynomials of the second kind* denoted U_n;
- $(a, b) = (0, \infty)$ and $\rho(x) = x^{\alpha}e^{-x}$, $\sigma(x) = x$, $\tau(x) = -x + \alpha + 1$, $\forall x \in (0, \infty)$ (where $\alpha > -1$), we obtain *the Laguerre polynomials* denoted L_n^{α};
- $(a, b) = (-\infty, \infty)$ and $\rho(x) = e^{-x^2}$, $\sigma(x) = 1$, $\tau(x) = -2x$, $\forall x \in (-\infty, \infty)$, we obtain *the Hermite polynomials* denoted H_n.

Remark 3. Also, the expression of the Chebyshev polynomials of the first, respectively second kind, can be given by:

$$T_n(x) = \cos(n \arccos x), \quad \text{respectively} \quad U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}, \ \forall x \in (-1, 1), \ \forall n \in \mathbb{N}.$$

Now we emphasize some properties of the Legendre, Chebyshev, Laguerre and Hermite polynomials (see (Bochner, 1929), (Szego, 1939)).

Proposition 3. *The Legendre polynomials satisfy the following:*

(i) $P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_{n-k}^k \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-2k-1)}{(n-k)! 2^k} x^{n-2k}$, $\forall x \in (-1, 1)$, $\forall n \in \mathbb{N}$;

(ii) $P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}(x^2 - 1)^n$, $\forall x \in (-1, 1)$, $\forall n \in \mathbb{N}$;

(iii) $(1 - x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$, $\forall x \in (-1, 1)$, $\forall n \in \mathbb{N}$;

(iv) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$, $\forall x \in (-1, 1)$, $\forall n \in \mathbb{N}^*$;

(v) $\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n. \end{cases}$

Proposition 4. *The Chebyshev polynomials of the first kind verify the following:*

(i) $T_n(x) = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k C_n^{2k} x^{n-2k}(1-x^2)^k,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N};$

(ii) $T_n(x) = \frac{(-1)^n\sqrt{1-x^2}}{(2n-1)!!} \cdot \frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}},\ \forall x \in (-1,1),\ \forall n \in \mathbb{N};$

(iii) $(1-x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N};$

(iv) $T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*.$

(v) $\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\pi}{2}, & \text{if } m = n \neq 0 \\ \pi, & \text{if } m = n = 0. \end{cases}$

Proposition 5. *The Chebyshev polynomials of the second kind verify the following:*

(i) $U_n(x) = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k C_{n+1}^{2k+1} x^{n-2k}(1-x^2)^k,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N};$

(ii) $U_n(x) = \frac{(-1)^n(n+1)}{(2n+1)!!\sqrt{1-x^2}} \cdot \frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}},\ \forall x \in (-1,1),\ \forall n \in \mathbb{N};$

(iii) $(1-x^2)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N};$

(iv) $U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*.$

(v) $\int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\pi}{2}, & \text{if } m = n. \end{cases}$

Proposition 6. *The Laguerre polynomials verify the following:*

(i) $L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} C_{n+\alpha}^{n-k} x^k,\ \forall x \in (0,\infty), \alpha > -1,\ \forall n \in \mathbb{N};$

(ii) $L_n^\alpha(x) = \frac{1}{n!}x^{-\alpha}e^x \cdot \frac{d^n}{dx^n}(x^{n+\alpha}e^{-x}),\ \forall x \in (0,\infty), \alpha > -1,\ \forall n \in \mathbb{N};$

(iii) $x(L_n^\alpha(x))'' + (1+\alpha-x)(L_n^\alpha(x))' + nL_n^\alpha(x) = 0,\ \forall x \in (0,\infty), \alpha > -1,\ \forall n \in \mathbb{N};$

(iv) $(n+1)L_{n+1}^\alpha(x) + (x-2n-1-\alpha)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0,\ \forall x \in (0,\infty), \alpha > -1,\ \forall n \in \mathbb{N}^*.$

(v) $\int_{0}^{\infty} L_m^\alpha(x)L_n^\alpha(x)x^\alpha e^{-x}\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{1}{n!}\Gamma(n+1+\alpha), & \text{if } m = n. \end{cases}$

Proposition 7. *The Hermite polynomials satisfy the following:*

(i) $H_n(x) = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k C_n^{2k} \cdot (k+1) \cdot (k+2) \cdot \ldots \cdot (2k) \cdot (2x)^{n-2k},\ \forall x \in (-\infty,\infty),\ \forall n \in \mathbb{N};$

(ii) $H_n(x) = (-1)^n e^{x^2} \cdot \frac{d^n}{dx^n}(e^{-x^2}),\ \forall x \in (-\infty,\infty),\ \forall n \in \mathbb{N};$

(*iii*) $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0,\ \forall x \in (-\infty, \infty),\ \forall n \in \mathbb{N}$;

(*iv*) $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0,\ \forall x \in (-\infty, \infty),\ \forall n \in \mathbb{N}^*$;

(*v*) $\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ 2^n n!\sqrt{\pi}, & \text{if } m = n. \end{cases}$

Further, we show other connections between the classical orthogonal polynomials indicated in Remark 2.

Proposition 8. *For the Legendre polynomials, the following relations hold:*

(*i*) $xP_{n+1}'(x) - P_n'(x) = (n+1)P_{n+1}(x),\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*$;

(*ii*) $P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x),\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*$.

Proof. Using Proposition 3, we deduce

$$P_n'(x) = \frac{nP_{n-1}(x) - nxP_n(x)}{1 - x^2},\ \ \forall x \in (-1, 1),\ \ \forall n \in \mathbb{N}^*.$$

Thus, making the computations, we obtain:
(*i*)

$$\begin{aligned} & xP_{n+1}'(x) - P_n'(x) = \\ & = x \cdot \frac{(n+1)P_n(x) - (n+1)xP_{n+1}(x)}{1-x^2} - \frac{nP_{n-1}(x) - nxP_n(x)}{1-x^2} = \\ & = \frac{(2n+1)xP_n(x) - nP_{n-1}(x) - nx^2P_{n+1}(x) - x^2P_{n+1}(x)}{1-x^2} = \\ & = \frac{(n+1)P_{n+1}(x) - x^2(n+1)P_{n+1}(x)}{1-x^2} = (n+1)P_{n+1}(x),\ \ \forall x \in (-1,1),\ \ \forall n \in \mathbb{N}^*; \end{aligned}$$

(*ii*)

$$\begin{aligned} P_{n+1}'(x) - xP_n'(x) & = \frac{(n+1)P_n(x) - (n+1)xP_{n+1}(x)}{1-x^2} - x \cdot \frac{nP_{n-1}(x) - nxP_n(x)}{1-x^2} = \\ & = \frac{(n+1)P_n(x) - nxP_{n+1}(x) + nx^2P_n(x) - xP_{n+1}(x) - nxP_{n-1}(x)}{1-x^2} = \\ & = \frac{(n+1)P_n(x) - xP_{n+1}(x) - nxP_{n-1}(x) + xP_{n+1}(x) - nx^2P_n(x) - x^2P_n(x) + nxP_{n-1}(x)}{1-x^2} = \\ & = \frac{(n+1)P_n(x) - x^2(n+1)P_n(x)}{1-x^2} = (n+1)P_n(x),\ \ \forall x \in (-1,1),\ \ \forall n \in \mathbb{N}^*. \end{aligned}$$

□

Proposition 9. *The Chebyshev polynomials of the first, respectively second kind, verify:*

(*i*) $U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0,\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*$;

(*ii*) $U_n(x) - 2T_n(x) - U_{n-2}(x) = 0,\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*$.

Proof. From Remark 3, after some computations we obtain:

$$U_n(x) = \frac{T'_{n+1}(x)}{n+1}, \forall x \in (-1,1),\ \forall n \in \mathbb{N}.$$

(i) We have:

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = \frac{T'_{n+1}(x)}{n+1} - 2x\frac{T'_n(x)}{n} + \frac{T'_{n-1}(x)}{n-1} =$$

$$= \frac{1}{\sqrt{1-x^2}}[\sin((n+1)\arccos x) + \sin((n-1)\arccos x) - 2x\sin(n\arccos x)] =$$

$$= \frac{1}{\sqrt{1-x^2}}[2x\sin(n\arccos x) - 2x\sin(n\arccos x)] = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*.$$

(ii) Similarly with (i), we deduce:

$$U_n(x) - 2T_n(x) - U_{n-2}(x) = \frac{T'_{n+1}(x)}{n+1} - \frac{T'_{n-1}(x)}{n-1} - 2T_n(x) =$$

$$= \frac{1}{\sqrt{1-x^2}}[\sin((n+1)\arccos x) - \sin((n-1)\arccos x)] - 2\cos(n\arccos x) =$$

$$= \frac{1}{\sqrt{1-x^2}}2\sin(\arccos x)\cos(n\arccos x) - 2\cos(n\arccos x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*.$$

□

Proposition 10. *For the Hermite polynomials, the following relations are checked:*

(i) $H'_{n+1}(x) + H'_{n-1}(x) = (2n+1)H_n(x) + 2xH_{n-1}(x),\ \forall x \in (-\infty,\infty),\ \forall n \ge 2;$

(ii) $H'_{n+1}(x) - 2xH'_n(x) = 2H_n(x) - 4n(n-1)H_{n-2}(x),\ \forall x \in (-\infty,\infty),\ \forall n \ge 2.$

Proof. From Proposition 7, $\forall x \in (-\infty,\infty)$, $\forall n \in \mathbb{N}^*$ it results

$$H'_n(x) = 2nH_{n-1}(x)$$

and then
(i)

$$H'_{n+1}(x) + H'_{n-1}(x) = 2(n+1)H_n(x) + 2(n-1)H_{n-2}(x) =$$

$$= (2n+1)H_n(x) + H_n(x) + 2(n-1)H_{n-2}(x) = (2n+1)H_n(x) + 2xH_{n-1}(x);$$

(ii)

$$H'_{n+1}(x) - 2xH'_n(x) = 2(n+1)H_n(x) - 4nxH_{n-1}(x) =$$

$$= 2H_n(x) + 2n(H_n(x) - 2xH_{n-1}(x)) = 2H_n(x) - 4n(n-1)H_{n-2}(x).$$

□

Proposition 11. *The classical orthogonal polynomials from Remark 2 satisfy:*

(i) $(n+1)P_{n+1}(x) + (1-x^2)P'_n(x) - x(n+1)P_n(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*;$

(ii) $nT_{n+1}(x) + (1-x^2)T'_n(x) - nxT_n(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*;$

(*iii*) $nU_{n+1}(x) + (1 - x^2)U'_n(x) - nxU_n(x) - U_{n-1}(x) = 0,\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*$;

(*iv*) $(n+1)L^{\alpha+1}_{n+1}(x) - (L^{\alpha}_n(x))' - (\alpha - x + 2n + 2)L^{\alpha+1}_n(x) + (n+\alpha)L^{\alpha+1}_{n-1}(x) = 0,\ \forall x \in (0, \infty), \alpha > -1,\ \forall n \in \mathbb{N}^*$;

(*v*) $H_{n+1}(x) + H'_n(x) - 2xH_n(x) = 0,\ \forall x \in (-\infty, \infty),\ \forall n \in \mathbb{N}^*$.

Proof. (*i*) From Proposition 3, (*ii*), respectively (*iv*), we obtain

$$(1 - x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x),$$

respectively

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x)$$

and then

$$(n + 1)P_{n+1}(x) + (1 - x^2)P'_n(x) + (nx - 2n - 1)P_n(x) =$$
$$= (2n + 1)xP_n(x) - nP_{n-1}(x) + nP_{n-1}(x) - nxP_n(x) - x(n + 1)P_n(x) = 0,\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*.$$

(*ii*) From Proposition 4, (*ii*), respectively (*iv*), we deduce

$$(1 - x^2)T'_n(x) = -nxT_n(x) + nT_{n-1}(x),$$

respectively

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

and then

$$nT_{n+1}(x) + (1 - x^2)T'_n(x) - nxT_n(x) =$$
$$= 2nxT_n(x) - nT_{n-1}(x) - nxT_n(x) + nT_{n-1}(x) - nxT_n(x) = 0,\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*.$$

(*iii*) From Proposition 5, (*ii*), respectively (*iv*), we observe

$$(1 - x^2)U'_n(x) = -nxU_n(x) + (n + 1)U_{n-1}(x),$$

respectively

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

and then

$$nU_{n+1}(x) + (1 - x^2)U'_n(x) - nxU_n(x) - U_{n-1}(x) =$$
$$= 2nxU_n(x) - nU_{n-1}(x) - nxU_n(x) + (n + 1)U_{n-1}(x) - nxU_n(x) - U_{n-1}(x) = 0, \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*.$$

(*iv*) From Proposition 6, (*ii*), respectively (*iv*), it results

$$(L^{\alpha}_n)'(x) = -L^{\alpha+1}_{n-1}(x),$$

respectively

$$(n + 1)L^{\alpha}_{n+1} = (\alpha - x + 2n + 1)L^{\alpha}_n(x) - (n + \alpha)L^{\alpha}_{n-1}(x)$$

and then

$$(n + 1)L^{\alpha+1}_{n+1}(x) - (L^{\alpha}_n(x))' - (\alpha - x + 2n + 2)L^{\alpha+1}_n(x) + (n + \alpha)L^{\alpha+1}_{n-1}(x) =$$
$$= (\alpha - x + 2n + 2)L^{\alpha+1}_n(x) - (n + \alpha + 1)L^{\alpha+1}_{n-1}(x) + L^{\alpha+1}_{n-1}(x) -$$
$$-(\alpha - x + 2n + 2)L^{\alpha+1}_n(x) + (n + \alpha)L^{\alpha+1}_{n-1}(x) = 0,\ \forall x \in (0, \infty),\ \alpha > -1,\ \forall n \in \mathbb{N}^*.$$

(v) From Proposition 7, (ii), respectively (iv), it follows

$$H_n'(x) = 2nH_{n-1}(x),$$

respectively

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

and then

$$H_{n+1}(x) + H_n'(x) - 2xH_n(x) = 2xH_n(x) - 2nH_{n-1}(x) + 2nH_{n-1}(x) - 2xH_n(x) = 0,\ \forall x \in (-\infty, \infty),\ \forall n \in \mathbb{N}^*.$$

□

Open problem: An important open problem is to define other orthogonal polynomials that fulfill similar conditions with those emphasized for the classical orthogonal polynomials in this article.

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