



Fixed Points of Generalized Kannan Type α -admissible Mappings in Cone Metric Spaces with Banach Algebra

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Abstract

In this paper, we introduce the generalized Kannan type α -admissible mappings in the setting of cone metric spaces equipped with Banach algebra. Our results generalize and extend the fixed point result for Kannan type mappings in metric and cone metric spaces. An example is presented which illustrates our main result.

Keywords: Cone metric space, α -admissible mapping, Kannan's contraction, Solid cone, Banach algebra, Fixed point.

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1. Introduction

Huang and Zhang (Huang & Zhang, 2007) introduced the notion of cone metric spaces as a generalization of metric spaces. They replaced the set of nonnegative real numbers by a subset of a Banach space called the cone; and defined the metric as a vector-valued function. They obtained some fixed point results in the setting of cone metric spaces with the assumption that the cone is normal. Later, the assumption of normality of cone was removed by Rezapour and Hamlbarani (Rezapour & Hamlbarani, 2008). Liu and Xu (Liu & Xu, 2013a) defined the cone metric spaces with Banach algebra and defined the vector-valued metric into a subset of a Banach algebra. The motivation for the work of Liu and Xu (Liu & Xu, 2013a) can be found in (Cakalli *et al.*, 2012; Kadelburg *et al.*, 2011; Du, 2010; Feng & Mao, 2010). The results proved by Liu and Xu (Liu & Xu, 2013a) demands the normality of the underlying cone. Later on, Xu and Radenović (Xu

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& Radenović, 2014) showed that the condition of normality of cone can be removed, and so, the results of Liu and Xu (Liu & Xu, 2013a) are also true in case of a non-normal cone.

Let (X, d) be a metric space and $T : X \rightarrow X$ be a mapping satisfying the following condition: there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \lambda d(x, y) \text{ for all } x, y \in X. \quad (1.1)$$

Then the mapping T is called a Banach contraction. The Banach's contraction principle states that a Banach contraction on a complete metric space has a unique fixed point, i.e., there exists a unique point $x^* \in X$ such that $x^* = Tx^*$. Kannan (Kannan, 1968, 1969) introduced the following contractive condition: there exists $\lambda \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \lambda [d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X. \quad (1.2)$$

Kannan (Kannan, 1968, 1969) showed that the conditions (1.1) and (1.2) are independent of each other, and proved a fixed point result for the mapping satisfying the condition (1.2) instead the condition (1.1).

Samet et al. (Samet et al., 2012) introduced a new type of mappings called α -admissible mappings, and with the help of this new class of mappings they generalized several known results of metric spaces. Very recently, Malhotra et al. (Malhotra et al., 2015) introduced the α -admissible mappings in the setting of cone metric spaces equipped with Banach algebra and solid cones. They generalized and extended several known results of metric and cone metric spaces by proving a fixed point result for generalized Lipschitz contraction over cone metric spaces. The main result of (Malhotra et al., 2015) was a generalization of Banach's fixed point theorem. In this paper, we introduce the notion of generalized Kannan type α -admissible mappings in the setting of cone metric spaces equipped with Banach algebra which extend the concept introduced in (Malhotra et al., 2015) and generalize the result of Kannan (Kannan, 1968, 1969) in cone metric spaces equipped with Banach algebra.

2. Preliminaries

First, we state some known definitions and results which will be used in the sequel.

Let A be a real Banach algebra, i.e., A is a real Banach space in which an operation of multiplication is defined, subject to the following properties: for all $x, y, z \in A, a \in \mathbb{R}$

1. $x(yz) = (xy)z$;
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
3. $a(xy) = (ax)y = x(ay)$;
4. $\|xy\| \leq \|x\|\|y\|$.

In this paper, we shall assume that the Banach algebra A has a unit, i.e., a multiplicative identity e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1} . For more details we refer to (Rudin, 1991).

The following proposition is well known (Rudin, 1991).

Proposition 2.1. Let A be a real Banach algebra with a unit e and $x \in A$. If the spectral radius $\rho(x)$ of x is less than one, i.e.,

$$\rho(x) = \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}} = \inf_{n \geq 1} \|x^n\|^{\frac{1}{n}} < 1$$

then $e - x$ is invertible. Actually,

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

A subset P of A is called a cone if

1. P is non-empty, closed and $\{\theta, e\} \subset P$, where θ is the zero vector of A ;
2. $a_1P + a_2P \subset P$ for all non-negative real numbers a_1, a_2 ;
3. $P^2 = PP \subset P$
4. $P \cap (-P) = \{\theta\}$.

For a given cone $P \subset A$, we can define a partial ordering \leq with respect to P by $x \leq y$ if and only if $y - x \in P$. The notation $x \ll y$ will stand for $y - x \in P^\circ$, where P° denotes the interior of P .

The cone P is called normal if there exists a number $K > 0$ such that for all $a, b \in A$,

$$a \leq b \text{ implies } \|a\| \leq K\|b\|.$$

The least positive value of K satisfying the above inequality is called the normal constant (see (Huang & Zhang, 2007)). Note that, for any normal cone P we have $K \geq 1$ (see (Rezapour & Hambarani, 2008)). In the following we always assume that P is a cone in a real Banach algebra A with $P^\circ \neq \emptyset$ (i.e., the cone P is a solid cone) and \leq is the partial ordering with respect to P .

The following lemmas and remark will be useful in the sequel.

Lemma 2.1 (See (Kadelburg et al., 2010)). If E is a real Banach space with a cone P and if $a \leq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

Lemma 2.2 (See (Radenović & Rhoades, 2009)). If E is a real Banach space with a solid cone P and if $\theta \leq u \ll c$ for each $\theta \ll c$, then $u = \theta$.

Lemma 2.3 (See (Radenović & Rhoades, 2009)). If E is a real Banach space with a solid cone P and if $\|x_n\| \rightarrow 0$ as $n \rightarrow \infty$, then for any $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that, $x_n \ll c$ for all $n > n_0$.

Remark (See (Xu & Radenović, 2014)). If $\rho(x) < 1$ then $\|x^n\| \rightarrow 0$ as $n \rightarrow \infty$.

Definition 2.1 (See (Liu & Xu, 2013a,b; Huang & Zhang, 2007)). Let X be a non-empty set. Suppose that the mapping $d: X \times X \rightarrow A$ satisfies:

1. $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X , and (X, d) is called a cone metric space over the Banach algebra A .

Definition 2.2 (See (Huang & Zhang, 2007)). Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X . Then:

1. The sequence $\{x_n\}$ converges to x whenever for each $c \in A$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$.
2. The sequence $\{x_n\}$ is a Cauchy sequence whenever for each $c \in A$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$.
3. (X, d) is a complete cone metric space if every Cauchy sequence is convergent in X .

It is obvious that the limit of a convergent sequence in a cone metric space is unique. A mapping $T: X \rightarrow X$ is called continuous at $x \in X$, if for every sequence $\{x_n\}$ in X such that $x_n \rightarrow x$ as $n \rightarrow \infty$, we have $Tx_n \rightarrow Tx$ as $n \rightarrow \infty$.

Definition 2.3 (See (Samet et al., 2012)). Let X be a nonempty set and $\alpha: X \times X \rightarrow [0, \infty)$ be a function. We say that T is α -admissible if $(x, y) \in X$, $\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1$.

Now, we define the generalized Lipschitz contractions on the cone metric spaces with a Banach algebra (see also, (Liu & Xu, 2013a)).

Definition 2.4. (Malhotra et al., 2015) Let (X, d) be a complete cone metric space over a Banach algebra A , P the underlying solid cone and $\alpha: X \times X \rightarrow [0, \infty)$ be a function. Then the mapping $T: X \rightarrow X$ is said to be generalized Lipschitz contraction if there exists $k \in P$ such that $\rho(k) < 1$ and,

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$ with $\alpha(x, y) \geq 1$. Here, the vector k is called the Lipschitz vector of T .

Malhotra et al. (Malhotra et al., 2015) proved a fixed point result for such generalized contraction. Here, we prove a Kannan's version of the result of Malhotra et al. (Malhotra et al., 2015).

Now we can state our main results.

3. Main results

First, we define generalized Kannan type contractions in cone metric spaces with Banach algebra.

Definition 3.1. Let (X, d) be a complete cone metric space over a Banach algebra A , P the underlying solid cone and $\alpha: X \times X \rightarrow [0, \infty)$ be a function. Then the mapping $T: X \rightarrow X$ is said to be generalized Kannan type contraction if there exists $k \in P$ such that $\rho(k) < \frac{1}{2}$ and,

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \quad (3.1)$$

for all $x, y \in X$ with $\alpha(x, y) \geq 1$. Here, the vector k is called the Kannan-Lipschitz vector of T .

The following theorem is the main result of this paper.

Theorem 3.1. Let (X, d) be a complete cone metric space over a Banach algebra A , P be the underlying solid cone and $\alpha: X \times X \rightarrow [0, \infty)$ be a function. Suppose, $T: X \rightarrow X$ be a generalized Kannan type contraction with Kannan-Lipschitz vector k and the following conditions are satisfied:

- (i) T is α -admissible;
- (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
- (iii) T is continuous.

Then T has a fixed point $x^* \in X$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and define a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point for T . Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is α -admissible we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \implies \alpha(Tx_0, T^2x_0) = \alpha(x_1, x_2) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Since T is generalized Kannan type contraction with Kannan-Lipschitz vector k , we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq k[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ &= k[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \end{aligned}$$

i.e.,

$$(e - k)d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n).$$

Since $\rho(k) < \frac{1}{2} < 1$, $e - k$ is invertible, therefore it follows from the above inequality that

$$d(x_n, x_{n+1}) \leq k(e - k)^{-1}d(x_{n-1}, x_n) = \lambda d(x_{n-1}, x_n) \leq \lambda^n d(x_0, x_1) \tag{3.3}$$

where $\lambda = k(e - k)^{-1}$. Since $(e - k)^{-1} = \sum_{i=0}^{\infty} k^i$ we have

$$\rho((e - k)^{-1}) = \rho\left(\sum_{i=0}^{\infty} k^i\right) \leq \sum_{i=0}^{\infty} \rho(k^i) \leq \sum_{i=0}^{\infty} [\rho(k)]^i = \frac{1}{1 - \rho(k)}.$$

Therefore,

$$\begin{aligned} \rho(\lambda) &= \rho(k(e - k)^{-1}) \leq \rho(k)\rho((e - k)^{-1}) \\ &\leq \frac{\rho(k)}{1 - \rho(k)} < 1 \quad \left(\text{since } \rho(k) < \frac{1}{2}\right). \end{aligned}$$

Thus, for $n < m$ it follows from the inequality (3.3) that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \lambda^n d(x_0, x_1) + \lambda^{n+1} d(x_0, x_1) + \cdots + \lambda^{m-1} d(x_0, x_1) \\ &= (e + \lambda + \cdots + \lambda^{m-n-1}) \lambda^n d(x_0, x_1) \\ &\leq \left(\sum_{i=0}^{\infty} \lambda^i \right) \lambda^n d(x_0, x_1) \\ &= (e - \lambda)^{-1} \lambda^n d(x_0, x_1). \end{aligned}$$

Since $\rho(\lambda) < 1$, by Remark 2 we have $\|\lambda^n\| \rightarrow 0$ as $n \rightarrow \infty$. Therefore, by Lemma 2.3 it follows that: for every $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) \leq (e - \lambda)^{-1} \lambda^n d(x_0, x_1) \ll c$$

for all $n > n_0$. It implies that $\{x_n\}$ is a Cauchy sequence. By completeness of X , there exists $x^* \in X$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Since T is continuous, it follows that $x_{n+1} = Tx_n \rightarrow Tx^*$ as $n \rightarrow \infty$. By the uniqueness of limit we get $x^* = Tx^*$, that is x^* is a fixed point of T . \square

In the above theorem, we use the continuity of the mapping T . We now show that the assumption of continuity can be replaced by another condition.

Theorem 3.2. *Let (X, d) be a complete cone metric space over a Banach algebra A , P be the underlying solid cone and $\alpha: X \times X \rightarrow [0, \infty)$ be a function. Suppose, $T: X \rightarrow X$ be a generalized Kannan type contraction with Kannan-Lipschitz vector k and the following conditions are satisfied:*

- (i) T is α -admissible;
- (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
- (iii) if x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x \in X$ as $n \rightarrow \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then T has a fixed point $x^* \in X$.

Proof. By proof of theorem 3.1, we know that the sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \in \mathbb{N}$ is a Cauchy sequence in complete cone metric space (X, d) . Then, there exists $x^* \in X$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. On the other hand, from (3.2) and hypothesis (iii), we have

$$\alpha(x_n, x^*) \geq 1, \quad \text{for all } n \in \mathbb{N}. \quad (3.4)$$

Since T is a generalized Kannan type contraction, using (3.4) we obtain

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\ &= d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\ &\leq d(x^*, x_{n+1}) + k[d(x_n, Tx_n) + d(x^*, Tx^*)] \end{aligned}$$

i.e.,

$$\begin{aligned} d(x^*, Tx^*) &\leq (e - k)^{-1}[d(x^*, x_{n+1}) + kd(x_n, Tx_n)] \\ &= (e - k)^{-1}d(x^*, x_{n+1}) + \lambda d(x_n, Tx_n). \end{aligned}$$

By (3.3) we have $d(x_n, Tx_n) = d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1)$, therefore

$$d(x^*, Tx^*) \leq (e - k)^{-1}d(x^*, x_{n+1}) + \lambda^{n+1}d(x_0, x_1).$$

As $x_n \rightarrow x^*$ as $n \rightarrow \infty$ and $\rho(\lambda) < 1$, for every $c \in P$ with $\theta \ll c$ and for every $m \in \mathbb{N}$ there exists $n(m)$ such that $d(x_{n+1}, x^*) \ll \frac{(e-k)c}{2m}$ and $\lambda^{n+1}d(x_0, x_1) \ll \frac{c}{2m}$ for all $n > n(m)$. Therefore, it follows from the above inequality that

$$d(x^*, Tx^*) \leq \frac{c}{2m} + \frac{c}{2m} = \frac{c}{m} \text{ for all } n > n(m), m \in \mathbb{N}.$$

It implies that $\frac{c}{m} - d(x^*, Tx^*) \in P$ for all $m \in \mathbb{N}$. Since P is closed, letting $m \rightarrow \infty$ we obtain $\theta - d(x^*, Tx^*) \in P$. By definition, we must have $d(x^*, Tx^*) = \theta$, i.e., $Tx^* = x^*$. Thus, x^* is a fixed point of T . □

Next, we give an example which illustrates the above result.

Example 3.1. Let $A = \mathbb{R}^2$ with the norm

$$\|(x_1, x_2)\| = |x_1| + |x_2|.$$

Define the multiplication on A by

$$xy = (x_1y_1, x_1y_2 + x_2y_1) \text{ for all } x = (x_1, x_2), y = (y_1, y_2) \in A.$$

Then, A is a Banach algebra with unit $e = (1, 0)$. Let $P = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$. Then P is a positive cone.

Let $X = [0, 1] \times [0, 1]$ and define the cone metric $d: X \times X \rightarrow P$ by

$$d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1|, |x_2 - y_2|) \in P.$$

Then, (X, d) is a complete cone metric space. Let $\mathbb{Q} \cap [0, 1) = \mathbb{Q}_1$ and define the mappings $T: X \rightarrow X$ and $\alpha: X \times X \rightarrow [0, \infty)$ by:

$$T(x_1, x_2) = \begin{cases} \left(\frac{1}{2}, \frac{1}{2}\right), & \text{if } x_1, x_2 \in \mathbb{Q}_1; \\ \left(\frac{1}{4}, \frac{1}{4}\right), & \text{if } x_1 = x_2 = 1; \\ (x_1, x_2), & \text{otherwise.} \end{cases}$$

and

$$\alpha((x_1, x_2), (y_1, y_2)) = \begin{cases} 1, & \text{if } (x_1, x_2, y_1, y_2 \in \mathbb{Q}_1) \text{ or } (x_1, x_2 \in \mathbb{Q}_1, y_1 = y_2 = 1); \\ 0, & \text{otherwise.} \end{cases}$$

Then, T is a generalized Kannan type contraction with Kannan-Lipschitz vector $k = \left(\frac{1}{3}, 0\right)$, where $\rho(k) = \frac{1}{3} < \frac{1}{2}$. Indeed, $x_1, x_2, y_1, y_2 \in \mathbb{Q}_1$ then (3.1) is satisfied trivially. If $x_1, x_2 \in \mathbb{Q}_1$ and $y_1 = y_2 = 1$ then we have

$$\begin{aligned} d(T(x_1, x_2), T(y_1, y_2)) &= d\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{4}, \frac{1}{4}\right)\right) \\ &= \left(\frac{1}{4}, \frac{1}{4}\right) \\ &\leq \left(\frac{1}{3}, 0\right) [d((x_1, x_2), T(x_1, x_2)) + d((y_1, y_2), T(y_1, y_2))]. \end{aligned}$$

T is obviously an α -admissible mapping, and for every $x_1, x_2, y_1, y_2 \in \mathbb{Q}_1$ we have

$$\alpha((x_1, x_2), T(x_1, x_2)) = 1.$$

Therefore, the conditions (i) and (ii) of Theorem 3.2 are satisfied. Finally, one can see that the condition (iii) of Theorem 3.2 is satisfied. Thus, all the conditions of Theorem 3.2 are satisfied and we conclude the existence of at least one fixed point of T . Indeed, $\left(\frac{1}{2}, \frac{1}{2}\right)$ and all the points $(x, 1), x \in \mathbb{Q}_1$ and $(1, x), x \in \mathbb{Q}_1$ are fixed points of T .

Remark. Notice that, in the above example the results of Malhotra et al. (Malhotra et al., 2015) are not applicable. Indeed, if $x_1 = x_2 = \frac{3}{4} \in \mathbb{Q}_1$ and $y_1 = y_2 = 1$, then $\alpha((x_1, x_2), (y_1, y_2)) = 1$ and

$$d(T(x_1, x_2), T(y_1, y_2)) = d\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{4}, \frac{1}{4}\right)\right) = \left(\frac{1}{4}, \frac{1}{4}\right).$$

Now

$$d((x_1, x_2), (y_1, y_2)) = d\left(\left(\frac{3}{4}, \frac{3}{4}\right), (1, 1)\right) = \left(\frac{1}{4}, \frac{1}{4}\right).$$

Therefore, there exists no $k \in P$ such that $\rho(k) < 1$ and the following inequality is satisfied:

$$d(T(x_1, x_2), (y_1, y_2)) \leq kd((x_1, x_2), (y_1, y_2)).$$

This shows that T is not a generalized Lipschitz contraction, and so, the results of Malhotra et al. (Malhotra et al., 2015) are not applicable here.

In the Example 3.1 we can see that the mapping T may have more than one fixed points. Let us denote the set of all fixed points of T by $\text{Fix}(T)$.

Next, to assure the uniqueness of fixed point of a generalized Kannan type contraction we use the following property (see (Samet et al., 2012)):

$$\forall x, y \in \text{Fix}(T) \exists z \in X: \alpha(x, z) \geq 1, \alpha(y, z) \geq 1. \quad (\text{H})$$

Theorem 3.3. Adding condition (H) to the hypothesis of Theorem 3.1 (resp. Theorem 3.2) we obtain the uniqueness of the fixed point of T .

Proof. Following similar arguments to those in the proof of Theorem 3.1 (resp. Theorem 3.2) we obtain the existence of fixed point. Let the condition (H) is satisfied and $x^*, y^* \in \text{Fix}(T)$ and $x^* \neq y^*$. By (H) there exists $z \in X$ such that

$$\alpha(x^*, z) \geq 1 \quad \text{and} \quad \alpha(y^*, z) \geq 1. \tag{3.5}$$

Since T is α -admissible and $x^*, y^* \in \text{Fix}(T)$, therefore from (3.5) we obtain

$$\alpha(x^*, T^n z) \geq 1 \quad \text{and} \quad \alpha(y^*, T^n z) \geq 1. \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

Since T is generalized Kannan type contraction, using (3.6), we have

$$\begin{aligned} d(x^*, T^n z) &= d(Tx^*, T(T^{n-1}z)) \\ &\leq k[d(x^*, Tx^*) + d(T^{n-1}z, T(T^{n-1}z))] \\ &= kd(T^{n-1}z, T^n z) \\ &\leq k[d(x^*, T^{n-1}z) + d(x^*, T^n z)] \end{aligned}$$

i.e.,

$$d(x^*, T^n z) \leq k(e - k)^{-1}d(x^*, T^{n-1}z) = \lambda d(x^*, T^{n-1}z) \quad \text{for all } n \in \mathbb{N}.$$

Repetition of this process we obtain

$$d(x^*, T^n z) \leq \lambda^n d(x^*, Tz) \quad \text{for all } n \in \mathbb{N}.$$

where $\lambda = k(e - k)^{-1}$ and $\rho(\lambda) < 1$. Since $\rho(\lambda) < 1$, by Remark 2 we have $\|\lambda^n\| \rightarrow 0$ as $n \rightarrow \infty$, and so,

$$\|\lambda^n d(x^*, Tz)\| \leq \|\lambda^n\| \|d(x^*, Tz)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, by Lemma 2.3 it follows that: for every $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x^*, T^n z) \leq \lambda^n d(x^*, Tz) \ll c.$$

it implies that

$$T^n z \rightarrow x^* \quad \text{as } n \rightarrow \infty.$$

Similarly we get

$$T^n z \rightarrow y^* \quad \text{as } n \rightarrow \infty.$$

Therefore, by uniqueness of the limit we obtain $x^* = y^*$. This finishes the proof. □

4. Some consequences

In this section, we give some consequences of the results of previous section. The following corollary is Theorem 3.3 of Xu and Radenović (Liu & Xu, 2013a).

Corollary 4.1 (Theorem 3.3, Xu and Radenović (Liu & Xu, 2013a)). *Let (X, d) be a complete cone metric space over a Banach algebra A and P be the underlying solid cone with $k \in P$ where $\rho(k) < \frac{1}{2}$. Suppose the mapping $T: X \rightarrow X$ satisfies the following condition :*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \quad \text{for all } x, y \in X.$$

Then T has a unique fixed point in X . Moreover, for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point of X .

Proof. Define the function $\alpha: X \times X \rightarrow [0, \infty)$ by $\alpha(x, y) = 1$ for all $x, y \in X$. Then, all the conditions of Theorem 3.3 are satisfied, and so, the mapping T has a unique fixed point in X . \square

Next, we derive the ordered and cyclic versions of Kannan's contraction principle. In the next theorems, we prove results of Ran and Reurings (Ran & Reurings, 2003), Liu and Xu (Liu & Xu, 2013a) and Nieto, Rodríguez-López (Nieto & Rodríguez-López, 2005) and Kirk et al. (Kirk et al., 2003) for Kannan's mappings.

The following theorem is the Kannan's version of the result of Ran and Reurings (Ran & Reurings, 2003) in cone metric spaces when the cone metric is endowed with a Banach algebra.

Theorem 4.1. *Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) be a complete cone metric space over a Banach algebra A with P the underlying solid cone. Let $T: X \rightarrow X$ be a continuous nondecreasing mapping with respect to \sqsubseteq . Suppose that the following two assumptions hold:*

(i) *there exists $k \in P$ such that $\rho(k) < \frac{1}{2}$ and*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \quad \text{for all } x, y \in X \text{ with } x \sqsubseteq y;$$

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$.*

Then, T has a fixed point in X .

Proof. Define the mapping $\alpha_r: X \times X \rightarrow [0, \infty)$ by

$$\alpha_r(x, y) = \begin{cases} 1, & \text{if } x \sqsubseteq y; \\ 0, & \text{otherwise.} \end{cases}$$

Note that, the condition (i) implies that the mapping T a generalized Kannan type contraction with Kannan-Lipschitz vector k , where $\rho(k) < \frac{1}{2}$. Since T is nondecreasing it is an α_r -admissible mapping. The condition (ii) implies that, there exists $x_0 \in X$ such that $\alpha_r(x_0, Tx_0) = 1$. Therefore, all the conditions of Theorem 3.1 are satisfied, and so, the mapping T has a fixed point in X . \square

The following theorem is the Kannan’s version of the result of Nieto, Rodríguez-López (Nieto & Rodríguez-López, 2005) when the cone metric is endowed with a Banach algebra.

Theorem 4.2. Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) be a complete cone metric space over a Banach algebra A with P the underlying solid cone. Let $T : X \rightarrow X$ be a nondecreasing mapping with respect to \sqsubseteq . Suppose that the following three assumptions hold:

- (i) there exists $k \in P$ such that $\rho(k) < \frac{1}{2}$ and

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X \text{ with } x \sqsubseteq y;$$

- (ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$;
- (iii) if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \rightarrow x \in X$ as $n \rightarrow \infty$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$.

Then, T has a fixed point in X .

Proof. Define the mapping $\alpha_r : X \times X \rightarrow [0, \infty)$ similar to that as in the proof of Theorem 4.1. Now, the proof follows from the Theorem 3.2. □

Next, we define the cyclic contractions (see (Kirk et al., 2003)) in cone metric spaces.

Let X be a nonempty set, $T : X \rightarrow X$ a mapping and A_1, A_2, \dots, A_m be subsets of X . Then $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to T if

1. $A_i, i = 1, 2, \dots, m$ are nonempty sets;
2. $T(A_1) \subset A_2, \dots, T(A_{m-1}) \subset T(A_m), T(A_m) \subset T(A_1)$.

Remark. (See (Kirk et al., 2003)) If $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to T , then $\text{Fix}(T) \subset \bigcap_{i=1}^m A_i$.

A cyclic contraction on a cone metric space is defined as follows.

Definition 4.1. Let (X, d) be a complete cone metric space over a Banach algebra A and P be the underlying solid cone. Suppose, A_1, A_2, \dots, A_m be subsets of X and $Y = \bigcup_{i=1}^m A_i$. A mapping $T : Y \rightarrow Y$ is called a generalized cyclic Kannan type contraction with Kannan-Lipschitz vector k if following conditions hold:

1. $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T ;
2. there exists $k \in P$ such that $\rho(k) < \frac{1}{2}$ and

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \tag{4.1}$$

for any $x \in A_i, y \in A_{i+1}$ ($i = 1, 2, \dots, m$ where $A_{m+1} = A_1$).

The following theorem is the Kannan's version of the result Kirk et al. (Kirk et al., 2003) when the cone metric is endowed with a Banach algebra.

Theorem 4.3. *Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) be a complete cone metric space over a Banach algebra A with P the underlying solid cone. Suppose, A_1, A_2, \dots, A_m be closed subsets of X and $Y = \bigcup_{i=1}^m A_i$ and $T: Y \rightarrow Y$ be a generalized cyclic Kannan type contraction with Kannan-Lipschitz vector k . Then, T has a unique fixed point in X .*

Proof. Define the mapping $\alpha_c: X \times X \rightarrow [0, \infty)$ by:

$$\alpha_c(x, y) = \begin{cases} 1, & \text{if } (x, y) \in A_i \times A_{i+1} \text{ (} i = 1, 2, \dots, m \text{ where } A_{m+1} = A_1\text{);} \\ 0, & \text{otherwise.} \end{cases}$$

First, by definition of the function α and the cyclic representation, T is α_c -admissible. Again, by definition of the function α_c , T is a generalized cyclic Kannan type contraction with Kannan-Lipschitz vector k . Suppose, for a sequence $\{x_n\}$ we have $\alpha_c(x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x \in X$ as $n \rightarrow \infty$. Then, as $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation with respect to T , we must have $x \in \bigcap_{i=1}^m A_i$. Therefore, $\alpha_c(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Now, the proof of existence of fixed point of T follows from Theorem 3.2. For uniqueness, if $x^*, y^* \in \text{Fix}(T)$, then by Remark 4 we have $x^*, y^* \in \bigcap_{i=1}^m A_i$. Since each $A_i, i \in \{1, 2, \dots, m\}$ is nonempty, there exists $z \in Y$ such that $x^*, y^* \in A_i, z \in A_{i+1}$ for some $i \in \{1, 2, \dots, m\}$, and so $\alpha_c(x^*, z) = \alpha_c(y^*, z) = 1$. Thus, the condition (H) is satisfied and the uniqueness of fixed point follows from Theorem 3.3. \square

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On the Growth of Solutions of Higher Order Complex Differential Equations with finite $[p, q]$ -Order

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Abstract

In this paper, we study the growth of entire solutions of higher order linear complex differential equations with entire coefficients of finite $[p, q]$ -order. We give another conditions that generalize some results due to (Belaïdi, 2015), (Liu *et al.*, 2010) and (Li & Cao, 2012).

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1. Introduction

In this article, we use the standard notation and fundamental results of the Nevanlinna value distribution theory of meromorphic functions, see (Hayman, 1964; Laine, 1993; Yang & Yi, 2003). We define, for $r \in [0, +\infty)$, $\exp_0 r := r$, $\exp_1 r := e^r$ and $\exp_{n+1} r := \exp(\exp_n r)$, $n \in \mathbb{N}$. For all r sufficiently large, we define $\log_0 r := r$, $\log_1 r := \log r$ and $\log_{n+1} r := \log(\log_n r)$, $n \in \mathbb{N}$. Moreover, we denote by $\exp_{-1} r := \log r$ and $\log_{-1} r := \exp_1 r$.

For a meromorphic function f in complex plane \mathbb{C} , the order of growth is defined by

$$\sigma(f) = \limsup_{r \rightarrow +\infty} \frac{\log T(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f . The exponents of convergence of sequence of the zeros and distinct zeros of f are respectively defined by

$$\lambda(f) = \limsup_{r \rightarrow +\infty} \frac{\log N\left(r, \frac{1}{f}\right)}{\log r}, \quad \bar{\lambda}(f) = \limsup_{r \rightarrow +\infty} \frac{\log \bar{N}\left(r, \frac{1}{f}\right)}{\log r},$$

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where $N\left(r, \frac{1}{f}\right)$ (resp. $\bar{N}\left(r, \frac{1}{f}\right)$) is the integrated counting function of zeros (resp. distinct zeros) of $f(z)$ in the disc $\{z : |z| \leq r\}$.

(Juneja et al., 1976, 1977) have investigated some properties of entire functions of $[p, q]$ -order and obtained some results about their growth. In order to maintain accordance with general definitions of the entire function f of iterated p -order¹, (Liu et al., 2010) gave a minor modification of the original definition of the $[p, q]$ -order given by (Juneja et al., 1976, 1977).

We recall the following definition,

Definition 1.1. (Kinnunen, 1998) Let $p \geq 1$ be an integer. The iterated p -order $\sigma_p(f)$ of a meromorphic function f is defined by

$$\sigma_p(f) = \limsup_{r \rightarrow +\infty} \frac{\log_p T(r, f)}{\log r}.$$

Now, we shall introduce the definition of meromorphic functions of $[p, q]$ -order, where p, q are positive integers satisfying $p \geq q \geq 1$ or $2 \leq q = p + 1$. In order to keep accordance with Definition 1.1, (Li & Cao, 2012; Belaïdi, 2015) have gave a minor modification to the original definition of $[p, q]$ -order (e.g. see, (Juneja et al., 1976, 1977)). We recall the following definitions

Definition 1.2. (Belaïdi, 2015; Li & Cao, 2012; Liu et al., 2010) Let $p \geq q \geq 1$ or $2 \leq q = p + 1$ be integers. If $f(z)$ is a transcendental meromorphic function, then the $[p, q]$ -order is defined by

$$\sigma_{[p,q]}(f) = \limsup_{r \rightarrow +\infty} \frac{\log_p T(r, f)}{\log_q r}.$$

It is easy to see that $0 \leq \sigma_{[p,q]}(f) \leq +\infty$. If $f(z)$ is rational, then $\sigma_{[p,q]}(f) = 0$ for any $p \geq q \geq 1$. By Definition 1.2, we note that $\sigma_{[1,1]}(f) = \sigma(f)$ (order of growth), $\sigma_{[2,1]}(f) = \sigma_2(f)$ (hyper-order), $\sigma_{[1,2]}(f) = \sigma_{\log}(f)$ (logarithmic order) and $\sigma_{[p,1]}(f) = \sigma_p(f)$ (iterated p -order).

Definition 1.3. (Belaïdi, 2015; Li & Cao, 2012) Let $p \geq q \geq 1$ or $2 \leq q = p + 1$ be integers. The $[p, q]$ convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$\lambda_{[p,q]}(f) = \limsup_{r \rightarrow +\infty} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log_q r}.$$

Similarly, the $[p, q]$ convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\bar{\lambda}_{[p,q]}(f) = \limsup_{r \rightarrow +\infty} \frac{\log_p \bar{N}\left(r, \frac{1}{f}\right)}{\log_q r}.$$

¹see (Kinnunen, 1998), for the definition of the iterated p -order.

We recall also the following definitions. The linear measure of a set $E \subset (0, +\infty)$ is defined as

$$m(E) = \int_0^{+\infty} \chi_E(t) dt$$

and the logarithmic measure of a set $F \subset (1, +\infty)$ is defined as

$$\ell m(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt,$$

where $\chi_H(t)$ is the characteristic function of the set H . The upper density of a set $E \subset (0, +\infty)$ is defined by

$$\overline{\text{dens}}(E) = \limsup_{r \rightarrow +\infty} \frac{m(E \cap [0, r])}{r}.$$

The upper logarithmic density of a set $F \subset (1, +\infty)$ is defined by

$$\overline{\log \text{dens}}(F) = \limsup_{r \rightarrow +\infty} \frac{\ell m(F \cap [1, r])}{\log r}.$$

Proposition 1.1. (Belaïdi, 2015) *For all $H \subset [1, +\infty)$ the following statements hold :*

- (i) *If $\overline{\log \text{dens}}(H) = \infty$, then $m(H) = \infty$,*
- (ii) *if $\overline{\text{dens}}(H) > 0$, then $m(H) = \infty$,*
- (iii) *if $\overline{\log \text{dens}}(H) > 0$, then $\ell m(H) = \infty$.*

For $a \in \overline{\mathbb{C}}$, the deficiency of a with respect to a meromorphic function f is defined by

$$\delta(a, f) = \liminf_{r \rightarrow +\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r, f)} = 1 - \limsup_{r \rightarrow +\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

Consider the differential equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_1f' + A_0f = 0. \tag{1.1}$$

(Liu et al., 2010) studied the growth of solutions of the homogeneous differential equation (1.1) with coefficients that are entire functions of finite $[p, q]$ -order and obtained following result

Theorem 1.1. (Liu et al., 2010) *Let $A_j(z)$ ($j = 0, 1, \dots, k - 1$) be entire functions satisfying $\max\{\sigma_{[p,q]}(A_j) : j \neq s\} < \sigma_{[p,q]}(A_s) < \infty$. Then every solution $f(z)$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A_s)$. Furthermore, at least one solution of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_s)$.*

Theorem 1.2. (Liu et al., 2010) *Let A_0, A_1, \dots, A_{k-1} be entire functions, and let $s \in \{0, \dots, k - 1\}$ be the largest index for which $\sigma_{[p,q]}(A_s) = \max_{0 \leq j \leq k-1} \sigma_{[p,q]}(A_j)$. Then there are at least $k - s$ linearly independent solutions $f(z)$ of (1.1) such that $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_s)$. Moreover, all solutions of (1.1) satisfy $\sigma_{[p+1,q]}(f) \leq \rho$ if and only if $\sigma_{[p,q]}(A_j) \leq \rho$ for all $j = 0, 1, \dots, k - 1$.*

Theorem 1.3. (Liu et al., 2010) Let H be a set of complex numbers satisfying $\overline{\text{dens}} \{ |z| : z \in H \} > 0$ and let $A_j(z)$ ($j = 0, 1, \dots, k - 1$) be entire functions satisfying

$$\max \{ \sigma_{[p,q]}(A_j) : j = 0, 1, \dots, k - 1 \} \leq \alpha.$$

Suppose that there exists a positive constant β satisfying $\beta < \alpha$ such that any given ε ($0 < \varepsilon < \alpha - \beta$), we have

$$|A_0(z)| \geq \exp_{p+1} \{ (\alpha - \varepsilon) \log_q r \}$$

and

$$|A_j(z)| \leq \exp_{p+1} \{ \beta \log_q r \} \quad (j = 1, \dots, k - 1)$$

for $z \in H$. Then, every solution $f \not\equiv 0$ of the equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \alpha$.

Recently, (Belaïdi, 2015) has obtained the following results which generalize and improve Theorem 1.3 and also improve some results due to (Li & Cao, 2012).

Theorem 1.4. (Belaïdi, 2015) Let H be a set of complex numbers satisfying $\overline{\log \text{dens}} \{ |z| : z \in H \} > 0$ and let $A_j(z)$ ($j = 0, 1, \dots, k - 1$) be meromorphic functions satisfying

$$\max \{ \sigma_{[p,q]}(A_j) : j = 0, 1, \dots, k - 1 \} \leq \rho, \quad 0 < \rho < +\infty.$$

Suppose that there exist two real numbers α and β satisfying $0 \leq \beta < \alpha$ such that

$$|A_0(z)| \geq \exp_p \left(\alpha \left[\log_{q-1} r \right]^\rho \right) \tag{1.2}$$

and

$$|A_j(z)| \leq \exp_p \left(\beta \left[\log_{q-1} r \right]^\rho \right), \quad (j = 1, \dots, k - 1) \tag{1.3}$$

as $|z| = r \rightarrow +\infty$ for $z \in H$. Then the following statements hold :

- (i) If $p \geq q \geq 2$ or $3 \leq q = p + 1$, then every meromorphic solution $f \not\equiv 0$ whose poles are uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \rho$.
- (ii) If $p = 1, q = 2$, then every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies $\sigma_{[2,2]}(f) \geq \rho$.

Theorem 1.5. (Belaïdi, 2015) Let H be a set of complex numbers satisfying $\overline{\log \text{dens}} \{ |z| : z \in H \} > 0$ and let $A_j(z)$ ($j = 0, 1, \dots, k - 1$) be meromorphic functions satisfying

$$\max \{ \sigma_{[p,q]}(A_j) : j = 0, 1, \dots, k - 1 \} \leq \rho, \quad 0 < \rho < +\infty.$$

Suppose that there exist two positive constants α and β such that, we have

$$m(r, A_0) \geq \exp_{p-1} \left(\alpha \left[\log_{q-1} r \right]^\rho \right) \tag{1.4}$$

and

$$m(r, A_j) \leq \exp_{p-1} \left(\beta \left[\log_{q-1} r \right]^\rho \right), \quad (j = 1, \dots, k - 1) \tag{1.5}$$

as $|z| = r \rightarrow +\infty$ for $z \in H$. Then the following statements hold :

- (i) If $p \geq q \geq 2$ and $0 \leq \beta < \alpha$, then every meromorphic solution $f \not\equiv 0$ whose poles are uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \rho$.
- (ii) If $3 \leq q = p + 1, 0 \leq \beta < \alpha$ and $\rho > 1$, then every meromorphic solution $f \not\equiv 0$ whose poles are uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\sigma_{[p+1,p+1]}(f) = \rho$.
- (iii) If $p = 1, q = 2, 0 \leq (k - 1)\beta < \alpha$ and $\rho > 1$, then every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies $\sigma_{[2,2]}(f) \geq \rho$.

2. Main results

Now, a natural question is whether somewhat similar results to Theorem 1.4 and Theorem 1.5 could be obtained for the differential equation (1.1), where $A_j(z)$ ($j = 0, 1, \dots, k$) are entire functions and the dominant coefficient is some $A_s(z)$ ($0 \leq s \leq k - 1$) instead of $A_0(z)$? The main purpose of this article is to answer the above question and improving and generalizing the previous results.

Theorem 2.1. *Let H be a set of complex numbers satisfying $\overline{\log \text{dens}} \{|z| : z \in H\} > 0$. Let $A_j(z)$ ($j = 0, 1, \dots, k - 1$) be entire functions satisfying*

$$\max \left\{ \sigma_{[p,q]}(A_j) : j = 0, 1, \dots, k - 1 \right\} \leq \rho, \quad 0 < \rho < +\infty.$$

Suppose that there exist two real numbers α and β satisfying $0 \leq \beta < \alpha$ and let $s \in \{0, \dots, k - 1\}$ be an integer for which

$$|A_s(z)| \geq \exp_p \left(\alpha \left[\log_{q-1} r \right]^\rho \right), \quad 0 \leq s \leq k - 1 \tag{2.1}$$

and

$$|A_j(z)| \leq \exp_p \left(\beta \left[\log_{q-1} r \right]^\rho \right), \quad j \neq s, \tag{2.2}$$

as $|z| = r \rightarrow +\infty, z \in H$. Then,

- (i) *If $p \geq q \geq 1$, then every polynomial solution $f \neq 0$ of equation (1.1) is of $\deg f \leq s - 1$ ($s \geq 1$) and every transcendental solution f of equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \rho$.*
- (ii) *If $2 \leq q = p + 1, \rho > 1$, then every polynomial solution $f \neq 0$ of equation (1.1) is of $\deg f \leq s - 1$ ($s \geq 1$) and every transcendental solution f of equation (1.1) satisfies $\rho \leq \sigma_{[p+1,p+1]}(f) \leq \rho + 1$.*

Corollary 2.1. *Let H be a set of complex numbers satisfying $\overline{\log \text{dens}} \{|z| : z \in H\} > 0$. Let $F(z) \neq 0, A_j(z)$ ($j = 0, 1, \dots, k - 1$) be entire functions. Suppose that $H, A_j(z)$ ($j = 0, 1, \dots, k - 1$) satisfy the hypotheses in Theorem 2.1. Consider the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_1f' + A_0f = F. \tag{2.3}$$

- (i) *Let $p \geq q \geq 1$, if $\sigma_{[p+1,q]}(F) \leq \rho$, then every transcendental solution f of equation (2.3) satisfies $\bar{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \rho$ with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \rho$; if $\rho_{[p+1,q]}(F) > \rho$, then every transcendental solution f of equation (2.3) satisfies $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.*
- (ii) *Let $2 \leq q = p + 1$ and $\rho > 1$, if $\sigma_{[p+1,p+1]}(F) \leq \rho$, then every transcendental solution f of equation (2.3) satisfies $\bar{\lambda}_{[p+1,p+1]}(f) = \lambda_{[p+1,p+1]}(f) = \sigma_{[p+1,p+1]}(f) = \rho$ with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \rho$; if $\rho_{[p+1,p+1]}(F) > \rho$, then every transcendental solution f of equation (2.3) satisfies $\rho_{[p+1,p+1]}(f) = \rho_{[p+1,p+1]}(F)$.*

Theorem 2.2. *Let H be a set of complex numbers satisfying $\overline{\log \text{dens}} \{|z| : z \in H\} > 0$. Let $A_j(z)$ ($j = 0, 1, \dots, k - 1$) be entire functions satisfying*

$$\max \left\{ \sigma_{[p,q]}(A_j) : j = 0, 1, \dots, k - 1 \right\} \leq \rho, \quad 0 < \rho < +\infty.$$

Suppose that there exist two real numbers α and β satisfying $0 \leq \beta < \alpha$ and let $s \in \{0, \dots, k - 1\}$ be an integer for which

$$m(r, A_s) \geq \exp_{p-1} \left(\alpha \left[\log_{q-1} r \right]^\rho \right), \quad 0 \leq s \leq k - 1 \tag{2.4}$$

and

$$m(r, A_j) \leq \exp_{p-1} \left(\beta \left[\log_{q-1} r \right]^\rho \right), \quad j \neq s, \tag{2.5}$$

as $|z| = r \rightarrow +\infty, z \in H$. Then the following statements hold :

(i) If $p \geq q \geq 1$ and $0 \leq \beta < \alpha$, then every polynomial solution $f \neq 0$ of (1.1) is of $\deg f \leq s - 1$ ($s \geq 1$), and every transcendental solution f satisfies $\sigma_{[p,q]}(f) \geq \rho \geq \sigma_{[p+1,q]}(f)$.

(ii) If $2 \leq q = p + 1$ and $0 \leq (k - 1)\beta < \alpha$, then every polynomial solution $f \neq 0$ of (1.1) is of $\deg f \leq s - 1$ ($s \geq 1$), and every transcendental solution f satisfies $\rho \leq \sigma_{[p,p+1]}(f)$ and $\sigma_{[p+1,p+1]}(f) \leq \rho + 1$.

3. Some preliminary lemmas

Lemma 3.1. (Gundersen, 1988) Let f be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exists a set $E_1 \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and $s, j(0 \leq s < j)$, such that for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$

$$\left| \frac{f^{(j)}(z)}{f^{(s)}(z)} \right| \leq B \left[\frac{T(\alpha r, f)}{r} (\log^\alpha r) \log T(\alpha r, f) \right]^{j-s}.$$

Lemma 3.2. (Gundersen, 1988) Let f be a meromorphic function, and let j be a given positive integer, and let $\alpha > 1$ be a real constant. Then there exists a constant $R > 0$ such that for all $r \geq R$ we have

$$T(r, f^{(j)}) \leq (j + 2) T(\alpha r, f).$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, $\mu_f(r)$ be the maximum term, i.e., $\mu_f(r) = \max\{|a_n| r^n; n = 0, 1, \dots\}$, and let $\nu_f(r)$ be the central index of f , i.e., $\nu_f(r) = \max\{m; \mu_f(r) = |a_m| r^m\}$.

Lemma 3.3. (Hayman, 1974) Let $f(z)$ be a transcendental entire function, and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then for all $|z| = r$ outside a set E_2 of r of finite logarithmic measure, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z} \right)^j (1 + o(1)), \quad j \in \mathbb{N},$$

where $\nu_f(r)$ is the central index of $f(z)$.

Lemma 3.4. (Juneja et al., 1976) Let $f(z)$ be an entire function of $[p, q]$ -order, and let $\nu_f(r)$ be the central index of $f(z)$. Then

$$\sigma_{[p,q]}(f) = \limsup_{r \rightarrow +\infty} \frac{\log_p \nu_f(r)}{\log_q r}.$$

Lemma 3.5. Let $A_0(z), \dots, A_{k-1}(z)$ be entire functions of finite $[p, q]$ -order. Then,
 (i) If $p \geq q \geq 1$, then every solution $f \not\equiv 0$ of equation (1.1) satisfies

$$\sigma_{[p+1,q]}(f) \leq \max \left\{ \sigma_{[p,q]}(A_j) : j = 0, 1, \dots, k-1 \right\}.$$

(ii) If $2 \leq q = p + 1$, then every solution $f \not\equiv 0$ of equation (1.1) satisfies

$$\sigma_{[p+1,p+1]}(f) \leq \max \left\{ \sigma_{[p,p+1]}(A_j) : j = 0, 1, \dots, k-1 \right\} + 1.$$

Proof. We prove only (ii). For the proof of (i) see (Liu et al., 2010). Let $f \not\equiv 0$ be a solution of equation (1.1). By (1.1), we have

$$\left| \frac{f^{(k)}}{f} \right| \leq |A_{k-1}| \left| \frac{f^{(k-1)}}{f} \right| + |A_{k-2}| \left| \frac{f^{(k-2)}}{f} \right| + \dots + |A_1| \left| \frac{f'}{f} \right| + |A_0|. \tag{3.1}$$

Set $\max \left\{ \sigma_{[p,p+1]}(A_j) : j = 0, 1, \dots, k-1 \right\} = \rho$. For any given $\varepsilon > 0$, when r is sufficiently large, we have

$$|A_j(z)| \leq \exp_{p+1} \left((\rho + \varepsilon) \left[\log_{p+1} r \right] \right), \quad j = 0, 1, \dots, k-1. \tag{3.2}$$

By Lemma 3.3, there exists a set $E_2 \subset [1, +\infty)$ with logarithmic measure $\ell m E_2 < \infty$, we can choose z satisfying $|z| = r \notin [0, 1] \cup E_2$ and $|f(z)| = M(r, f)$, such that

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z} \right)^j (1 + o(1)), \quad j = 1, \dots, k \tag{3.3}$$

holds. Substituting (3.2) and (3.3) into (3.1), we obtain

$$\left(\frac{\nu_f(r)}{|z|} \right)^k |1 + o(1)| \leq k \exp_{p+1} \left((\rho + \varepsilon) \left[\log_{p+1} r \right] \right) \left(\frac{\nu_f(r)}{|z|} \right)^{k-1} |1 + o(1)|, \tag{3.4}$$

where z satisfies $|z| = r \notin [0, 1] \cup E_2$ and $|f(z)| = M(r, f)$. By (3.4), we get

$$\nu_f(r) |1 + o(1)| \leq kr |1 + o(1)| \exp_{p+1} \left((\rho + \varepsilon) \left[\log_{p+1} r \right] \right). \tag{3.5}$$

So, from (3.5), we obtain

$$\limsup_{r \rightarrow +\infty} \frac{\log_{p+1} \nu_f(r)}{\log_{p+1} r} \leq \rho + 1 + \varepsilon. \tag{3.6}$$

Since $\varepsilon > 0$ is arbitrary, by (3.6) and Lemma 3.4 we have $\sigma_{[p+1,p+1]}(f) \leq \rho + 1$. □

Remark. Lemma 3.5 (ii) has been proved for $p = 1$ and $q = 2$ by (Cao et al., 2013).

Lemma 3.6. (Chen & Shon, 2004) Let $f(z)$ be a transcendental entire function. Then there is a set $E_3 \subset (1, +\infty)$ having finite logarithmic measure such that when we take a point z satisfying $|z| = r \notin [0, 1] \cup E_3$ and $|f(z)| = M(r, f)$, we have

$$\left| \frac{f(z)}{f^{(s)}(z)} \right| \leq 2r^s, \quad s \in \mathbb{N}.$$

Lemma 3.7. *Let f be a transcendental meromorphic function of finite $[p, q]$ -order. Then the following statements hold :*

- (i) *If $p \geq q \geq 1$, then $\rho_{[p,q]}(f') = \rho_{[p,q]}(f)$.*
- (ii) *If $2 \leq q = p + 1$, then $\rho_{[p,p+1]}(f') = \rho_{[p,p+1]}(f)$.*

Proof. We prove only (ii). For the proof of (i) see (Belaïdi, 2015). Let f be a transcendental meromorphic function of finite $[p, q]$ -order. By lemma of logarithmic derivative ², we have

$$\begin{aligned} T(r, f') &= m(r, f') + N(r, f') \leq m(r, f) + m\left(r, \frac{f'}{f}\right) + 2N(r, f) \\ &\leq 2T(r, f) + m\left(r, \frac{f'}{f}\right) \leq 2T(r, f) + O(\log T(r, f) + \log r) \end{aligned} \tag{3.7}$$

holds outside of an exceptional set $E_4 \subset (0, +\infty)$ with finite linear measure. By (3.7), it is easy to see that $\rho_{[p,p+1]}(f') \leq \rho_{[p,p+1]}(f)$ if $2 \leq q = p + 1$. On the other hand, by (Chuang, 1951), ((Yang & Yi, 2003), p. 35), we have for $r \rightarrow +\infty$

$$T(r, f) < O(T(2r, f') + \log r). \tag{3.8}$$

Hence, by using (3.8) we obtain $\rho_{[p,p+1]}(f) \leq \rho_{[p,p+1]}(f')$ if $2 \leq q = p + 1$. Thus, $\rho_{[p,p+1]}(f') = \rho_{[p,p+1]}(f)$ if $2 \leq q = p + 1$. □

Remark. Lemma 3.7 (ii) has been proved for $p = 1$ and $q = 2$ by (Chern, 2006).

Lemma 3.8. (Belaïdi, 2015) *Let A_j ($j = 0, 1, \dots, k - 1$), $F \neq 0$ be meromorphic functions. Then the following statements hold :*

- (i) *If $p \geq q \geq 1$, then every every meromorphic solution f of equation (2.3) such that*

$$\max \left\{ \sigma_{[p,q]}(A_j); \sigma_{[p,q]}(F) : j = 0, 1, \dots, k - 1 \right\} < \sigma_{[p,q]}(f)$$

satisfies $\bar{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \sigma_{[p,q]}(f)$.

- (ii) *If $2 \leq q = p + 1$, then every meromorphic solution f of equation (2.3) such that*

$$\max \left\{ 1; \sigma_{[p,q]}(A_j); \sigma_{[p,q]}(F) : j = 0, 1, \dots, k - 1 \right\} < \sigma_{[p,q]}(f)$$

satisfies $\bar{\lambda}_{[p,p+1]}(f) = \lambda_{[p,p+1]}(f) = \rho_{[p,p+1]}(f)$.

4. Proofs of main results

Proof of Theorem 2.1 It's should be noticed that the case $s = 0$ returns to Theorem 1.4. So, we will prove Theorem 2.1 in case $s > 0$.

² see, (Hayman, 1964; Yang & Yi, 2003).

(i) Case : $p \geq q \geq 1$. Suppose that $f \neq 0$ is a polynomial solution of the equation (1.1), let $f(z) = a_n z^n + \dots + a_0, a_n \neq 0$ and suppose that $n \geq s$, i.e., $f^{(s)}(z) \neq 0$. Then from (1.1), we have

$$|A_s| A_n^s |a_n| r^{n-s} (1 + o(1)) \leq |A_s| |f^{(s)}(z)| \leq \sum_{\substack{j=0 \\ j \neq s}}^k |A_j| |f^{(j)}(z)| \leq \sum_{\substack{j=0 \\ j \neq s}}^k |A_j| A_n^j |a_n| r^{n-j} (1 + o(1)), \quad (4.1)$$

where $A_k \equiv 1$ and $A_n^j = n(n-1) \dots (n-j+1)$. It follows from (4.1), (2.1) and (2.2) that

$$\exp_p(\alpha [\log_{q-1} r]^\rho) r^{-s} \leq O(\exp_p(\beta [\log_{q-1} r]^\rho)). \quad (4.2)$$

Since $\alpha > \beta$, we see that (4.2) is a contradiction as $r \rightarrow +\infty$. Then $\deg f \leq s - 1$.

Now, suppose that f is a transcendental solution of the equation (1.1). From the conditions of Theorem 2.1, there is a set H of complex numbers satisfying $\log \text{dens} \{|z| : z \in H\} > 0$, and there exists A_s ($0 \leq s \leq k - 1, k \geq 2$) such that for all $z \in H$ we have (2.1) and (2.2) as $|z| \rightarrow +\infty$. Set $H_1 = \{|z| : z \in H\}$, since $\log \text{dens} \{|z| : z \in H\} > 0$, then H_1 is a set with $\ell m(H_1) = \infty$.

From (1.1), we have

$$\begin{aligned} -A_s &= \frac{f}{f^{(s)}} \left(\frac{f^{(k)}}{f} + A_{k-1} \frac{f^{(k-1)}}{f} + \dots + A_{s+1} \frac{f^{(s+1)}}{f} \right. \\ &\quad \left. + A_{s-1} \frac{f^{(s-1)}}{f} + \dots + A_1 \frac{f'}{f} + A_0 \right). \end{aligned} \quad (4.3)$$

By Lemma 3.1, there exists a set $E_1 \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq B [T(2r, f)]^{j+1}, \quad j = 1, 2, \dots, k - 1. \quad (4.4)$$

By Lemma 3.6, there is a set $E_3 \subset (1, +\infty)$ having finite logarithmic measure such that when we take a point z satisfying $|z| = r \notin [0, 1] \cup E_3$ and $|f(z)| = M(r, f)$, we have

$$\left| \frac{f(z)}{f^{(s)}(z)} \right| \leq 2r^s. \quad (4.5)$$

It follows from (4.3) – (4.5), (2.1) and (2.2) that

$$\exp_p(\alpha [\log_{q-1} r]^\rho) \leq 2kB [T(2r, f)]^{k+1} r^s \exp_p(\beta [\log_{q-1} r]^\rho). \quad (4.6)$$

for all $|z| = r \in H_1 \setminus ([0, 1] \cup E_1 \cup E_3)$ and $|f(z)| = M(r, f)$. Then by (4.6), we obtain $\rho \leq \sigma_{[p+1, q]}(f)$. On the other hand, by Lemma 3.5 (i), we have $\sigma_{[p+1, q]}(f) \leq \rho$. Hence, every transcendental solution f of the equation (1.1) satisfies $\sigma_{[p+1, q]}(f) = \rho$.

(ii) Case : $2 \leq q = p + 1, \rho > 1$. Suppose that $f \neq 0$ is a polynomial solution of the equation (1.1), let $f(z) = a_n z^n + \dots + a_0, a_n \neq 0$ and suppose that $n \geq s$, i.e. $f^{(s)}(z) \neq 0$. From (4.2), we have

$$\exp_p(\alpha [\log_p r]^\rho) r^{-s} \leq O(\exp_p(\beta [\log_p r]^\rho)). \quad (4.7)$$

Since $\alpha > \beta$, we see that (4.7) is a contradiction as $r \rightarrow +\infty$. Then $\deg f \leq s - 1$.
 Now, suppose that f is a transcendental. Then from (4.6) we have

$$\exp_p \left(\alpha \left[\log_p r \right]^\rho \right) \leq 2kBr^s [T(2r, f)]^{k+1} \exp_p \left(\beta \left[\log_p r \right]^\rho \right) \tag{4.8}$$

holds for all z satisfying $|z| = r \in H_1 \setminus ([0, 1] \cup E_1 \cup E_3)$, as $r \rightarrow +\infty$. By (4.8), every transcendental solution f of equation (1.1) satisfies $\sigma_{[p+1, p+1]}(f) \geq \rho$, and by Lemma 3.5 (ii), we have $\sigma_{[p+1, p+1]}(f) \leq \rho + 1$, thus $\rho \leq \sigma_{[p+1, p+1]}(f) \leq \rho + 1$.

Proof of Corollary 2.1 (i) (a) Let $p \geq q \geq 1$. Let f be a transcendental solution of the equation (2.3) and $\{f_1, f_2, \dots, f_k\}$ is a solution base of the corresponding homogeneous equation (1.1) of (2.3). By Theorem 2.1, we know that for $j = 1, 2, \dots, k$

$$\sigma_{[p+1, q]}(f_j) = \rho.$$

Then f can be expressed in the form

$$f(z) = B_1(z) f_1(z) + B_2(z) f_2(z) + \dots + B_k(z) f_k(z), \tag{4.9}$$

where B_1, B_2, \dots, B_k are suitable meromorphic functions satisfying

$$B'_j = F \cdot G_j(f_1, f_2, \dots, f_k) \cdot (W(f_1, f_2, \dots, f_k))^{-1}, \quad j = 1, 2, \dots, k, \tag{4.10}$$

where $G_j(f_1, f_2, \dots, f_k)$ are differential polynomials in f_1, f_2, \dots, f_k and their derivatives with constant coefficients, thus

$$\sigma_{[p+1, q]}(G_j) \leq \max_{j=1, 2, \dots, k} \sigma_{[p+1, q]}(f_j) = \rho, \quad j = 1, 2, \dots, k. \tag{4.11}$$

Since the Wronskian $W(f_1, f_2, \dots, f_k)$ is a differential polynomial in f_1, f_2, \dots, f_k , it is easy to deduce also that

$$\sigma_{[p+1, q]}(W) \leq \max_{j=1, 2, \dots, k} \sigma_{[p+1, q]}(f_j) = \rho. \tag{4.12}$$

Since $\sigma_{[p+1, q]}(F) \leq \rho$, then by using Lemma 3.7 (i) and (4.10) – (4.12) we get for $j = 1, 2, \dots, k$

$$\sigma_{[p+1, q]}(B_j) = \sigma_{[p+1, q]}(B'_j) \leq \max \{ \sigma_{[p+1, q]}(F); \rho \} = \rho. \tag{4.13}$$

Then by (4.9) and (4.13), we obtain

$$\sigma_{[p+1, q]}(f) \leq \max_{j=1, 2, \dots, k} \{ \sigma_{[p+1, q]}(f_j); \sigma_{[p+1, q]}(B_j) \} = \rho. \tag{4.14}$$

Now, we assert that every transcendental solution f of (2.3) satisfies $\sigma_{[p+1, q]}(f) = \rho$ with at most one exceptional solution f_0 satisfying $\sigma_{[p+1, q]}(f_0) < \rho$. In fact, if f^* is another transcendental solution with $\sigma_{[p+1, q]}(f^*) < \rho$ of (2.3), then $\sigma_{[p+1, q]}(f_0 - f^*) < \rho$, but $f_0 - f^*$ is a solution of the corresponding homogeneous equation (1.1), and this is a contradiction with the results of Theorem 2.1. Then, $\sigma_{[p+1, q]}(f) = \rho$ holds for every transcendental solution f of (2.3) with at

most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \rho$. By Lemma 3.8, every transcendental solution f of (2.3) with $\sigma_{[p+1,q]}(f) = \rho$ satisfies $\lambda_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \rho$.

(b) If $\rho < \rho_{[p+1,q]}(F)$, then by using Lemma 3.7 (i), (4.11) and (4.12), we have from (4.10) for $j = 1, 2, \dots, k$

$$\begin{aligned} \rho_{[p+1,q]}(B_j) &= \rho_{[p+1,q]}(B'_j) \\ &\leq \max \{ \rho_{[p+1,q]}(F), \rho_{[p+1,q]}(f_j) : j = 1, 2, \dots, k \} = \rho_{[p+1,q]}(F). \end{aligned} \tag{4.15}$$

Then from (4.15) and (4.9), we get

$$\rho_{[p+1,q]}(f) \leq \max \{ \rho_{[p+1,q]}(f_j), \rho_{[p+1,q]}(B_j) : j = 1, 2, \dots, k \} \leq \rho_{[p+1,q]}(F). \tag{4.16}$$

On the other hand, if $\rho < \rho_{[p+1,q]}(F)$, it follows from equation (2.3) that a simple consideration of $[p, q]$ -order implies $\rho_{[p+1,q]}(f) \geq \rho_{[p+1,q]}(F)$. By this inequality and (4.16) we obtain $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.

(ii) For $2 \leq q = p+1, \rho > 1$, by the similar proof in case (i), we can also obtain that the conclusions of case (ii) hold.

Proof of Theorem 2.2 Suppose that $f \not\equiv 0$ is a solution of the equation (1.1). From the conditions the Theorem 2.2, there is a set H of complex numbers satisfying $\log \text{dens} \{ |z| : z \in H \} > 0$, and there exists A_s ($0 \leq s \leq k-1, k \geq 2$) such that for all $z \in H$ we have (2.4) and (2.5) as $|z| \rightarrow +\infty$. Set $H_1 = \{ |z| : z \in H \}$, since $\log \text{dens} \{ |z| : z \in H \} > 0$ then H_1 is a set with $\ell m(H_1) = \infty$.

(i) Let $p \geq q \geq 1$ and $0 \leq \beta < \alpha$. Suppose that $f \not\equiv 0$ is a polynomial with $\deg f = n \geq s$, then $f^{(s)} \not\equiv 0$, implies that $\frac{f^{(j)}}{f^{(s)}} (j = 0, 1, \dots, k)$ is a rational, hence $T(r, \frac{f^{(j)}}{f^{(s)}}) = O(\log r)$ for r sufficiently large. From (4.3) we have

$$T(r, A_s) \leq \sum_{\substack{j=0 \\ j \neq s}}^{k-1} T(r, A_j) + O(\log r). \tag{4.17}$$

It follows by (4.17), (2.4) and (2.5) that

$$\exp_{p-1}(\alpha [\log_{q-1} r]^\rho) \leq O(\exp_{p-1}(\beta [\log_{q-1} r]^\rho)) \tag{4.18}$$

which is a contradiction since $\alpha > \beta$ and $r \rightarrow +\infty$. Then, every polynomial solution $f \not\equiv 0$ of (1.1) is of $\deg f \leq s-1$.

Now, suppose that f is a transcendental solution of (1.1). By using the first main theorem of Nevanlinna and properties of the characteristic function, we obtain from (4.3)

$$\begin{aligned} T(r, A_s) &\leq T(r, f^{(k)}) + kT(r, f^{(s)}) + \sum_{j=0, j \neq s}^{k-1} T(r, f^{(j)}) \\ &\quad + \sum_{j=0, j \neq s}^{k-1} T(r, A_j) + O(1). \end{aligned} \tag{4.19}$$

By Lemma 3.2, there exists a constant $R > 0$ such that for all z satisfying $|z| = r > R$, we rewrite (4.19) as follows

$$\begin{aligned} m(r, A_s) &= T(r, A_s) \leq \left(\frac{3}{2}k^2 + \frac{7}{2}k\right)T(2r, f) + \sum_{j=0, j \neq s}^{k-1} T(r, A_j) + O(1) \\ &= \left(\frac{3}{2}k^2 + \frac{7}{2}k\right)T(2r, f) + \sum_{j=0, j \neq s}^{k-1} m(r, A_j) + O(1). \end{aligned} \tag{4.20}$$

It follows by (4.20), (2.4) and (2.5) that

$$\begin{aligned} \exp_{p-1}(\alpha [\log_{q-1} r]^\rho) &\leq \left(\frac{3}{2}k^2 + \frac{7}{2}k\right)T(2r, f) \\ &\quad + (k-1) \exp_{p-1}(\beta [\log_{q-1} r]^\rho) + O(1) \end{aligned} \tag{4.21}$$

holds for all z satisfying $|z| = r \in H_1$ as $r \rightarrow +\infty$. Then, by (4.21), every transcendental solution f of equation (1.1) satisfies $\sigma_{[p,q]}(f) \geq \rho$, and by Lemma 3.5 (i), we have $\sigma_{[p+1,q]}(f) \leq \rho$. Thus, $\sigma_{[p,q]}(f) \geq \rho \geq \sigma_{[p+1,q]}(f)$.

(ii) Let $2 \leq q = p+1$ and $0 \leq (k-1)\beta < \alpha$. Suppose that $f \not\equiv 0$ is a polynomial with $\deg f = n \geq s$, then $f^{(s)} \not\equiv 0$. By the same reasoning as in the proof in case (i), it is clear that $f(z)$ is a polynomial with $\deg f \leq s-1$.

Now, suppose that f is a transcendental solution of (1.1). Then by (4.21)

$$\begin{aligned} \exp_{p-1}(\alpha [\log_p r]^\rho) &\leq \left(\frac{3}{2}k^2 + \frac{7}{2}k\right)T(2r, f) \\ &\quad + (k-1) \exp_{p-1}(\beta [\log_p r]^\rho) + O(1) \end{aligned} \tag{4.22}$$

holds for all z satisfying $|z| = r \in H_1$ as $r \rightarrow +\infty$. Then, by (4.22), every transcendental solution f of equation (1.1) satisfies $\sigma_{[p,p+1]}(f) \geq \rho$, and by Lemma 3.5 (ii), we have $\sigma_{[p+1,p+1]}(f) \leq \rho + 1$. Hence, $\rho \leq \sigma_{[p,p+1]}(f)$ and $\sigma_{[p+1,p+1]}(f) \leq \rho + 1$.

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Fuzzy Differential Superordination

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Abstract

S.S. Miller and P.T. Mocanu in (Miller & Mocanu, 2003) the notion of differential superordination as a dual concept of differential subordination (Miller & Mocanu, 2000). In (Oros & Oros, 2011) The authors define the notion of fuzzy subordination, in (Oros & Oros, 2012b) they define the notion of fuzzy differential subordination and in (Oros & Oros, Jun2012a) they determine conditions for a function to be a dominant of the fuzzy differential subordination and they also gave the best dominant. In this paper, we introduced the concept of fuzzy differential superordination and we set conditions for a function to be subordinant of fuzzy differential superordination and we also give the best subordinant.

Keywords: fuzzy set, fuzzy differential subordination, fuzzy differential superordination, fuzzy subordinant, fuzzy best subordinant, sandwich theorem.
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1. Introduction and Preliminaries

The general form of differential superordination method can be presented as follows: Let Ω and Δ be any set in C , let p be analytic in the unit disk U and let $\varphi(r, s, t; z) : C^3 \times U \rightarrow C$. The problem is to study the following implication:

$$\Omega \subset \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow \Delta \subset p(z). \quad (1.1)$$

If Δ is a simply connected domain containing the point a and $\Delta \neq C$, then there is a conformal mapping q of U onto Δ such that $q(0) = a$. In this case, relation (1.1) can be rewritten as

$$\Omega \subset \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) < p(z). \quad (1.2)$$

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If Ω is also a simply connected domain and $\Omega \neq C$, then there is conformal mapping h of U onto Ω such that $h(0) = \varphi(a, 0, 0; 0)$. If in addition, the function $\varphi(p(z), zp'(z), z^2p''(z); z)$ is univalent in U , then (1.2) can be rewritten as

$$h(z) < \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) < p(z). \tag{1.3}$$

For further details on the differential superordination method, the valuable monograph (Miller & Mocanu, 2003) can be seen.

Let U denote the unit disc of the complex plane

$$U = \{z \in C : |z| < 1\}, \bar{U} = \{z \in C : |z| \leq 1\}$$

and $H(U)$ denote the class of analytic function in U . For $a \in C$ and $n \in N$, we denote by

$$H[a, n] = \left\{ f \in H(U) : f(z) = a + a_{(n+1)}z^{(n+1)} + \dots, z \in U \right\},$$

$$A_n = \left\{ f \in H(U) : f(z) = z + a_{(n+1)}z^{(n+1)} + \dots, z \in U \right\}$$

with $A_1 = A$. Let $S = \{f \in A : f \text{ univalent in } U\}$ be the class of analytic and univalent functions in the open unit disk U , with condition $f(0) = 0, f'(0) = 1$, that is the analytic and univalent functions with the following power series development

$$f(z) = z + a_2z^2 + \dots, z \in U.$$

Denote by

$$S^* = \left\{ f \in A : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0, z \in U \right\}, \text{ the class of normalized starlike functions in } U, \text{ and}$$

$$C = \left\{ f \in A : \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) > 0, z \in U \right\}, \text{ the class of normalized convex functions in } U, \text{ and}$$

$$K = \left\{ f \in A : \operatorname{Re} \left(\frac{f'(z)}{g'(z)} \right) > 0, g(z) \in C, z \in U \right\}, \text{ the class of normalized close to convex functions in } U.$$

In order to introduce the notation of fuzzy differential superordination, we use the following definitions and lemmas:

Definition 1.1 (Miller & Mocanu, 2000) We denote by Q the set of functions q that are analytic and injective on $\bar{U} \setminus E(q)$, where $E(q) = \left\{ \zeta \in \partial U : \lim_{\zeta \rightarrow \infty} q(z) = \infty \right\}$, and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. The set $E(q)$ is called exemption set.

Definition 1.2 (Zadeh, 1965) Let X be a non-empty set. An application $F : X \rightarrow [0, 1]$ is called fuzzy subset. An alternate definition, more precise, would be the following:

A pair (A, F_A) , where

$$F_A : X \rightarrow [0, 1] \quad \text{and} \quad A = \{x \in X : 0 < F_A \leq 1\} = \operatorname{supp}(A, F_A),$$

is called fuzzy subset. The function F_A is called membership function of the fuzzy set (A, F_A) .

Definition 1.3 (Zadeh, 1965) Let two fuzzy subsets of $X, (M, F_M)$ and (N, F_N) . We say that fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x), x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \leq$

$F_N(x)$, $x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.

Definition 1.4 (Oros & Oros, 2011) Let $D \subseteq C$, $z_0 \in D$ be a fixed point, and let the functions $f, g \in H(D)$. The function f is said to be fuzzy subordinate to g , written $f <_F g$ or $f(z) <_F g(z)$ if the following conditions are satisfied:

1. $f(z_0) = g(z_0)$,
2. $F_{f(D)}(f(z)) \leq F_{g(D)}(g(z)), z \in U$.

Definition 1.5 (Oros & Oros, Jun2012a) A function $L(z, t), z \in U, t \geq 0$, is a fuzzy subordination chain if $L(\cdot, t)$ is analytic and univalent in U . For all $t \geq 0, L(z, t)$ is continuously differentiable on $[0, \infty)$ for all $z \in U$, and

$$F_{L[U \times [0, \infty)]}(L(z, t_1)) \leq F_{L[U \times [0, \infty)]}(L(z, t_2)), \quad t_1 \leq t_2.$$

Remark (Oros & Oros, 2011) Let the functions $f, g \in H(U)$ and g is an univalent function then $f < g$ if $f(0) = g(0)$ and $f(U) \subseteq g(U)$. From here if g is an univalent function, then $f <_F g$ if and only if $f < g$.

Lemma 1.6 (Miller & Mocanu, 2000) Let $q \in Q(a)$ and let $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$, be analytic in $U, q(z) \neq a$ and $n \geq 1$, if q is not subordinate to p , then there exist points $z_0 = r_0 e^i \in U$ and $\zeta_0 \in \partial U \setminus E(p)$ and $m \geq n \geq 1$ for which $q(U_{r_0}) \subset p(U)$,

1. $q(z_0) = p(\zeta_0)$,
2. $z_0 q'(z_0) = m \zeta_0 p'(\zeta_0)$, and
3. $Re \left(\frac{z_0 q''(z_0)}{q'(z_0)} + 1 \right) \geq m Re \left(\frac{\zeta_0 p''(\zeta_0)}{p'(\zeta_0)} + 1 \right)$.

Lemma 1.7 (Oros & Oros, 2012b) Let h be a convex function with $h(0) = a$, and let $\gamma \in C^*$ be a complex number with $Re(\gamma) \geq 0$. If $p \in H[a, n]$ with $p(0) = a$ and $\psi : C^2 \times U \rightarrow C$, $\psi(p(z), zp'(z)) = p(z) + \frac{1}{\gamma} zp'(z)$, is analytic in U , then

$$F_{\psi(C^2 \times U)}[p(z) + \frac{1}{\gamma} zp'(z)] \leq F_{h(U)}h(z), \text{ implies, } F_{p(U)}p(z) \leq F_{q(U)}q(z) \leq F_{h(U)}h(z), z \in U$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)t^{\frac{\gamma}{n}-1} dt.$$

The function q is convex and is the fuzzy best (a, n) dominant.

Lemma 1.8 (Oros & Oros, 2012b) Let h be starlike in U , with $h(0) = 0$. If $p \in H[0, 1] \cap Q$ is univalent in U , then $zp'(z) <_F h(z)$, implies $p(z) <_F q(z), z \in U$ where q is given by

$$q(z) = \int_0^z h(t)t^{-1} dt.$$

The function q is convex and is the fuzzy best dominant.

Lemma 1.9 (Pascu, 2006) If $L_\gamma : A \rightarrow A$ is the integral operator defined by $L_\gamma[f] = F$, given by $L_\gamma[f](z) = F(z) = \frac{\gamma+1}{z^\gamma} \int_0^z h(t)t^{\gamma-1} dt$ and $Re(\gamma) > 0$ then

1. $L_\gamma[S^*] \subset S^*$
2. $L_\gamma[K] \subset K$
3. $L_\gamma[C] \subset C$.

Lemma 1.10 (Oros & Oros, 2012b) The function $L(z, t) = a_1(t)z + a_2(t)z^2 + \dots$, with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$ is fuzzy subordination chain if and only if

$$Re \left\{ \frac{z \partial L(z, t) / \partial z}{\partial L(z, t) / \partial t} \right\} > 0, z \in U. \tag{1.4}$$

2. Main Results

Let

$$\Omega = \text{supp}(\Omega, F_\Omega) = \{z \in C : 0 < F_\Omega(z) \leq 1\},$$

$$\Delta = \text{supp}(\Delta, F_\Delta) = \{z \in C : 0 < F_\Delta(z) \leq 1\},$$

$$p(U) = \text{supp}(p(U), F_{p(U)}) = \{z \in C : 0 < F_{p(U)}(z) \leq 1\}$$

and

$$\varphi(C^3 \times U) = \text{supp}(\varphi(C^3 \times U), F_{\varphi(C^3 \times U)}) = \{\varphi(p(z), zp'(z), z^2 p''(z); z)\}$$

Definition 2.1 Let Ω be a set in C and $q \in H[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Phi_n[\Omega, q]$, consist of those functions $\varphi : C^3 \times U \rightarrow C$ that satisfy the admissibility condition:

$$F_{\varphi(C^3 \times U)}(\varphi(r, s, t); \zeta) \leq F_\Omega(z). \text{ i.e. } F_\Omega(\varphi(r, s, t; \zeta)) > 0, \tag{2.1}$$

Whenever

$$r = q(z), s = \frac{zq'(z)}{m}, Re \left(\frac{t}{s} + 1 \right) \geq \frac{1}{m} Re \left(\frac{zq''(z)}{q'(z)} + 1 \right),$$

where $\zeta \in \partial U$, $z \in U$ and $m \geq n \geq 1$. When $n = 1$ we write $\Phi_1[\Omega, q]$ as $\Phi[\Omega, q]$.

In the special case when h is an analytic mapping of U onto $\Omega \neq C$, we denote this class $\Phi_n[h(U), q]$ by $\Phi_n[h, q]$.

If $\varphi : C^2 \times U \rightarrow C$ and $q \in H[a, n]$, then the admissibility condition (2.1) reduces to

$$F_\Omega(\varphi(q(z), (zq'(z))/m; \zeta)) > 0, \text{ when } z \in U, \zeta \in \partial U \text{ and } m \geq n \geq 1$$

If $\varphi : C \times U \rightarrow C$, then the admissibility condition (2.1) reduces to $F_\Omega(\varphi(q(z); \zeta)) > 0$, when $z \in U, \zeta \in \partial U$.

Let (Ω, F_Ω) and (Δ, F_Δ) be any fuzzy sets in C , let p be an analytic function in the unit disc U and let $\varphi(r, s, t; z) : C^3 \times U \rightarrow C$. To study the following implication:

$$F_\Omega(z) \leq F_{\varphi(C^3 \times U)}(\varphi(p(z), zp'(z), z^2 p''(z); z)), \Rightarrow F_\Delta(z) \leq F_{p(U)}(p(z)). \tag{2.2}$$

There are three distinct cases to consider in analyzing this implication, which we list as the following Problems.

Problem 2.2 Given (Ω, F_Ω) and (Δ, F_Δ) any fuzzy sets in C , find condonations on the function φ so that (2.2) holds. We call such a φ an admissible function.

Problem 2.3 Given φ and (Ω, F_Ω) , find (Δ, F_Δ) so that (2.2) holds. Furthermore, find the "largest" such Δ .

Problem 2.4 Given φ and (Δ, F_Δ) , find (Ω, F_Ω) so that (2.2) holds. Furthermore find the "smallest" such Ω .

If either (Ω, F_Ω) or (Δ, F_Δ) in (2.2) is a simply connected domain. Then it may be possible to rephrase (2.2) in terms of fuzzy differential superordination. If p is univalent in U , and if (Δ, F_Δ) is simply connected domain with $\Delta \neq C$, then there is a conformal mapping q of U onto Δ such that $q(0) = p(0)$. In this case (2.2) can be rewritten as $F_\Omega(z) \leq F_{\varphi(C^3 \times U)}(\varphi(p(z), zp'(z), z^2 p''(z); z))$ implies

$$F_{q(U)}(q(z)) \leq F_{p(U)}p(z), z \in U, \text{ i.e. } q(z) <_F p(z). \tag{2.3}$$

If (Ω, F_Ω) is also a simply connected domain and $\Omega \neq C$. Then there is a conformal mapping h of U onto Ω such that $h(0) = (p(0), 0, 0; 0)$, if in addition, the function $\varphi(p(z), zp'(z), z^2 p''(z); z)$ is univalent in U , then (2.3) can be rewritten as

$$h(z) <_F \varphi(p(z), zp'(z), z^2 p''(z); z), \Rightarrow q(z) <_F p(z). \tag{2.4}$$

This implication also has meaning if h and q are analytic and not necessarily univalent.

Definition 2.5 Let $\varphi : C^3 \times U \rightarrow C$ and let h be analytic in U . If p and $\varphi(p(z), zp'(z), z^2 p''(z); z)$ are univalent in U and satisfy the (second-order) fuzzy differential superordination

$$F_{h(U)}h(z) \leq F_{\varphi(C^3 \times U)}(\varphi(p(z), zp'(z), z^2 p''(z); z))$$

.i.e.

$$h(z) <_F \varphi(p(z), zp'(z), z^2 p''(z); z),$$

then p is called a fuzzy solution of the fuzzy differential superordination. An analytic function q is called fuzzy subordination of the fuzzy differential superordination, or more simply a fuzzy subordination if $q(z) <_F p(z), z \in U$, for all p satisfying (2.4).

A univalent fuzzy subordination \tilde{q} that satisfies $q <_F \tilde{q}$ for all fuzzy subordinate q of (2.4) is said to be the fuzzy best subordinate of (2.4).

Note that the fuzzy best subordinate is unique uo to a relation of U . In the special case when the set inclusions of (2.2) can be replaced by the fuzzy superordination of (2.4) we can reinterpret the three problem referred to above as follows:

Problem 2.6 Given analytic functions h and q , find a class of admissible functions $\Phi[h, q]$ such that (2.4) holds.

Problem 2.7 Given the fuzzy differential superordination in (2.4), find a fuzzy subordination q , moreover, find the fuzzy best subordinate.

Problem 2.8 Given φ and fuzzy subordinate q , find the largest class of analytic function h such that holds. The next theorem is key results.

Theorem 2.9 Let $\varphi \in \Phi_n[\Omega, q]$ and let $q \in H[a, n]$. If $p \in Q(a)$ and $\varphi(p(z), zp'(z), z^2 p''(z); z)$ is univalent in U , then

$$F_\Omega(z) \leq F_{\varphi(C^3 \times U)}(\varphi(p(z), zp'(z), z^2 p''(z); z)), \quad z \in U \Rightarrow q(z) <_F p(z). \tag{2.5}$$

Proof. Form (2.5) and Definition (1.3). We have

$$\Omega \subset (\varphi(p(z), zp'(z), z^2 p''(z); z)). \quad (2.6)$$

Assume $q(z) \not\prec p(z)$. By Lemma (1.6), there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(p)$ and $m \geq n \geq 1$.

That satisfy

$$q(z_0) = p(\zeta_0), z_0 q'(z_0) = m \zeta_0 p'(\zeta_0) \quad \text{and} \quad \operatorname{Re} \left(\frac{z q''(z)}{q'(z)} + 1 \right) \geq m \operatorname{Re} \left(\frac{\zeta_0 p''(\zeta_0)}{p'(\zeta_0)} + 1 \right).$$

Using these condition with $r = p(\zeta_0)$, $s = \zeta_0 p'(\zeta_0)$, $t = \zeta_0^2 p''(\zeta_0)$ and $\zeta = \zeta_0$ in definition (2.1) we obtain

$$F_{\varphi(C^3 \times U)}(\varphi(p(\zeta_0), \zeta_0 p'(\zeta_0), \zeta_0^2 p''(\zeta_0); \zeta_0)) \leq F(\zeta_0) \quad (2.7)$$

Since this contradict (2.6) we must have $q(z) \prec_F p(z)$.

We next consider the special situation when h is analytic on U and $h(U) = \Omega \neq C$. In this case, the class $\Phi_n[h(U), q]$ is written as $\Phi_n[h, q]$ and the following result an immediate consequence of Theorem (2.9).

Theorem 2.10 Let $q \in H[a, n]$, let h be analytic in U and let $\varphi \in \Phi_n[h, q]$, if $p \in Q(a)$ and $\varphi(p(z), zp'(z), z^2 p''(z); z)$ is univalent in U , then

$$h(z) \prec_F \varphi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec_F p(z). \quad (2.8)$$

Theorem (2.9) and Theorem (2.10) can only used to obtain fuzzy subordinates of a fuzzy differential superordination of the form (2.6) or (2.8) the following theorem proves the existence of the fuzzy best subordinate of q for certain φ and also provides a method for finding the fuzzy best subordinant.

Theorem 2.11 Let h be analytic in U and let $\varphi : C^3 \times U \rightarrow C$ suppose that the differential equation

$$\varphi(p(z), zp'(z), z^2 p''(z); z) = h(z), \quad (2.9)$$

has a solution $q \in Q(a)$. If $\varphi \in \Phi_n[h, q]$, $p \in Q(a)$ and $\varphi(p(z), zp'(z), z^2 p''(z); z)$ is univalent in U , then

$$h(z) \prec_F \varphi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec_F p(z), \quad (2.10)$$

and q is the best subordinate.

Proof. Since $\varphi \in \Phi[h, q]$, by applying Theorem (2.10) we deduce that q is a fuzzy subordinant, of (2.10), since q also satisfies (2.9), it is also a solution of the fuzzy differential superordination (2.10) and therefore all subordinates of (2.10) will be fuzzy subordinant to q . Hence q will be the fuzzy best subordinant of (2.10). From this theorem we see that the problem of finding the fuzzy best subordinant of (2.10) essentially reduces to showing that differential equation 2.9 has a univalent solution and checking that $\varphi \in \Phi_n[h, q]$. The conclusion of the theorem can written in the symmetric form.

$$\varphi(q(z), zq'(z), z^2 q''(z); z) \prec_F \varphi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec_F p(z).$$

We can simplify Theorems (2.9), (2.10) and (2.11) for the case of first-order fuzzy differential subordination. The following results are immediately obtained by using these theorems and admissibility condition (2.1).

Theorem 2.12 Let $\Omega \subset C, q \in H[a, n], \varphi : C^2 \times U \rightarrow C$ and suppose that $F_{\varphi(C^2 \times U)}(\varphi(q(z), zp'(z); \zeta)) \leq F_{\Omega}(z)$, for $z \in U, \zeta \in \partial U$ and $0 < t \leq \frac{1}{n} < 1$, if $p \in Q(a)$ and $\varphi(p(z), zp'(z); z)$ is univalent in U , then

$$F_{\Omega}(z) \leq F_{\varphi(C^2 \times U)}(\varphi(p(z), zp'(z); z)) \Rightarrow F_{q(U)}q(z) \leq F_{p(U)}p(z) \quad \text{i.e.} \quad q(z) <_F p(z).$$

Theorem 2.13 Let h be univalent in $U, q \in H[a, n], \varphi : C^2 \times U \rightarrow C$ and suppose that $F_{\varphi(C^2 \times U)}(\varphi(q(z), zp'(z); z)) \leq F_{h(U)}h(z)$, for $z \in U, \zeta \in \partial U$ and $0 < t \leq \frac{1}{n} < 1$, if $p \in Q(a)$ and $\varphi(p(z), zp'(z); z)$ is univalent in U , then

$$h(z) <_F \varphi(p(z), zp'(z); z) \Rightarrow q(z) <_F p(z). \tag{2.11}$$

Furthermore if $\varphi(p(z), zp'(z); z) = h(z)$ has a univalent solution $q \in Q(a)$ then q is fuzzy best subordinated.

Georgia and Gheorghe (Oros & Oros, 2012b) considered the fuzzy subordination

$$F_{\psi(C^2 \times U)}[p(z) + \frac{1}{\gamma}zp'(z)] \leq F_{h_2(U)}h_2(z), \tag{2.12}$$

where h_2 is convex function in $U, h_2(0) = a, \gamma \neq 0$ and $Re(\gamma) \geq 0$. They showed if $p \in H[a, 1]$ satisfies (2.12), then

$$F_{p(U)}p(z) \leq F_{q_2(U)}q_2(z) \leq F_{h_2(U)}h_2(z), z \in U, \tag{2.13}$$

where

$$q_2(z) = \frac{1}{nz^\gamma} \int_0^z h_2(t)t^{\gamma-1} dt.$$

The function q is convex and is the fuzzy best dominant of (2.12).

We next prove an analogous result for the corresponding fuzzy differential subordination.

Theorem 2.14 Let h_1 be convex in U , with $h_1(0) = a, \gamma \neq 0$, with $Re(\gamma) \geq 0$, and $p \in H[a, 1] \cap Q$ if $p(z) + \frac{1}{\gamma}zp'(z)$ is univalent in U ,

$$h_1(z) <_F \frac{1}{\gamma}zp'(z), \tag{2.14}$$

and

$$q_1(z) = \frac{\gamma}{z^\gamma} \int_0^z h_1(t)t^{\gamma-1} dt, \tag{2.15}$$

then $q_1(z) <_F p(z)$, and the function q_1 is convex and is the fuzzy best subordinate.

Proof. If we let $\varphi : C^2 \times U \rightarrow C, \varphi(r, s) = r + \frac{1}{\gamma}s$, for $r = p(z), s = zp'(z), z \in U$, then relation (2.14) becomes

$$h_1(z) <_F \varphi(p(z), zp'(z); z).$$

The integral given by (2.15), with the exception of a different normalization $q(0) = a$ has the form

$$\begin{aligned} q_1(z) &= \frac{\gamma}{nz^\gamma} \int_0^z h(t)t^{\gamma-1} dt = \frac{\gamma}{nz^\gamma} \int_0^z (a + a_n z^n + \dots)t^{\gamma-1} dt \\ &= a + \frac{a_1}{\gamma+1} z + \dots, z \in U, \end{aligned}$$

which gives $q \in [a, 1]$, since h is convex and $Re(\gamma) \geq 0$, we deduce from (2) of Lemma (1.10) that q is convex and univalent. A simple colocation shows that q_1 also satisfies the differential equation.

$$q_1(z) + \frac{1}{\gamma} z q_1'(z) = \varphi(p(z), z p'(z)) = h_1(z). \quad (2.16)$$

Since q_1 is the univalent solution of the differential equation (2.16) associated with fuzzy differential subordination (2.14), we can prove that it is the fuzzy best subordinate of (2.14) by applying Theorem (2.13). Without loss of generality, we can assume that h_1 and q_1 are analytic and univalent on \bar{U} and $q_1'(\zeta)$ for $|\zeta| = 1$. If not, then we could replace h_1 with $h_1(\rho z)$ and q_1 with $q_1(\rho z)$, where $0 < \rho < 1$. These new function would then have the desired properties and we would prove the Theorem by using Theorem (2.14) and then letting $\varphi \rightarrow 1$ with our assumptions, to apply Theorem (2.13) only need to show that $\varphi \in \Phi[h_1, q_1]$. This is equivalent to showing that

$$\varphi_0 = \varphi(q_1(z) + tz q_1'(z)) = q_1(z) + \frac{1}{\gamma} z q_1'(z) \in h_1(U),$$

for $z \in U$ and $t \in (0, 1]$. From (2.16) we see that (2.12) is satisfied with p, h replaced by q_1, h_1 . Hence, from (2.11) we obtain

$$q_1(z) <_F p(z).$$

Since $h_1(U)$ is convex domain and $t \in (0, 1]$. We conclude that $\varphi_0 \in h_1(U)$ which proves that q_1 is the fuzzy best subordinate.

Theorem 2.15 Let $q \in H[a, 1], \varphi : C^2 \times U \rightarrow C$ and set $\varphi(q(z), z q'(z)) = h(z)$. If $L(z, t) = \varphi(q(z), tz q'(z))$ is fuzzy subordination chain and $p \in H[a, 1] \cap Q$, then

$$h(z) <_F \varphi(p(z), z p'(z)) \Rightarrow (z) <_F p(z).$$

Furthermore, if $\varphi(q(z), z q'(z)) = h(z)$, has a univalent solution $q \in Q$, then q is the fuzzy best subordinate.

Proof. Since L is a fuzzy subordination chain $L(z, t) <_F L(z, 1)$, or equivalently, $(p(z), z p'(z)) <_F h(z)$, for all $z \in U$ and $t \in (0, 1]$. Since this implies that (2.11) is satisfied, we obtain the desired conclusion by applying of this result by again considering the fuzzy differential superordination

$$F_{h(U)}(h(z)) \leq F_{\varphi(C^2 \times U)}(p(z) + \frac{1}{\gamma} z p'(z)), \quad (2.17)$$

with corresponding differential equation

$$q(z) + \frac{1}{\gamma} z q'(z) = h(z). \quad (2.18)$$

In Theorem (2.14), we assumed that h in (2.18) was convex, which implied that solution q was convex. On the other hand if we assuming that q is convex and h is defined by (2.18) and by simple calculation we have

$$Re \left\{ \frac{h'(z)}{q'(z)} \right\} = Re \left\{ \frac{(\gamma + 1)}{\gamma} + \frac{1}{\gamma} \frac{zq''(z)}{q'(z)} \right\} > 0,$$

then h is close to convex, therefore h is univalent function. By using the fuzzy subordination chain as given in Theorem (2.15) to obtain fuzzy best subordinant for (2.17).

In next theorem we introduce an example of a solution of problem (2.4),(2.8) referred to the introduction.

Theorem 2.16 Let q be convex in U and let h be defined by $h(z) = q(z) + \frac{1}{\gamma}zq'(z)$, with $Re(\gamma) > 0$. If $p \in [a, 1] \cap Q$, $p(z) + \frac{1}{\gamma}zp'(z)$ is univalent in U and

$$F_{h(U)}(h(z)) \leq F_{(C^2 \times U)}(p(z) + \frac{1}{\gamma}zp'(z))$$

then

$$q(z) \prec_F p(z),$$

where

$$q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t)t^{\gamma-1} dt.$$

The function q is the fuzzy best subordinant.

Proof . Let $L(z, t) = \varphi(q(z), tzq'(z)) = q(z) + \frac{1}{\gamma}zq'(z)$. By simple calculation, we get

$$Re \left\{ \frac{z\partial L(z, t)/\partial z}{\partial L(z, t)/\partial t} \right\} = Re \left\{ \gamma + t \frac{zq''(z)}{q'(z)} \right\},$$

since q convex function, $Re(\gamma) > 0$ and $t \in (0, 1]$, we obtain

$$Re \left\{ \frac{z\partial L(z, t)/\partial z}{\partial L(z, t)/\partial t} \right\}$$

Using lemma (1.9), we deduce that L is fuzzy subordination chain. By Theorem (2.15), we conclude that q is fuzzy subordinant of fuzzy differential superordination

$$F_{h(U)}(h(z)) \leq F_{\varphi(C^2 \times U)}(p(z) + \frac{1}{\gamma}zp'(z)),$$

Furthermore, since q is a univalent solution of (2.18), it also is the fuzzy best subordinant of

$$F_{h(U)}(h(z)) \leq F_{\varphi(C^2 \times U)}(p(z) + \frac{1}{\gamma}zp'(z)).$$

Example 2.17 . Let $h(z) = \frac{1-z}{1+z}$ and $p(z) = 1+z, z \in U$, it is clear to show that $h(0) = 1, h'(z) = \frac{-2}{(1+z)^2}$, $h''(z) = \frac{4}{(1+z)^3}$ and $p(z) + zp'(z) = 1 + 2z$. Since

$$\begin{aligned} Re \left\{ \frac{zh''(z)}{h'(z)} + 1 \right\} &= Re \left\{ \frac{1-z}{1+z} \right\} = Re \left\{ \frac{(1-r(\cos \theta + i \sin \theta))}{(1+r(\cos \theta + i \sin \theta))} \right\} \\ &= \frac{1-r^2}{1+2r \cos \theta + r^2} > 0, \end{aligned}$$

where $r = |z| < 1, \theta \in R$. Then the function h is convex in U .

We have

$$q(z) = \frac{1}{z} \int_0^z \frac{1-t}{1+t} dt = \frac{2 \ln(1+z)}{z} - 1.$$

Using Theorem (2.14), we obtain $\frac{1-z}{1+z} <_F 1 + 2z$, induce $\frac{2 \ln(1+z)}{z} - 1 <_F 1 + z, z \in U$.

In next result, we introduce fuzzy differential superordination for which the fuzzy subordinant function h is a starlike function.

Theorem 2.18 Let h be starlike in U , with $h(0) = 0$, If $p \in [0, 1] \cap Q$ and $zp'(z)$ is univalent in U , then

$$F_{h(U)}(h(z)) \leq F_{\varphi(C^2 \times U)}(zp'(z)) \Rightarrow F_{q(U)}q(z) \leq F_{p(U)}p(z), z \in U. \quad (2.19)$$

Where

$$q(z) = \int_0^z h(t)t^{-1} dt, \quad (2.20)$$

The function q is convex and is the fuzzy best subordinant.

Proof. Differentiating (2.20), we have $zq'(z) = h(z)$, if we let $\varphi : C^2 \times U \rightarrow C, \varphi(s) = s$, for $s = zp'(z), z \in U$, relation (2.19) becomes

$$F_{h(U)}h(z) \leq F_{\varphi(C^2 \times U)}(\varphi(zp'(z))),$$

the function q is the solution of $\varphi(zq'(z)) = zq'(z) = h(z)$. Since h is starlike, we deduce from Alexander's theorem that q is convex and univalent. As in the previous theorem we can assume that h and q are analytic and univalent on \bar{U} and $q'(\zeta) \neq 0$ for $|\zeta| = 1$, the conclusion of this theorem follows from Theorem (2.13), if we show that $\varphi \in \Phi[h, q]$, we get this immediately since $h(U)$ is starlike domain and

$$\varphi(tzq'(z)) = tzq'(z) = th(z) \in h(U), \quad z \in U \quad \text{and} \quad 0 < t \leq 1 \quad (2.21)$$

Form (2.21), we have

$$F_{\varphi(C^2 \times U)}(tzq'(z)) \leq F_{h(U)}(h(z))$$

Using Definition (2.1), we obtain $\varphi \in \Phi[h, q]$, and applying Theorem (2.13), we conclude that q is the fuzzy best subordinant.

Example 2.19 Let $h(z) = z + z^2, z \in U$ It is clear to show that

$$\begin{aligned} \operatorname{Re} \left\{ \frac{zh'(z)}{h(z)} \right\} &= \operatorname{Re} \left\{ 1 + \frac{z}{1+z} \right\} = \operatorname{Re} \left\{ 1 + \frac{r(\cos \theta + i \sin \theta)}{1 + r(\cos \theta + i \sin \theta)} \right\} \\ &= 1 + \frac{1 + r \cos \theta}{1 + 2r \cos \theta + r^2} > 0. \end{aligned}$$

If $p \in H[0, 1] \cap Q$ and $zp'(z)$ is univalent in U , then

$$z + z^2 <_F zp'(z) \Rightarrow z + z^2 <_F p(z).$$

In the next section, we will combine some theorems for the Sandwich theorem.

3. Sandwich theorem

We can combine Theorem (2.14) to gather with Lemma (1.7) to obtain the following fuzzy differential "sandwich theorem".

Theorem 3.1 Let h_1 and h_2 be convex in U , with $h_1(z) = h_2(z) = a$. Let $\gamma \neq 0$, with $Re(\gamma) > 0$ and let the function q_i be defined by

$$q_i = \frac{\gamma}{z^\gamma} \int_0^z h_i(t)t^{\gamma-1} dt,$$

for $i = 1, 2$. If $p \in H[a, 1] \cap Q$ and $p(z) + \frac{1}{\gamma}zp'(z)$ is univalent, then

$$h_1(z) <_F p(z) + \frac{1}{\gamma}zp'(z) <_F h_2(z) \Rightarrow q_1(z) <_F p(z) <_F q_2(z), \quad z \in U. \quad (3.1)$$

The function, q_1 and q_2 are convex and they are respectively the fuzzy best subordinant and fuzzy best dominant.

If we set $f(z) = p(z) + \frac{1}{\gamma}zp'(z)$, then (3.1), can be expressed as the following "sandwich theorem" involving fuzzy subordination preserving integral operator.

Corollary 3.2 Let h_1 and h_2 be convex in U and f be univalent function in U , with $h_1(0) = h_2(0) = f(0)$, Let $\gamma \neq 0$, with $Re(\gamma) > 0$. If

$$h_1(z) <_F f(z) <_F h_2(z),$$

then

$$\frac{\gamma}{z^\gamma} \int_0^z h_1(t)t^{\gamma-1} dt <_F \frac{\gamma}{z^\gamma} \int_0^z f(t)t^{\gamma-1} dt <_F \frac{\gamma}{z^\gamma} \int_0^z h_2(t)t^{\gamma-1} dt$$

When the middle integral is univalent. If we combine Theorem (2.18), with Lemma (1.8), we obtain the following "sandwich result".

Theorem 3.3 Let h_1 and h_2 be starlike functions in U , with $h_1(0) = h_2(0) = 0$ and let the function q_i be defined by

$$q_i = \int_0^z h_i(t)t^{-1} dt,$$

for $i = 1, 2$. If $p \in H[0, 1] \cap Q$ and $zp'(z)$ is univalent in U , then

$$h_1(z) <_F zp'(z) <_F h_2(z) \Rightarrow q_1(z) <_F p(z) <_F q_2(z), \quad z \in U.$$

The functions q_1 and q_2 are convex and they are respectively the fuzzy best subordinant and fuzzy best dominant. If we set $f(z) = zp'(z)$, then this last theorem can be expressed as the following "sandwich theorem" involving fuzzy subordination preserving integral operator.

Corollary 3.3 Let h_1 and h_2 be starlike functions in U and f be univalent in U , with $h_1(0) = h_2(0) = 0$. If

$$h_1(z) <_F f(z) <_F h_2(z), \quad z \in U,$$

then

$$\int_0^z h_1(t)t^{-1} dt <_F \int_0^z f(t)t^{-1} dt <_F \int_0^z h_2(t)t^{-1} dt <_F$$

when the middle integral is univalent.

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Evaluation of Classification Models in Machine Learning

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Abstract

We study the problem of evaluation of different classification models that are used in machine learning. The reason of the model evaluation is to find the optimal solution from various classification models generated in an iterated and complex model building process. Depending on the method of observing, there are different measures for evaluation the performance of the model. To evaluate classification models the most direct criterion that can be measured quantitatively is the classification accuracy. The main disadvantages of accuracy as a measure for evaluation are as follows: neglects the differences between the types of errors and it dependent on the distribution of class in the dataset. In this paper we discussed selection of the most appropriate measures depends on the characteristics of the problem and the various ways it can be implemented.

Keywords: accuracy, confusion matrix, costs of misclassification, F-measure, ROC graph.

2010 MSC: 68T01, 68T05.

1. Introduction

Machine learning is a field of artificial intelligence that deals with the construction of adaptive computing systems that are able to improve their performance by using information from experience. Machine learning is the discipline that studies the generalization and construction and analysis of algorithms that can generalize. But as much as the applications of machine learning were diverse, there are tasks that are repetitive. Therefore, it is possible to talk about the types of learning tasks that often occur. One of the most common tasks of learning that occurs in practice is classification. Classification is an important recognition of object types, for example whether a particular tissue represents malignant tissue or not.

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Classification is one of the most common tasks of machine learning, and is a problem of classification unknown instance in one of the pre-offered categories - classes. The important observation in classification is that target functions are discrete. In general, the class label can't be meaningfully assigned numerical or some other values. This means that the class attribute, whose value should be determined, categorical attribute.

The classification of an object is based on finding similarities with predetermined objects that are members of different classes, with the similarity of the two objects is determined by analyzing their characteristics. In classifying every object is classified into one of the classes with certain accuracy. The task is that on the characteristics of objects whose classification is known in advance, make a model by which will be performed classification of new objects (Fawcett, 2003; Marzban, 2004; Vardhan *et al.*, 2012). In problem of classification, the number of classes is known in advance and limited.

A wide range of algorithms for classification is available, each with their own strengths and weaknesses. There is no such a learning algorithm which works best with all the problems of supervised learning. Machine learning involves a large number of algorithms such as: artificial neural networks, genetic algorithms, rule induction, decision trees, statistical and pattern recognition methods, k-nearest neighbors, Naïve Bayes classifiers and discriminatory analysis.

The main objective of this paper is to discuss the various classification models that can be used in the problem of classification. This paper presents the advantages and disadvantages of these models. For this purpose we have organized the paper in the following way. In the second part of this paper we present evaluation of classification models, in the third part of the paper we present measures for the evaluation of classification models. In the last part of the paper, we discuss the results and give directions for further research.

2. Evaluation of classification models

For modeling regularity in the data there are a number of methods. Also, methods on the same set of examples for learning result in different models changing the parameters of the method. Due to the same problem and the same set of training data can produce a higher number of different models; it emphasizes the need for the evaluation of the quality model with respect to the given problem. That is why the evaluation of discovered knowledge is one of the essential components of the process of intelligent data analysis. Since this work deals with the classification problems, hereinafter will discuss the evaluation of classification models.

The task of evaluating classification models is to measure the degree to which the classification suggested using the model corresponding to the actual classification of the case. Depending on the method of observing, there are different measures for evaluation the performance of the model. Selection of the most appropriate measures shall be done depending on the characteristics of the problem and ways of its implementation.

3. Measures for the evaluation of classification models

In the evaluation of classification models basic concept is the notion of fault. If the application of the classification models in selected case leading to the prediction of a class that is different from

the actual class examples then there is an error in classification. If any mistake is equally important, then the total number of errors in the observed set can be an indicator of work a classifier.

This approach is based on accuracy as a measure for evaluating the quality of the classification model. This measure can be defined as the ratio of the number of correctly classified examples according to the total number of classified examples.

$$Accuracy = \frac{\text{number of correctly classified examples}}{\text{total number of cases}} \tag{3.1}$$

The main disadvantages of accuracy as a measure for evaluation are as follows: (1) neglects the differences between the types of errors; (2) dependent on the distribution of class in the dataset.

It is often important in practical problem solving distinguish certain types of errors. It is often the case in medicine, for example detecting the existence of disease in a patient. If system needs to classify breast tissue on malignant and benign based on mammography image, then if the system incorrectly marked diseased tissue as healthy tissue, the error is more important, because it will not notice the existence of the disease and will not apply the appropriate therapy. In case that the system recognizes healthy tissue as sick, error has less importance because it will further surgery and diagnosis to determine that the patient is not diseased.

In cases where it is necessary to distinguish more types of errors result of the classification is shown in the form of two-dimensional matrix, where each row of the matrix corresponds to one class and record number of examples where it is forecasted class, and each column of the matrix is also marked by a class and x h

+ c+aald number of examples where it is an actual class. llvvvv mI f vf we look for example classification problem with five classes, where we need to classify the emotional state of the person appearing in the video in five different emotional categories: happy, sad, angry, gentle and frightened, then we confusion matrix display as in Figure 1.

		Actual class				
		happy	sad	angry	gentle	frightened
Predicted class	happy	51	2	1	1	1
	sad	3	23	1	1	0
	angry	2	2	17	0	0
	gentle	0	1	2	9	1
	frightened	1	0	1	1	18

Figure 1. Illustration of confusion matrix for the classification problem of recognizing emotional states.

On the diagonal of the matrix is the number of correct classified examples, while other elements of the matrix indicate the number of examples that were incorrectly classified as some of the other classes. Figure 1 shows that the six examples of class *happy* wrongly classified as follows: three are classified as class *sad*, two in class *angry*, zero in class *gentle*, and one in class *frightened*. It can be concluded that the use of a confusion matrix allows better analysis of different types of errors.

The largest number of measures for evaluation of classification models related to classification problems with two classes. This is not a particular limitation for the use of these measures, given

that problems with larger number of classes can be displayed as a series of problems with two classes. Each of these measures in particular stands out one of the class as a target class, with the data set is divided into positive and negative examples of the target class. The negative examples include examples of all other classes. That is why below we consider a classification problem with two classes.

Confusion matrix in classification problem with two classes is shown in Figure 2. It can be concluded from the figure that there are possible four different results forecasts. Really positive and really negative outcomes are correct classification, while the false positive and false negative outcomes are two possible types of errors.

False positive example is a negative example class that is wrongly classified as positive and false negative is a positive example of the class who is wrongly classified as negative. In the context of our research entrance to confusion matrix have the following meanings (Kohavi & Provost, 1998):

- a is the number of correct predictions that instances are negative,
- b is the number of incorrect predictions that instances are positive,
- c is the number of incorrect predictions that instances are negative,
- d is the number of correct predictions that instances are positive.

		Predicted class	
		Negatives	Positives
Actual class	Negatives	a	b
	Positives	c	d

Figure 2. Confusion matrix in classification problem with two classes.

A few standard terms are defined in a matrix with two classes: accuracy, true positive rate, false positive rate, true negative rate, false negative rate and precision. The accuracy is the proportion of true results (both true positives and true negatives) among the total number of cases examined. Accuracy may be determined using the equation:

$$Accuracy = \frac{a + b}{a + b + c + d}. \tag{3.2}$$

True positive rate is the proportion of positive cases that are properly identified and can be calculated using equation:

$$True\ positive\ rate = \frac{d}{c + d}. \tag{3.3}$$

The false positive rate is the proportion of negative cases that were incorrectly classified as positive, and calculated with equation:

$$False\ positive\ rate = \frac{b}{a + b}. \tag{3.4}$$

The true negative rate was defined as the proportion of negatives cases which are classified correctly, and is calculated using the equation:

$$\text{True negative rate} = \frac{a}{a + b}. \quad (3.5)$$

The false negative rate is the proportion of positive cases that were incorrectly classified as negative, and are calculated using equation:

$$\text{False negative rate} = \frac{c}{c + d}. \quad (3.6)$$

Finally, precision or positive predictive value presents the fraction of predictive positive cases that are accurate, and is calculated using the equation:

$$\text{Precision} = \frac{d}{b + d}. \quad (3.7)$$

There are cases when accuracy is not adequate measures. The accuracy is determined by the equation (3.2) can't be an adequate measure of performance when the number of negative cases is much higher than the number of positive cases (Kubat *et al.*, 1998). If there are two classes and one is significantly smaller than the other, it is possible to obtain high accuracy if all instances are classified in larger class.

Suppose that there are 1000 cases, 995 negative cases and five cases which are positive. If the system classifies all of them negative, accuracy will be 99.5%, although classifier missed all positive cases. Or, for example, in tests which establish whether the patient is suffering from some disease, and the disease has only 1% of people in the population, a test should always reported that the patient has no disease would have an accuracy of 99%, but is unusable. In such cases, the accuracy as a measure of model quality is not adequate measure. In these cases the sensitivity of the classifier is an important measure and his ability to observe instances that are required, in this case ill patients.

In machine learning, most classifiers assumes equal importance of classes in terms of the number of instances and the level of importance, which means that all classes have the same significance. Standard techniques in machine learning are not successful when predicting a minority class in an unbalanced data set or when the false negatives are considered more important than false positives. In practical terms, unequal costs of inaccurate classifications are common, especially in medical diagnostics, so that the asymmetric misclassification costs must be taken into account as an important factor.

Cost-sensitive classifiers adapting models to costs of misclassification in the learning phase, with the objectives to reduce the costs of misclassification rather than to maximize the accuracy of classification. Because many practical problems of classifications have different costs associated with different types of errors, various algorithms for the evaluation of the sensitivity of classification is used.

Complementarity is one of the important characteristics of the evaluation of classification models. Using the pairs measures can be displayed specific accuracy of classification models with somewhat opposed positions. For example, by varying the parameter selected modeling techniques can be at the expense of one of the specific measures to increase the accuracy of the model

shown in another measure. This is an optimization problem in which the selection with the appropriate settings based on the one measure, maximize other measures. In some cases, the quality of the classifier needs expressed by a number, not a pair of dependent measures, which is achieved by using pairs measures. Using the pairs value of measures, one measure is fixed and is observed only second measure. Thus, for example, can be considered measures of accuracy with fixed value of the response to 20% and in this case the derived measure is called the precision of 20%.

Besides derived measure, there are measures that are not based on fixing one component of a pair of measure, for example *F-measure*, which is defined as follows:

$$F\text{-measure} = \frac{2 \times \text{response} \times \text{accuracy}}{\text{response} + \text{accuracy}}. \quad (3.8)$$

Another way to test the performance of the classifier is the ROC graph (Swets, 1988). ROC graph is the two-dimensional representation which on the X axis represents the false positive rate and the Y axis represents true positive rate. Item (0,1) is the perfect classifier: classifies all positive and all negative cases correctly. This is (0, 1), because the false positive rate is 0 (zero), a positive real rate is 1 (all). Point (0, 0) is a classifier that predicts all cases to be negative, while point (1, 1) corresponds to the classifier which provides that every case is positive. Point (1, 0) is a classifier that is incorrect for all classifications. In many cases, the classifier has a parameter which can be adjusted increasing the real positive rates at the cost of increasing false positive rates or reducing the false positive rate based on the dropping value of real positive rates.

Each setting parameters gives par value for a false positive rate and positive real rates and the number of such pairs can be used to represent the ROC curves. Nonparametric classifier is presented ROC to one point, which corresponds to the par value of the false positive rate and positive real rate.

Figure 3 shows an example of a ROC graph with two ROC curves and two ROC points marked P1 and P2. Nonparametric algorithms produce a single ROC point for a particular data set. Characteristics of ROC graph are:

- ROC curve or point is independent of the distribution of the class or the cost of errors (Kohavi & Provost, 1998).
- ROC graph contains all the information contained in the matrix of errors (Swets, 1988).
- ROC curve provides a visual tool for testing the ability of the classifier to correctly identify positive cases and negative cases that were incorrectly classified.

The area under of the one ROC curve can be used as a measure of accuracy in many applications, and it is called the measurement accuracy based on the surface (Swets, 1988).

Provost and Fawcett in 1997 (Provost & Fawcett, 1997) argued that the use of the classification accuracy of the classifier comparison is not adequate measure unless the cost classification and distribution of class unknown, but one classifier must be chosen for each situation. They propose a method of assessing the classifier using the ROC graph, imprecise costs and distribution of class.

Another way of comparing ROC points is the equation that balances accuracy with Euclidean distances from perfect classifier, i.e. from the point (0, 1) on the graph. In this way we include

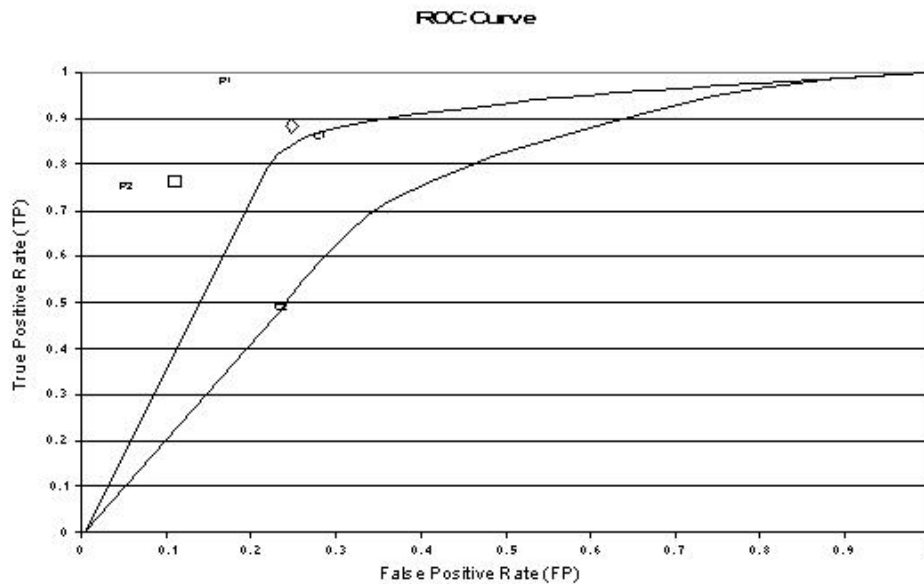


Figure 3. ROC graph

<http://www2.cs.uregina.ca/~dbd/cs831/notes/ROC/ROC.html>

weighting factors that allow us to define the relative cost of improper classification, if such data are available.

4. Conclusions

This research discusses the various classification models that can be used in the problem of classification. This research could help in future works, such as the implementation of an adequate classification model in different classification problems. There are many questions and issues that remain to be addressed and that we intend to investigate in future work. These conclusions and recommendations will be used in classification problems in the near future.

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On Threshold Voltage Variation-Tolerant Designs

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Abstract

Scaling CMOS transistors has been used to achieve *smaller, faster, and cheaper* integrated circuits. However, with CMOS transistors moving deep towards the nanometer range, the effects *threshold voltage* (V_{TH}) variations (besides other variations and noises) play on their reliabilities and that of the gates they are forming are worrying. For mitigating against this trend, *sizing* can be used to improve on the reliability of the CMOS gates. Simultaneously, sizing can also reduce power or maintain speed while only marginally affecting area. For evaluating the advantages sizing still holds, inverters of different sizings are compared in this paper with reliability enhanced inverters using well-known redundancy schemes like triple modular redundancy and hammock networks. Simulation results show that, *at the same reliability, sizing can lead to designs outperforming those obtained by the other methods on any of the design parameters (i.e., area, power or delay)*. These are reinforcing previous reports showing that *space redundancy* applied at the device-level outperform gate-level solutions.

Keywords: CMOS, sizing, reliability, redundancy, area, delay, power.
2010 MSC: 34M10, 30D35.

1. Introduction

Over half a century the semiconductor industry has relied on CMOS scaling as the basis for its growth, implementing *smaller, faster, and cheaper* integrated circuits (ICs). However, with sizes approaching *10nm*, industry is facing several fundamental limitations. One of these is the randomness of the number and locations of doping atoms (Asenov, 1998), (Asenov *et al.*, 2003), which together with imprecisions in fabrication are leading to device-to-device fluctuations/variations in key parameters, including V_{TH} .

When adding intrinsic and extrinsic noises (on top of variations), reliability looks like one of the greatest threats to the design of future ICs (SIA, 2014). The expected higher *probabilities of failures* (PFs), due to higher sensitivity to noises and variations, could make future ICs

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prohibitively unreliable. In this context, ITRS (SIA, 2014) predicted that CMOS scaling would become difficult when trying to go beyond 10nm as more "errors [will] arise from the difficulty of providing highly precise dimensional control needed to fabricate the devices and also from interference from the local environment." That is why VLSI designers should consider reliability as an extra design parameter, in addition to area, power, and delay.

The well-established approach for improving reliability is to add *redundancy* (von Neumann, 1956), (Moore & Shannon, 1956), (Winograd & Cowan, 1963), (Wakerly, 1976). Redundancy can be either in *space*, *time*, *information*, or a combination of some of these. *Space (hardware) redundancy* can be most easily understood in relation to voting and includes: modular redundancy (von Neumann, 1956), (Wakerly, 1976), (Abraham & Siewiorek, 1974), cascaded modular redundancy (Lee et al., 2007), (Hamamatsu et al., 2010), as well as multiplexing (e.g., von Neumann multiplexing (von Neumann, 1956), enhanced von Neumann multiplexing (Roy & Beiu, 2004), (Roy & Beiu, 2005), and parallel restitution (Sadek et al., 2004)). Still, voters are not necessarily needed. In fact, besides multiplexing, others schemes which do without voting include: quadded logic (Tryon, 1960), (Jensen, 1963); interwoven logic (Pierce, 1964); radial logic (Klaschka, 1967), (Klaschka, 1969); n -safe-logic (Mine & Koga, 1967), (Das & Chuang, 1972); dotted logic (Freeman & Metze, 1972); as well as solutions at the device/transistor level. *Time redundancy* is trading space for time (e.g., alternating logic, re-computing with shifted operands or with swapped operands, etc.), while *information redundancy* is based on error detection and error correction codes.

The focus of this paper is on space redundancy. Space redundancy can be applied at the system-, module-, gate-, or device-level. Applying space redundancy at the device-level is much more efficient than applying it at higher levels (as explained in (Moore & Shannon, 1956); see also (Beiu & Ibrahim, 2011)), while the common expectation is that spatial redundancy should always degrade performances, i.e., increase *area*, *power*, and *delay*. In this paper we will show that redundancy applied at the device-level can improve redundancy without increasing *area*, and even while reducing *power* or *delay*.

Sizing has already been suggested as a way to enhance tolerance to variations (Sulieman et al., 2010), (Ibrahim et al., 2011), (Keller et al., 2011), (Ibrahim & Beiu, 2011). In fact, sizing gives the VLSI designer options for optimizing the trade-offs between reliability and *area-power-delay*, while, in particular, it can enhance reliability and reduce power within the same area. For getting a better understanding of the advantages sizing can bring to reliability, the performances of differently sized inverters will be weighted against those obtained by using reliability improvement schemes including triple modular redundancy (TMR) and four-transistor hammock networks (H_{22}). The paper is organized as follows. The effect sizing plays on tolerating V_{TH} variations is discussed in Section 2. A brief review of space redundancy methods is presented in Section 3. Sizing is revisited in Section 4, followed by simulation results in Section 5 and concluding remarks in Section 6.

2. How Sizing Affects Variations

VLSI designers have normally adjusted the sizing of nMOS and pMOS transistors (i.e., width W and length L) in order to balance the driving currents when either the pMOS (I_{pMOS}) or the

nMOS (I_{nMOS}) stacks are switched ON. This requires the balancing of the ON resistances of the pMOS and nMOS stacks (R_{pMOS} , R_{nMOS}), which is achieved by adjusting (sizing) the transistors because pMOS conduction relies on holes which have slower mobility than electrons. Fig. 1 shows four different sizing options for a transistor. Although all of them have the same area $W \times L = 6a^2$, they have different ON resistances. In case of Fig. 1(a) there are 6 squares (a^2) connected in parallel, therefore $R_{ON} = R_{\square}/6$ (where R_{\square} is the resistance of a square, e.g., $W = a$, $L = a$). In case of Fig. 1(d), the six squares are connected in series, hence $R_{ON} = 6R_{\square}$. Similarly, R_{ON} for Fig. 1(b) and 1(c) can be estimated as $3R_{\square}/2$ and $2R_{\square}/3$.

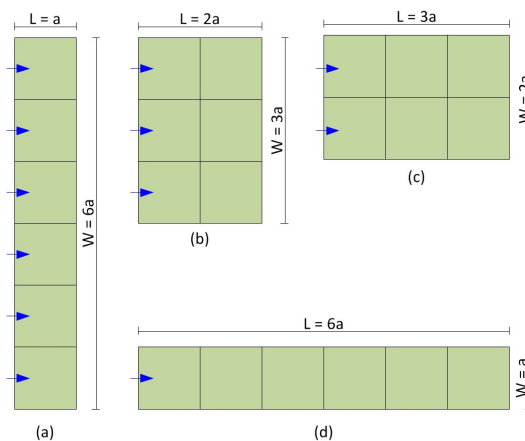


Figure 1. Four different sizing options having the same area ($A = 6a^2$).

With CMOS scaling approaching $10nm$, it becomes difficult to reproduce V_{TH} over the large number of transistors in a chip. This is due to the random fluctuations of both the number of dopants and of their physical locations. V_{TH} variations can be approximated (see (Asenov et al., 2003)) by a normal distribution with standard deviation:

$$\sigma_{V_{TH}} \approx 3.19 \times 10^{-8} t_{ox} N_A^{0.4} (L_{eff} \times W_{eff})^{-0.5} [V] \tag{2.1}$$

where t_{ox} is the oxide thickness, N_A is the channel doping, W_{eff} is the effective channel width, and L_{eff} is the effective channel length. In the following we will use normalized dimensions for L and W .

Eq.(2.1) shows that increasing the transistors area (by increasing L and/or W) will always reduce V_{TH} variations. While the four sizing options in Fig. 1 are expected to exhibit similar probabilities of switching (meaning that the transistor fails to open/close, see (Beiu & Ibrahim, 2011), (Ibrahim & Beiu, 2011), (Ibrahim et al., 2012)) as they have the same area, they will lead to very different performances, as their R_{ON} is between $R_{\square}/6$ and $6R_{\square}$.

For classical sizing VLSI designers set $L_{nMOS} = L_{pMOS} = min$ and $W_{nMOS} = 2 \times L_{nMOS}$ (i.e., $R_{nMOS} = R_{\square}/2$). To balance I_{pMOS} and I_{nMOS} , W_{pMOS} is then increased, such as R_{pMOS} matches R_{nMOS} . This also increases the area of the pMOS ($A_{pMOS} = L_{pMOS} \times W_{pMOS}$), improving their reliability. Fig. 2(a) shows that a pMOS transistor is more reliable than an nMOS. As classical sizing increases the area of the pMOS transistors it makes them even more reliable than nMOS transistors.

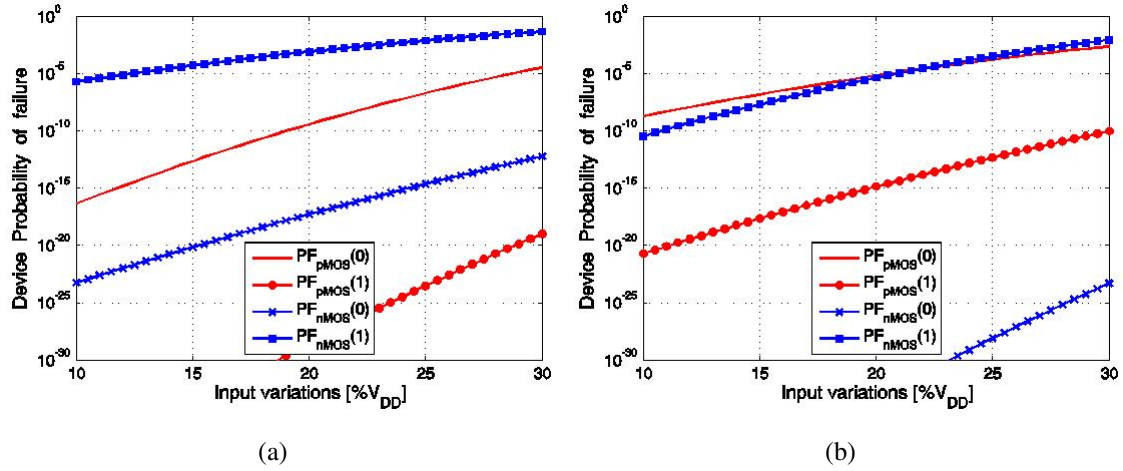


Figure 2. PF_{TRS} with respect to variations: (a) classical sized ($L_{nMOS} = L_{pMOS} = 1$, $W_{nMOS} = 2$, and $W_{pMOS} = 4$); (b) reverse sized ($W_{nMOS} = W_{pMOS} = 1$, $L_{nMOS} = 4$, and $L_{pMOS} = 2$).

For enhancing PF_{GATE} , it is essential to improve the reliability of the nMOS stack, ideally matching the reliability of the pMOS stack (similar to matching R_{pMOS} to R_{nMOS}). For doing this A_{nMOS} should be enlarged (see eq. (2.1)). Classical sizing is using $W/L > 1$ and $L = \min$, so it follows that W_{nMOS} has to be increased. Subsequently, this requires increasing W_{pMOS} (to compensate for the slower mobility of the holes). Hence, relying on classical sizing W_{nMOS} has to be increased, which leads to enlarging all transistors and degrading the gates *area*, *delay*, and *power* consumption.

3. Space Redundancy

Classical space redundancy schemes start from an unreliable system and use divide-and-conquer in a top-down fashion as follows. The unreliable system is divided into several sub-systems which are interconnected by a *network*. Each sub-system is further divided into several sub-sub-systems, which are also interconnected by a network. This continues down to the elementary transistors, and the level where redundancy will be applied has to be decided. Four levels are well-established: *system*, *module*, *gate*, and *device*. Redundancy can be applied simultaneously at more than one level even using different schemes at different levels. This implies that the optimization space is very large. Fundamentally, using space redundancy at any level translates into replicating all of the sub-systems at that level by a *redundancy factor* R . This R -times larger redundant system needs to be connected by a modified network. Most space redundancy methods are done at this point, while some space redundancy methods require additional blocks (e.g., voters) for connecting the original sub-systems. In the following we shall briefly review space redundancy methods by classifying them with respect to the need for voters, while also suggesting how complex is the connectivity pattern (modified network) they use.

3.1. Higher Level Methods

The most well-known high level redundancy methods are *triple modular redundancy* (TMR) and *n-modular redundancy* (NMR). TMR was proposed by von Neumann [4]. It divides a system into modules (sub-systems) and triplicates each module (Fig. 3). A voter is used to combine the outputs of the $R = 3$ modules operating in parallel (Lyons & Vanderkulk, 1962), (Gurzi, 1965), (Longden et al., 1966), (Wakerly, 1975), (Stroud, 1994), Morgan et al. (2007). TMR is able to mask failures that affect one module by taking the majority of three modules. The interconnectivity pattern is simple (Fig. 3(a)), while it might get slightly more complex if more voters are used in parallel (Fig. 3(b)).

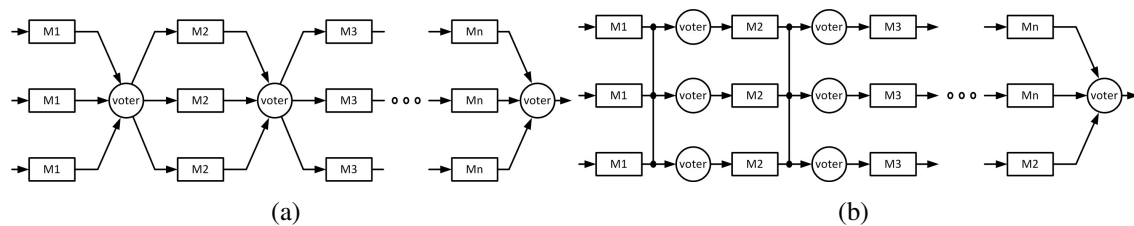


Figure 3. Triple modular redundancy: (a) one voter per stage; (b) three voters per stage.

NMR is an extension of TMR to any odd number n . It requires replicating all the modules n times ($R = n$), and also using larger voters (with n inputs), but it could tolerate $n/2$ module failures (Ness et al., 2007). The connectivity pattern gets more complex and the length of the wires increases as n is increased, and if more voters are being used in parallel. The early analyses have assumed that voters are very reliable. Later it was realized that even assuming that the reliability of a voter is independent of the number of inputs n is unrealistic, and could lead to wrong conclusions. This has motivated research into space redundancy methods which could do without voters.

A gate-level redundancy method without voting was also introduced by von Neumann in [4], and is known as *multiplexing*. Other gate-level space redundancy methods without voting are *quadded* (Tryon, 1960), (Jensen, 1963), *interwoven* (Pierce, 1964), *radial* (Klaschka, 1967), (Klaschka, 1969) and *n-fail-safe* (Mine & Koga, 1967), (Das & Chuang, 1972) logic. All of these exhibit simpler connectivity patterns than multiplexing, as being more regular and local, i.e., having shorter wires. Another gate-level method which does not require voting is *dotted logic* (Freeman & Metze, 1972), which is bridging the gap between gate- and device-level methods. The reason is that dotted logic took advantage of implementations which use wired AND and OR functions. This is not entirely gate-level anymore, but it is not yet device-level either.

3.2. Device-Level Methods

Device-level methods have also been introduced in a seminal article (Moore & Shannon, 1956) (relays in the original paper). The main conclusions of that work have been that:

- redundant relay (device-level) structures are able to outperform redundant gate-level schemes at significantly (orders of magnitude) smaller R ; and that
- the modified networks (there are different ways to connect the redundant relays) have a strong influence on reliability.

All the subsequent publications inspired by the original study of Moore and Shannon (Moore & Shannon, 1956) have detailed particular applications of those ideas. They rely on series-and-parallel networks of (a few) devices. The most widespread network used is a series-parallel network of 4 devices/relays/transistors which is the simplest hammock network (Moore & Shannon, 1956). This has been named a *quad configuration* by many of the later papers Suran (1964), Bolchini et al. (1996), Abid & El-Razouk (2006), Anghel & Nicolaidis (2007), El-Maleh et al. (2008). Here we shall use hammock network (hence the H abbreviation) as we do not want to create any confusion with respect to gate-level *quadded logic* (Tryon, 1960), (Jensen, 1963). A few papers have looked at simpler hammock networks (Djupdal & Haddow, 2007), or at hammock networks having more than four transistors (Anghel & Nicolaidis, 2007), (Aunet et al., 2005). It looks like this trend will be taking up due to developments on carbon nano tubes (Zarkesh-Ha & Shahi, 2010), (Zarkesh-Ha & Shahi, 2011).

4. Transistor Sizing Revisited

While sizing has been used for a very long time to balance driving currents, its use for enhancing reliability has only recently started to be explored for $W/L > 1$ (classical sizing) (Keller et al., 2011). Still, a *reverse sizing* ($W/L < 1$) has been proposed in (Suliman et al., 2010) for overcoming the problems mentioned in Section 2. This sizing method keeps all W minimum ($W_{nMOS} = W_{pMOS} = \min$), and increases L . Normally, this is not used for digital circuits, but has been used in analog circuits as "better matching can be obtained without consuming additional area, simply by changing the W/L aspect ratio" (Drennan & McAndrew, 2003). To make A_{nMOS} larger than A_{pMOS} , L_{pMOS} should be kept small ($L_{pMOS} = 2W_{pMOS}$), while L_{nMOS} should be increased. While occupying the same area, the reverse sizing method enhances the gates reliability but diminishes its performances.

This is because increasing L increases R_{ON} and hence the delay, but power is reduced as I_{ON} is reduced. Fig. 2(b) shows PF_{TRS} for reverse sizing. Increasing the area of the nMOS transistors improves their reliability and (more importantly) allows matching the reliability of the pMOS transistors (see $PF_{nMOS}(1)$ and $PF_{pMOS}(0)$ in Fig. 2(b)).

Aiming to simultaneously optimize *reliability* and *power-delay-area*, an exhaustive sizing search was suggested in (Ibrahim et al., 2011). Instead of using L_{min} (classical sizing) or W_{min} (reverse sizing), different sizings are obtained by analyzing all the possible A_{nMOS} and A_{pMOS} combinations, lower than a maximum area A_{max} , and achieving a PF_{GATE} lower than a target PF_{target} . This method is exhaustive as iterating through all the possible nMOS area combinations from $W_{nMOS} \times L_{nMOS} = 1 \times A_{max}$ to $A_{max} \times 1$. For each nMOS sizing combination, the algorithm tries to find all the corresponding pMOS sizing combinations ($W_{pMOS} \times L_{pMOS}$) such that R_{pMOS} matches R_{nMOS} and $A_{pMOS} \leq A_{max}$. If a pMOS sizing combination is found, Gate Reliability EDA (GREDA) (Ibrahim et al., 2012) is used to quickly and accurately estimate PF_{GATE} .

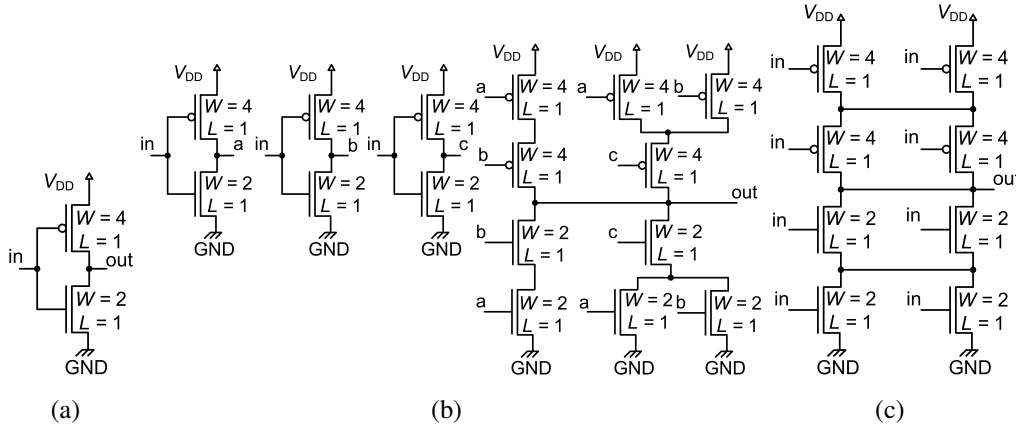


Figure 4. (a) Classical INV; (b) INV-TMR with a MIN-3 as voter; (c) H_{22} -INV.

If $PF_{GATE} \leq PF_{target}$, the method stores the identified nMOS and pMOS sizing combination in a list of candidate combinations. Finally, the method checks the list of candidate combinations. If the list is empty it means that PF_{GATE} cannot achieve PF_{target} with transistors of up to A_{max} . Otherwise, the design process is continued by using Spice to estimate the *delay*, *power*, and *power-delay-product (PDP)* for each candidate combination. The best combination that optimizes *delay*, *power* or *PDP* can then be selected. In all cases the *reliability* and the *area* constraints are always going to be satisfied.

5. Simulation Result

We have used 16nm PTM v2.1 incorporating high-*k/metal* gates and stress effects (Zhao & Cao, 2007), (PTM, 2011), as this is strongly affected by variations, and simulated at $V_{DD} = 700mV$ (nominal voltage) and $T = 27^\circ C$. The TMR-INV circuit has three INVs followed by a mirrored MIN-3 gate as voter. This MIN-3 implementation was preferred as it is considered the most reliable one (Suliman, 2009). All the transistors for TMR-INV (Fig. 4(b)) and H_{22} -INV (Fig. 4(c)) were sized using classical sizing, and the mobility of the electrons was assumed to be twice the mobility of the holes.

5.1. Reliability Results

In the first set of simulations GREDA was used to calculate the reliability of INVs with transistors of different sizings as well as TMR-INV (Fig. 4(b)) and H_{22} -INV (Fig. 4(c)). For all these simulations the input variations were assumed to be 15%, i.e., logic "1" = $0.85V_{DD} = 595mV$ and logic "0" = $0.15V_{DD} = 105mV$.

In the case of a classical INV, the simulation results show $PF_{INV}(0) = 7.25E - 21$ and $PF_{INV}(1) = 5.33E - 05$. The large difference between these values is due to $PF_{INV}(1)$ being dominated by PF_{nMOS} . For a reverse sized INV the simulation results show that increasing the area of the nMOS by increasing L_{nMOS} reduces $PF_{INV}(1)$ to $1.58E - 07$ (i.e., 2 orders of magnitude better than $PF_{INV}(1)$ for classical sizing).

For TMR-INV the simulations show $PF_{TMR-INV}(1) = 8.51E - 09$ (4 orders of magnitude better than classical) and $PF_{TMR-INV}(0) = 5.67E - 09$, which although 12 orders of magnitude worse than classical, is balanced with respect to $PF_{TMR-INV}(1)$. This is due to the fact that the output of the INV gate ($PF_{INV}(0) = 7.25E - 21$) is a logic "1" input for the MIN-3 gate, being significantly degraded (to $5.67E - 09$) as affected by $PF_{MIN-3}(1)$, which is determined by PF_{nMOS} .

For H_{22} -INV we have seen $PF_{H_{22}-INV}$ being improved significantly for both logic "0" and logic "1": from $PF_{INV}(0) = 7.25E - 21$ and $PF_{INV}(1) = 5.33E - 05$ (classical INV) to $PF_{H_{22}-INV}(0) = 2.10E - 40$ and $PF_{H_{22}-INV}(1) = 5.67E - 09$ respectively.

For a fair comparison of the performances of sizing versus the other space redundancy methods considered, the PF_{target} was set to $1.0E - 09$ (range achieved by TMR-INV and H_{22} -INV). The maximum transistor area was limited to $A_{max} = 10a^2$. Table 1 shows the seven different sizing combinations (with W/L aspect ratios above and below 1) satisfying both of these requirements and also matching R_{nMOS} to R_{pMOS} . All of them achieve reliabilities of the order $1E - 10$.

5.2. Performance Results

The second set of simulations has used Spice to estimate the performances of the different solutions. These are reported in Table 1, starting with the classically sized INV having an average delay of 5.54ps and an average power consumption of 0.24 μ W.

Table 1. Reliability enhanced INV circuits as well as differently sized INVs.

	A_{nMOS} ($W \times L$)	A_{pMOS} ($W \times L$)	Area ($\sum A_{trns}$)	Worst PF_{INV}	Delay [ps]	Power [μ W]	PDP [aJ]
Classical	2 \times 1	4 \times 1	6	5.33E-05	5.54	0.24	1.34
Reversed	1 \times 4	1 \times 2	6	1.58E-07	30.54	0.06	1.70
TMR	2 \times 1	4 \times 1	48	8.51E-09	20.01	3.55	71.03
H_{22}	2 \times 1	4 \times 1	24	5.67E-09	35.71	0.79	28.34
	1 \times 5	1 \times 3	8	4.47E-10	55.12	0.06	3.49
	1 \times 6	1 \times 3	9	1.89E-10	62.65	0.07	4.20
	3 \times 2	3 \times 1	9	1.89E-10	13.34	0.25	3.39
This paper	1 \times 7	1 \times 3	10	1.89E-10	70.71	0.07	5.01
	5 \times 1	9 \times 1	14	4.47E-10	5.45	0.64	3.49
	5 \times 1	10 \times 1	15	4.47E-10	5.62	0.69	3.89
	1 \times 5	2 \times 5	15	4.47E-10	153.04	0.15	22.57

Table 1 clearly shows that adding more gates (TMR-INV) or adding more transistors (H_{22} -INV), while improving the reliability over the classical INV by 4 orders of magnitude (from $1E - 5$ to $1E - 9$), significantly degrades both power and delay: TMR-INV increases the average delay by 3.6 \times , while the average power and PDP are increased by 14.8 \times and 53 \times respectively; H_{22} -INV is about 6.5 \times slower while consuming about 3.3 \times more power and having a 21 \times higher PDP.

The reverse sized INV improves redundancy by 2 orders of magnitude while also reducing power by 4 \times , but degrades delay and PDP by 5.5 \times and 1.3 \times respectively. Among other possible sizings, [3 \times 2, 3 \times 1] improves PF_{INV} by more than 5 orders of magnitude (over the classical [2 \times 1, 4 \times 1] sizing) at about the same power, while increasing delay and PDP by only 2.4 \times . For high-performance applications, one should select [5 \times 1, 9 \times 1] which improves reliability by 5 orders of magnitude while being as fast as a classical INV (in fact it is a shy 2% faster), and

consumes about $2.7\times$ more power. Alternatively, $[1 \times 5, 1 \times 3]$ could be selected for low-power applications, with reliability being improved by 5 orders of magnitude, and power being reduced $4\times$, while delay is increased $10 \times$.

6. Conclusions

This paper has compared the performances of different sized inverters with classical and reverse sized inverters, as well as with two redundancy methods at the gate-level (TMR) and device-level (H_{22}) (Mukherjee & Dhar, 2015) (Sheikh et al., 2016), (Robinett et al., 2007), (El-Maleh et al., 2009). The main conclusions are:

- Sizing can outperform both TMR and H_{22} methods with respect to reliability.
- Improving the reliability of a CMOS gate can be achieved without increasing *area*.
- Improving the reliability of a CMOS gate should not necessarily lead to penalties in *power* or *delay*, or even on the contrary, i.e., there are reliability enhanced solutions which can achieve either lower *power* or shorter *delays* but not both.

Sizing can be used to improve tolerance to variations, and it is possible to design CMOS gates trading reliability versus *area-power-delay*. The disadvantages are represented by very large libraries of gates and a much more complex design.

Future work will analyze re-sized solutions for other CMOS gates (e.g., NAND, NOR, XOR, etc.), of different *fan-ins* (see (Gemmeke & Ashouei, 2012), (Gemmeke et al., 2013)). These should be compared not only with classical sized gates, TMR, and H_{22} , but also with quadded, interwoven, radial, *n*-safe, and dotted logic solutions, and evaluated jointly with advanced CMOS (Berge & Aunet, 2009) (Liu & Moroz, 2007), (Maly, 2007), (Geppert, 2002), and even beyond-CMOS technologies (Courtland, 2016), (Desai et al., 2016).

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Matrix Representations of Fuzzy Quaternion Numbers

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Abstract

In this paper, firstly we discuss basic arithmetic operations with fuzzy quaternion numbers. Then, we introduce a noncommutative field, formed with 2×2 fuzzy complex matrices which is used for the matrix representation of fuzzy quaternion numbers as elements within this field. Finally, another way of representing fuzzy quaternion numbers is obtained by using 4×4 fuzzy real matrices.

Keywords: fuzzy complex numbers, fuzzy quaternion numbers.
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1. Introduction

The fuzzy quaternion numbers were defined in many ways by many authors. For example, in paper (Moura *et al.*, 2013) the authors proposed an extension for the set of fuzzy real numbers to the set of fuzzy quaternion numbers, defining the notion of fuzzy quaternion numbers by an application $h' : H \rightarrow [0, 1]$ such that $h'(a + bi + cj + dk) = \min\{\widetilde{A}(a), \widetilde{B}(b), \widetilde{C}(c), \widetilde{D}(d)\}$ for some $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ real fuzzy numbers. In paper (Moura *et al.*, 2014), the authors defined the fuzzy quaternion numbers using triangular fuzzy numbers.

Recently, in paper (Sida *et al.*, 2016) a new approach is proposed in order to introduce the fuzzy quaternion numbers concept, similar to the way that Fu and Shen (see (Fu & Shen, 2011)) have introduced the fuzzy complex numbers.

The study of fuzzy quaternion numbers is continued in this paper. More precisely, another approach for multiplication and division operation is presented. Then, following the ideas in (Moş & Popa, 2014), we will represent fuzzy quaternion numbers as a 2×2 fuzzy complex matrices, as well as 4×4 fuzzy real matrices.

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2. Preliminaries

For the concept of fuzzy real numbers and their arithmetic operations we make reference to the following papers: (Das & Mandal, 2002), (Dubois & Prade, 1980), (Dzitac, 2015), (Felbin, 1992), (Janfada et al., 2011), (Kaleva & Seikkala, 1984), (Mizumoto & Tanaka, 1979), (Xiao & Zhu, 2002).

Definition 2.1. (Dzitac, 2015) A fuzzy set in \mathbb{R} , namely a mapping

$$\tilde{A} : \mathbb{R} \rightarrow [0, 1],$$

with the following properties:

- (i) \tilde{A} is convex, i.e. $\tilde{A}(y) \geq \min\{\tilde{A}(x), \tilde{A}(z)\}$, for $x \leq y \leq z$;
- (ii) \tilde{A} is normal, i.e. $(\exists)x_0 \in \mathbb{R}; \tilde{A}(x_0) = 1$;
- (iii) \tilde{A} is upper semicontinuous, i.e.

$$(\forall)x \in \mathbb{R}, (\forall)\alpha \in (0, 1] : \tilde{A}(x) < \alpha,$$

$$(\exists)\delta > 0 \text{ such that } |y - x| < \delta \Rightarrow \tilde{A}(y) < \alpha$$

is called a fuzzy real number.

We will denote by \mathbb{R}_F - the set of all fuzzy real numbers.

Definition 2.2. (Mizumoto & Tanaka, 1979) The basic arithmetic operations $+$, $-$, \cdot , $/$ on \mathbb{R}_F are defined by:

1. *Addition:*

$$(\tilde{A} + \tilde{B})(x) = \bigvee_{y \in \mathbb{R}} \min\{\tilde{A}(y), \tilde{B}(x - y)\}, (\forall)x \in \mathbb{R} \quad (2.1)$$

2. *Subtraction:*

$$(\tilde{A} - \tilde{B})(x) = \bigvee_{y \in \mathbb{R}} \min\{\tilde{A}(y), \tilde{B}(y - x)\}, (\forall)x \in \mathbb{R} \quad (2.2)$$

3. *Multiplication:*

$$(\tilde{A} \cdot \tilde{B})(x) = \bigvee_{y \in \mathbb{R}^*} \min\{\tilde{A}(y), \tilde{B}(x/y)\}, (\forall)x \in \mathbb{R} \quad (2.3)$$

4. *Division:*

$$(\tilde{A}/\tilde{B})(x) = \bigvee_{y \in \mathbb{R}} \min\{\tilde{A}(x \cdot y), \tilde{B}(y)\}, (\forall)x \in \mathbb{R}. \quad (2.4)$$

Remark. A triangular fuzzy number is defined by its membership function

$$x(t) = \begin{cases} 0, & \text{if } t < a_1 \\ \frac{t-a_1}{a_2-a_1}, & \text{if } a_1 \leq t < a_2 \\ \frac{a_3-t}{a_3-a_2}, & \text{if } a_2 \leq t < a_3 \\ 0, & \text{if } t > a_3 \end{cases}, \text{ where } a_1 \leq a_2 \leq a_3, \quad (2.5)$$

and it is denoted $\tilde{x} = (a_1, a_2, a_3)$.

Remark. (Chang & Wang, 2009; Elomda & Hefny, 2013; Hanss, 2005; Nădăban et al., 2016) Let $\tilde{x} = (a_1, a_2, a_3)$, $\tilde{y} = (b_1, b_2, b_3)$ be two non negative triangular fuzzy numbers and $\alpha \in \mathbb{R}_+$. According to the extension principle, the arithmetic operations are defined as follows:

1. $\tilde{x} + \tilde{y} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
2. $\tilde{x} - \tilde{y} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$
3. $\alpha\tilde{x} = (\alpha a_1, \alpha a_2, \alpha a_3)$
4. $\tilde{x}^{-1} = (1/a_3, 1/a_2, 1/a_1)$
5. $\tilde{x} \times \tilde{y} \cong (a_1 b_1, a_2 b_2, a_3 b_3)$
6. $\tilde{x}/\tilde{y} \cong (a_1/b_3, a_2/b_2, a_3/b_1)$

We denote that the results of (4) – (6) are not triangular fuzzy numbers, but they can be approximated by triangular fuzzy numbers.

Definition 2.3. (Fu & Shen, 2011) An fuzzy complex number, \tilde{z} , is defined in the form of:

$$\tilde{z} = \tilde{A} + i\tilde{B}, \tag{2.6}$$

where $\tilde{A}, \tilde{B} \in \mathbb{R}_F$; \tilde{A} is the real part of \tilde{z} while \tilde{B} represents the imaginary part, i.e. $Re(\tilde{z}) = \tilde{A}$ and $Im(\tilde{z}) = \tilde{B}$.

We will denote by \mathbb{C}_F - the set of all fuzzy complex numbers. The operations on \mathbb{C}_F are a straightforward extension of those on real complex numbers.

Definition 2.4. (Fu & Shen, 2011) Let $\tilde{z} = \tilde{A} + i\tilde{B}$, $\tilde{z}_2 = \tilde{C} + i\tilde{D} \in \mathbb{C}_F$ where $\tilde{A}, \tilde{B}, \tilde{C}$ and \tilde{D} are fuzzy real numbers. The basic arithmetic operations are defined as follows:

1. *Addition:*

$$\tilde{z}_1 + \tilde{z}_2 = (\tilde{A} + \tilde{C}) + i(\tilde{B} + \tilde{D}), \tag{2.7}$$

where

$$(\tilde{A} + \tilde{C})(y) = \bigvee_{y=x_1+x_2} (\tilde{A}(x_1) \wedge \tilde{C}(x_2))$$

$$(\tilde{B} + \tilde{D})(y) = \bigvee_{y=x_1+x_2} (\tilde{B}(x_1) \wedge \tilde{D}(x_2))$$

2. *Subtraction:*

$$\tilde{z}_1 - \tilde{z}_2 = (\tilde{A} - \tilde{C}) + i(\tilde{B} - \tilde{D}), \tag{2.8}$$

where

$$(\tilde{A} - \tilde{C})(y) = \bigvee_{y=x_1-x_2} (\tilde{A}(x_1) \wedge \tilde{C}(x_2))$$

$$(\tilde{B} - \tilde{D})(y) = \bigvee_{y=x_1-x_2} (\tilde{B}(x_1) \wedge \tilde{D}(x_2))$$

3. *Multiplication:*

$$\tilde{z}_1 \times \tilde{z}_2 = (\widetilde{AC} - \widetilde{BD}) + i(\widetilde{BC} + \widetilde{AD}), \tag{2.9}$$

where

$$\begin{aligned} (\widetilde{AC} - \widetilde{BD})(y) &= \bigvee_{y=x_1x_2-x_3x_4} (\widetilde{A}(x_1) \wedge \widetilde{C}(x_2) \wedge \widetilde{B}(x_3) \wedge \widetilde{D}(x_4)) \\ (\widetilde{BC} + \widetilde{AD})(y) &= \bigvee_{y=x_1x_2+x_3x_4} (\widetilde{B}(x_1) \wedge \widetilde{C}(x_2) \wedge \widetilde{A}(x_3) \wedge \widetilde{D}(x_4)) \end{aligned}$$

4. *Division:*

$$\tilde{z}_1/\tilde{z}_2 = \left(\frac{\widetilde{AC} + \widetilde{BD}}{\widetilde{C}^2 + \widetilde{D}^2} \right) + i \left(\frac{\widetilde{BC} - \widetilde{AD}}{\widetilde{C}^2 + \widetilde{D}^2} \right), \tag{2.10}$$

where $\frac{\widetilde{AC} + \widetilde{BD}}{\widetilde{C}^2 + \widetilde{D}^2} = \tilde{t}_1$ and $\frac{\widetilde{BC} - \widetilde{AD}}{\widetilde{C}^2 + \widetilde{D}^2} = \tilde{t}_2$ are fuzzy real numbers:

$$\begin{aligned} \tilde{t}_1(y) &= \bigvee_{y=\frac{x_1x_3+x_2x_4}{x_3^2+x_4^2}, x_3^2+x_4^2 \neq 0} (\widetilde{A}(x_1) \wedge \widetilde{B}(x_2) \wedge \widetilde{C}(x_3) \wedge \widetilde{D}(x_4)) \\ \tilde{t}_2(y) &= \bigvee_{y=\frac{x_2x_3-x_1x_4}{x_3^2+x_4^2}, x_3^2+x_4^2 \neq 0} (\widetilde{A}(x_1) \wedge \widetilde{B}(x_2) \wedge \widetilde{C}(x_3) \wedge \widetilde{D}(x_4)) \end{aligned}$$

Remark. Fuzzy complex number $\tilde{z} = \widetilde{A} + i\widetilde{B}$ admits a matrix representation, namely:

$$\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ -\widetilde{B} & \widetilde{A} \end{pmatrix},$$

where $\widetilde{A}, \widetilde{B} \in \mathbb{R}_F$.

3. On arithmetic operation with fuzzy quaternion numbers

In paper (Sida et al., 2016) it was introduced fuzzy quaternion number as well as basic arithmetic operations with fuzzy quaternion numbers.

Definition 3.1. A fuzzy quaternion number is an element of the form

$$\tilde{q} = \widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k, \tag{3.1}$$

where $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D} \in \mathbb{R}_F$ and $i^2 = j^2 = k^2 = -1$; $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$.

We will denote by \mathbb{H}_F - the set of all fuzzy quaternion numbers.

Remark. \widetilde{A} is called the real part of \tilde{q} and sometimes denoted $\widetilde{A} = Re(\tilde{q})$, and $\widetilde{B}, \widetilde{C}, \widetilde{D}$, are called imaginary parts of \tilde{q} and denoted $\widetilde{B} = Im_1(\tilde{q}), \widetilde{C} = Im_2(\tilde{q}), \widetilde{D} = Im_3(\tilde{q})$.

Definition 3.2. If $\tilde{q}_1 = \widetilde{A}_1 + \widetilde{B}_1i + \widetilde{C}_1j + \widetilde{D}_1k$ and $\tilde{q}_2 = \widetilde{A}_2 + \widetilde{B}_2i + \widetilde{C}_2j + \widetilde{D}_2k \in \mathbb{H}_F$ the basic arithmetic operations are defined as follows:

(i)
$$\tilde{q}_1 + \tilde{q}_2 = (\tilde{A}_1 + \tilde{A}_2) + (\tilde{B}_1 + \tilde{B}_2)i + (\tilde{C}_1 + \tilde{C}_2)j + (\tilde{D}_1 + \tilde{D}_2)k \tag{3.2}$$

(ii)
$$\tilde{q}_1 - \tilde{q}_2 = (\tilde{A}_1 - \tilde{A}_2) + (\tilde{B}_1 - \tilde{B}_2)i + (\tilde{C}_1 - \tilde{C}_2)j + (\tilde{D}_1 - \tilde{D}_2)k \tag{3.3}$$

(iii)
$$\tilde{q}_1 \cdot \tilde{q}_2 = \tilde{A} + \tilde{B}i + \tilde{C}j + \tilde{D}k. \tag{3.4}$$

where

$$\begin{aligned} \tilde{A} &= (\tilde{A}_1 \cdot \tilde{A}_2 - \tilde{B}_1 \cdot \tilde{B}_2 - \tilde{C}_1 \cdot \tilde{C}_2 - \tilde{D}_1 \cdot \tilde{D}_2) \\ \tilde{B} &= (\tilde{A}_1 \cdot \tilde{B}_2 + \tilde{B}_1 \cdot \tilde{A}_2 + \tilde{C}_1 \cdot \tilde{D}_2 - \tilde{D}_1 \cdot \tilde{C}_2) \\ \tilde{C} &= (\tilde{A}_1 \cdot \tilde{C}_2 - \tilde{B}_1 \cdot \tilde{D}_2 + \tilde{C}_1 \cdot \tilde{A}_2 + \tilde{D}_1 \cdot \tilde{B}_2) \\ \tilde{D} &= (\tilde{A}_1 \cdot \tilde{D}_2 + \tilde{B}_1 \cdot \tilde{C}_2 - \tilde{C}_1 \cdot \tilde{B}_2 + \tilde{D}_1 \cdot \tilde{A}_2). \end{aligned}$$

Definition 3.3. If $\tilde{q}_1 = \tilde{A}_1 + \tilde{B}_1i + \tilde{C}_1j + \tilde{D}_1k$ and $\tilde{q}_2 = \tilde{A}_2 + \tilde{B}_2i + \tilde{C}_2j + \tilde{D}_2k \in \mathbb{H}_F$, then:

$$\frac{\tilde{q}_1}{\tilde{q}_2} = \tilde{A} + \tilde{B}i + \tilde{C}j + \tilde{D}k, \tag{3.5}$$

where

$$\begin{aligned} \tilde{A} &= \frac{\tilde{A}_1 \cdot \tilde{A}_2 + \tilde{B}_1 \cdot \tilde{B}_2 + \tilde{C}_1 \cdot \tilde{C}_2 + \tilde{D}_1 \cdot \tilde{D}_2}{\tilde{A}_2^2 + \tilde{B}_2^2 + \tilde{C}_2^2 + \tilde{D}_2^2} \\ \tilde{B} &= \frac{-\tilde{A}_1 \cdot \tilde{B}_2 + \tilde{B}_1 \cdot \tilde{A}_2 - \tilde{C}_1 \cdot \tilde{D}_2 + \tilde{D}_1 \cdot \tilde{C}_2}{\tilde{A}_2^2 + \tilde{B}_2^2 + \tilde{C}_2^2 + \tilde{D}_2^2} \\ \tilde{C} &= \frac{-\tilde{A}_1 \cdot \tilde{C}_2 + \tilde{B}_1 \cdot \tilde{D}_2 + \tilde{C}_1 \cdot \tilde{A}_2 - \tilde{D}_1 \cdot \tilde{B}_2}{\tilde{A}_2^2 + \tilde{B}_2^2 + \tilde{C}_2^2 + \tilde{D}_2^2} \\ \tilde{D} &= \frac{-\tilde{A}_1 \cdot \tilde{D}_2 - \tilde{B}_1 \cdot \tilde{C}_2 + \tilde{C}_1 \cdot \tilde{B}_2 + \tilde{D}_1 \cdot \tilde{A}_2}{\tilde{A}_2^2 + \tilde{B}_2^2 + \tilde{C}_2^2 + \tilde{D}_2^2}. \end{aligned}$$

Proposition 3.1. The expressing of the product $\tilde{q}_1\tilde{q}_2 = \tilde{A} + \tilde{B}i + \tilde{C}j + \tilde{D}k$ can be made as follows:

(i)

$$\begin{aligned} \tilde{A}(y) = \bigvee_{y=x_1x_2-x_3x_4-x_5x_6-x_7x_8} & \left(\tilde{A}_1(x_1) \wedge \tilde{A}_2(x_2) \wedge \tilde{B}_1(x_3) \wedge \right. \\ & \left. \wedge \tilde{B}_2(x_4) \wedge \tilde{C}_1(x_5) \wedge \tilde{C}_2(x_6) \wedge \tilde{D}_1(x_7) \wedge \tilde{D}_2(x_8) \right) \end{aligned}$$

(ii)

$$\begin{aligned} \tilde{B}(y) = \bigvee_{y=x_1x_2+x_3x_4+x_5x_6-x_7x_8} & \left(\tilde{A}_1(x_1) \wedge \tilde{B}_2(x_2) \wedge \tilde{B}_1(x_3) \wedge \right. \\ & \left. \wedge \tilde{A}_2(x_4) \wedge \tilde{C}_1(x_5) \wedge \tilde{D}_2(x_6) \wedge \tilde{D}_1(x_7) \wedge \tilde{C}_2(x_8) \right) \end{aligned}$$

(iii)

$$\begin{aligned} \tilde{C}(y) = \bigvee_{y=x_1x_2-x_3x_4+x_5x_6+x_7x_8} & \left(\tilde{A}_1(x_1) \wedge \tilde{C}_2(x_2) \wedge \tilde{B}_1(x_3) \wedge \right. \\ & \left. \wedge \tilde{D}_2(x_4) \wedge \tilde{C}_1(x_5) \wedge \tilde{A}_2(x_6) \wedge \tilde{D}_1(x_7) \wedge \tilde{B}_2(x_8) \right) \end{aligned}$$

(iv)

$$\begin{aligned} \tilde{D}(y) = \bigvee_{y=x_1x_2+x_3x_4-x_5x_6+x_7x_8} & \left(\tilde{A}_1(x_1) \wedge \tilde{D}_2(x_2) \wedge \tilde{B}_1(x_3) \wedge \right. \\ & \left. \wedge \tilde{C}_2(x_4) \wedge \tilde{C}_1(x_5) \wedge \tilde{B}_2(x_6) \wedge \tilde{D}_1(x_7) \wedge \tilde{A}_2(x_8) \right). \end{aligned}$$

Proposition 3.2. *The expressing of the quotient $\frac{\tilde{q}_1}{\tilde{q}_2} = \tilde{A} + \tilde{B}i + \tilde{C}j + \tilde{D}k$ can be made as follows:*

(i)

$$\begin{aligned} \tilde{A}(y) = \bigvee_{y=\frac{x_1x_2+x_3x_4+x_5x_6+x_7x_8}{x_2^2+x_4^2+x_6^2+x_8^2}} & \left(\tilde{A}_1(x_1) \wedge \tilde{A}_2(x_2) \wedge \tilde{B}_1(x_3) \wedge \right. \\ & \left. \wedge \tilde{B}_2(x_4) \wedge \tilde{C}_1(x_5) \wedge \tilde{C}_2(x_6) \wedge \tilde{D}_1(x_7) \wedge \tilde{D}_2(x_8) \right) \end{aligned}$$

(ii)

$$\begin{aligned} \tilde{B}(y) = \bigvee_{y=\frac{-x_1x_4+x_2x_3-x_5x_8+x_6x_7}{x_2^2+x_4^2+x_6^2+x_8^2}} & \left(\tilde{A}_1(x_1) \wedge \tilde{B}_2(x_2) \wedge \tilde{B}_1(x_3) \wedge \right. \\ & \left. \wedge \tilde{A}_2(x_4) \wedge \tilde{C}_1(x_5) \wedge \tilde{D}_2(x_6) \wedge \tilde{D}_1(x_7) \wedge \tilde{C}_2(x_8) \right) \end{aligned}$$

(iii)

$$\begin{aligned} \tilde{C}(y) = \bigvee_{y=\frac{-x_1x_6+x_3x_8+x_2x_5-x_4x_7}{x_2^2+x_4^2+x_6^2+x_8^2}} & \left(\tilde{A}_1(x_1) \wedge \tilde{C}_2(x_2) \wedge \tilde{B}_1(x_3) \wedge \right. \\ & \left. \wedge \tilde{D}_2(x_4) \wedge \tilde{C}_1(x_5) \wedge \tilde{A}_2(x_6) \wedge \tilde{D}_1(x_7) \wedge \tilde{B}_2(x_8) \right) \end{aligned}$$

(iv)

$$\begin{aligned} \tilde{D}(y) = \bigvee_{y=\frac{-x_1x_8-x_3x_6+x_4x_5+x_2x_7}{x_2^2+x_4^2+x_6^2+x_8^2}} & \left(\tilde{A}_1(x_1) \wedge \tilde{D}_2(x_2) \wedge \tilde{B}_1(x_3) \wedge \right. \\ & \left. \wedge \tilde{C}_2(x_4) \wedge \tilde{C}_1(x_5) \wedge \tilde{B}_2(x_6) \wedge \tilde{D}_1(x_7) \wedge \tilde{A}_2(x_8) \right). \end{aligned}$$

4. Matrix representations of fuzzy quaternion numbers

Just as the quaternion numbers are represented as matrices (Moț & Popa, 2014), so the fuzzy quaternion numbers have a matrix representations.

The first way in which we can represent the fuzzy quaternion numbers as a matrix is to use 2×2 fuzzy complex matrices and the representation given is one of a family of linearly related representations.

For this, we denote:

$$\begin{aligned} \mathbb{H}_{1F} &= \left\{ Q = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \mid z_1, z_2 \in \mathbb{C}_F \right\} = \\ &= \left\{ Q = \begin{pmatrix} \tilde{A} + \tilde{D}i & \tilde{C} + \tilde{B}i \\ -\tilde{C} + \tilde{B}i & \tilde{A} - \tilde{D}i \end{pmatrix} \mid \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in \mathbb{R}_F \right\}. \end{aligned} \tag{4.1}$$

Operations of addition and multiplication in \mathbb{H}_{1F} are made according to the rules of addition and multiplication of matrices.

For Q_1 and Q_2 in \mathbb{H}_{1F} we have:

$$\begin{aligned} Q_1 + Q_2 &= \begin{pmatrix} \tilde{A}_1 + \tilde{D}_1i & \tilde{C}_1 + \tilde{B}_1i \\ -\tilde{C}_1 + \tilde{B}_1i & \tilde{A}_1 - \tilde{D}_1i \end{pmatrix} + \begin{pmatrix} \tilde{A}_2 + \tilde{D}_2i & \tilde{C}_2 + \tilde{B}_2i \\ -\tilde{C}_2 + \tilde{B}_2i & \tilde{A}_2 - \tilde{D}_2i \end{pmatrix} = \\ &= \begin{pmatrix} \tilde{A}_1 + \tilde{A}_2 + (\tilde{D}_1 + \tilde{D}_2)i & \tilde{C}_1 + \tilde{C}_2 + (\tilde{B}_1 + \tilde{B}_2)i \\ -(\tilde{C}_1 + \tilde{C}_2) + (\tilde{B}_1 + \tilde{B}_2)i & \tilde{A}_1 + \tilde{A}_2 - (\tilde{D}_1 + \tilde{D}_2)i \end{pmatrix}, \end{aligned}$$

where $\tilde{A}_1 + \tilde{A}_2$, $\tilde{B}_1 + \tilde{B}_2$, $\tilde{C}_1 + \tilde{C}_2$ and $\tilde{D}_1 + \tilde{D}_2$ were defined in previous section. The product of two matrices Q_1 and Q_2 also follows the usual definition for matrix multiplication.

It is easy to verify that the \mathbb{H}_{1F} is closed under the operation of addition and multiplication. Moreover, any matrix of \mathbb{H}_{1F} admits the opposed matrix, namely

$$-Q = \begin{pmatrix} -z_1 & -z_2 \\ \bar{z}_2 & -\bar{z}_1 \end{pmatrix} = \begin{pmatrix} -\tilde{A} - \tilde{D}i & -\tilde{C} - \tilde{B}i \\ \tilde{C} - \tilde{B}i & -\tilde{A} + \tilde{D}i \end{pmatrix},$$

which belongs to \mathbb{H}_{1F} as well. For any non null matrix of \mathbb{H}_{1F} we have that:

$$\begin{vmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{vmatrix} = \begin{vmatrix} \tilde{A} + \tilde{D}i & \tilde{C} + \tilde{B}i \\ -\tilde{C} + \tilde{B}i & \tilde{A} - \tilde{D}i \end{vmatrix} = \tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2 + \tilde{D}^2,$$

which is equal to zero, if and only if $\tilde{A} = \tilde{B} = \tilde{C} = \tilde{D} = 0$. It result that any non null matrix of \mathbb{H}_{1F} admits the inverse matrix, namely:

$$Q^{-1} = \frac{1}{\tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2 + \tilde{D}^2} \begin{pmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & \bar{z}_1 \end{pmatrix} = \frac{1}{\tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2 + \tilde{D}^2} \begin{pmatrix} \tilde{A} - \tilde{D}i & -\tilde{C} - \tilde{B}i \\ \tilde{C} - \tilde{B}i & \tilde{A} + \tilde{D}i \end{pmatrix}$$

Therefore \mathbb{H}_{1F} has a field structure.

For commutativity we give the following counterexample: let $Q_1, Q_2 \in \mathbb{H}_{1F}$ where $\tilde{A}_1, \tilde{B}_1, \tilde{C}_1, \tilde{D}_1$ and $\tilde{A}_2, \tilde{B}_2, \tilde{C}_2, \tilde{D}_2$ are fuzzy triangular numbers defined by:

$$\begin{array}{llll} \tilde{A}_1 = [1, 2, 4] & \tilde{B}_1 = [0, 1, 3] & \tilde{C}_1 = [1, 2, 3] & \tilde{D}_1 = [0, 2, 4] \\ \tilde{A}_2 = [2, 4, 6] & \tilde{B}_2 = [2, 4, 5] & \tilde{C}_2 = [0, 1, 4] & \tilde{D}_2 = [1, 3, 4] \end{array}$$

We have

$$\tilde{Q}_1 \tilde{Q}_2 = \left(\begin{array}{ll} [-41, -4, 24] + [-9, 21, 55]i & [-18, 5, 46] + [-10, 8, 53]i \\ [-46, -5, 18] + [-10, 8, 53]i & [-41, -4, 24] + [-55, -21, 9]i \end{array} \right)$$

and

$$\tilde{Q}_2 \tilde{Q}_1 = \left(\begin{array}{ll} [-41, -4, 24] + [-14, 7, 50]i & [-10, 15, 54] + [-13, 16, 50]i \\ [-54, -15, 10] + [-13, 16, 50]i & [-41, -4, 24] + [-50, -7, 14]i \end{array} \right)$$

Based on the previous results we obtain the following theorem:

Theorem 4.1. \mathbb{H}_{1F} has a noncommutative field structure.

Theorem 4.2. \mathbb{H}_F and \mathbb{H}_{1F} are isomorphic fields.

Proof. We consider the mapping $\varphi : \mathbb{H}_F \rightarrow \mathbb{H}_{1F}$,

$$\varphi(\tilde{q}) = \varphi(\tilde{A} + \tilde{B}i + \tilde{C}j + \tilde{D}k) = \begin{pmatrix} \tilde{A} + \tilde{D}i & \tilde{C} + \tilde{B}i \\ -\tilde{C} + \tilde{B}i & \tilde{A} - \tilde{D}i \end{pmatrix},$$

which is a bijective application and it maintains the operations:

- (i) $\varphi(\tilde{q}_1 + \tilde{q}_2) = \varphi(\tilde{q}_1) + \varphi(\tilde{q}_2), \forall \tilde{q}_1, \tilde{q}_2 \in \mathbb{H}_F$
- (ii) $\varphi(\tilde{q}_1 \cdot \tilde{q}_2) = \varphi(\tilde{q}_1) \cdot \varphi(\tilde{q}_2), \tilde{q}_1, \tilde{q}_2 \in \mathbb{H}_F$

Hence it is an isomorphism of fields. Thus $\mathbb{H}_F \simeq \mathbb{H}_{1F}$. □

Remark. The isomorphism $\mathbb{H}_F \simeq \mathbb{H}_{1F}$ allows us to state that each fuzzy quaternion number of \mathbb{H}_F admits a matrix representation under the form of the elements of \mathbb{H}_{1F} , namely $\begin{pmatrix} \tilde{A} + \tilde{D}i & \tilde{C} + \tilde{B}i \\ -\tilde{C} + \tilde{B}i & \tilde{A} - \tilde{D}i \end{pmatrix}$.

Proposition 4.1. Any element of \mathbb{H}_{1F} admits:

$$Q = \tilde{A}\mathbf{1} + \tilde{B}I + \tilde{C}J + \tilde{D}K, \quad (4.2)$$

where $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ represent the matrix quaternion units.

Proof. It is verified by direct calculus:

$$\begin{aligned} Q &= \begin{pmatrix} \tilde{A} + \tilde{D}i & \tilde{C} + \tilde{B}i \\ -\tilde{C} + \tilde{B}i & \tilde{A} - \tilde{D}i \end{pmatrix} = \tilde{A} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \tilde{B} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \\ &\quad + \tilde{C} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \tilde{D} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \tilde{A}\mathbf{1} + \tilde{B}I + \tilde{C}J + \tilde{D}K. \end{aligned}$$

□

Remark. The matrix quaternion units I, J, K can be written with the help of Pauli matrix $\sigma_x, \sigma_y, \sigma_z$, namely:

$$I = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma_x, \quad J = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_y, \quad K = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_z.$$

Remark. If constraining any two of \tilde{B}, \tilde{C} and \tilde{D} to zero it results a representation of fuzzy complex numbers:

- i) If $\tilde{B} = \tilde{C} = \tilde{0}$, then it results a diagonal fuzzy complex matrix representation of fuzzy complex numbers:

$$Q = \begin{pmatrix} \tilde{A} + \tilde{D}i & \tilde{0} \\ \tilde{0} & \tilde{A} - \tilde{D}i \end{pmatrix}.$$

- ii) If $\tilde{B} = \tilde{D} = \tilde{0}$, then it results a fuzzy real matrix representation of fuzzy complex numbers:

$$Q = \begin{pmatrix} \tilde{A} & \tilde{C} \\ -\tilde{C} & \tilde{A} \end{pmatrix}.$$

Proposition 4.2. *The conjugate of a fuzzy quaternion corresponds to the Hermitian transpose (conjugate transpose) of the matrix*

$$Q^* = (\overline{Q})^T = \overline{Q^T} = \begin{pmatrix} \tilde{A} - \tilde{D}i & -\tilde{C} - \tilde{B}i \\ \tilde{C} - \tilde{B}i & \tilde{A} + \tilde{D}i \end{pmatrix},$$

where Q^T denotes the transpose of Q and \overline{Q} denotes the matrix with complex conjugated entries.

The second way in which we can represent the fuzzy quaternion numbers as a matrix is to use 4×4 fuzzy real matrices and the representation given is one of a family of linearly related representations.

In order to obtain such a representation we consider:

$$\mathbb{H}_{2F} = \left\{ Q_1 = \begin{pmatrix} \tilde{A} & -\tilde{B} & -\tilde{C} & -\tilde{D} \\ \tilde{B} & \tilde{A} & -\tilde{D} & \tilde{C} \\ \tilde{C} & \tilde{D} & \tilde{A} & -\tilde{B} \\ \tilde{D} & -\tilde{C} & \tilde{B} & \tilde{A} \end{pmatrix} \mid \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in \mathbb{R}_F \right\}.$$

Operations of addition and multiplication in \mathbb{H}_{2F} is made according to the rules of addition and multiplication of matrices.

It is easy to verify that the set of matrices \mathbb{H}_{2F} is closed under the operation of addition and multiplication of matrices. Moreover, any matrix of \mathbb{H}_{2F} , admits the opposed matrix and any non null matrix of \mathbb{H}_{2F} admits its inverse. Indeed, as

$$\begin{vmatrix} \tilde{A} & -\tilde{B} & -\tilde{C} & -\tilde{D} \\ \tilde{B} & \tilde{A} & -\tilde{D} & \tilde{C} \\ \tilde{C} & \tilde{D} & \tilde{A} & -\tilde{B} \\ \tilde{D} & -\tilde{C} & \tilde{B} & \tilde{A} \end{vmatrix} = (\tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2 + \tilde{D}^2)^2,$$

is equal to zero, if and only if $\tilde{A} = \tilde{B} = \tilde{C} = \tilde{D} = 0$, it results that for any non null matrix of \mathbb{H}_{2F} , there exists the inverse matrix:

$$Q_1^{-1} = \frac{1}{(\tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2 + \tilde{D}^2)^2} \begin{pmatrix} \tilde{A} & \tilde{B} & \tilde{C} & \tilde{D} \\ -\tilde{B} & \tilde{A} & -\tilde{D} & -\tilde{C} \\ -\tilde{C} & -\tilde{D} & \tilde{A} & -\tilde{B} \\ -\tilde{D} & \tilde{C} & -\tilde{B} & \tilde{A} \end{pmatrix}.$$

Thus \mathbb{H}_{2F} has a field structure.

For commutativity we give the following counterexample: let $Q_1, Q_2 \in \mathbb{H}_{2F}$ where $\tilde{A}_1, \tilde{B}_1, \tilde{C}_1, \tilde{D}_1$ and $\tilde{A}_2, \tilde{B}_2, \tilde{C}_2, \tilde{D}_2$ are fuzzy triangular numbers defined by:

$$\begin{aligned} \tilde{A}_1 &= [0, 2, 5], & \tilde{B}_1 &= [1, 3, 5], & \tilde{C}_1 &= [0, 1, 4], & \tilde{D}_1 &= [0, 2, 5] \\ \tilde{A}_2 &= [1, 2, 3], & \tilde{B}_2 &= [0, 3, 4], & \tilde{C}_2 &= [2, 4, 5], & \tilde{D}_2 &= [0, 2, 4]. \end{aligned}$$

We have

$$\tilde{Q}_1 \tilde{Q}_2 = \begin{pmatrix} [-60, -13, 15] & [-51, -6, 24] & [-57, -10, 20] & [-60, -17, 14] \\ [-24, 6, 51] & [-60, -13, 15] & [-60, -17, 14] & [-20, 10, 57] \\ [-20, 10, 57] & [-14, 17, 60] & [-60, -13, 15] & [-51, -6, 24] \\ [-14, 17, 60] & [-57, -10, 20] & [-24, 6, 51] & [-60, -13, 15] \end{pmatrix}$$

and

$$\tilde{Q}_2 \tilde{Q}_1 = \begin{pmatrix} [-60, -13, 15] & [-60, -18, 15] & [-57, -10, 20] & [-49, 1, 25] \\ [-15, 18, 60] & [-60, -13, 15] & [-49, 1, 25] & [-20, 10, 57] \\ [-20, 10, 57] & [-25, -1, 49] & [-60, -13, 15] & [-60, -18, 15] \\ [-25, -1, 49] & [-57, -10, 20] & [-15, 18, 60] & [-60, -13, 15] \end{pmatrix}.$$

Based on the previous results we obtain the following theorem:

Theorem 4.3. \mathbb{H}_{2F} has a noncommutative field structure.

Theorem 4.4. \mathbb{H}_F and \mathbb{H}_{2F} are isomorphic fields.

Proof. We consider the mapping $\psi : \mathbb{H}_F \rightarrow \mathbb{H}_{2F}$,

$$\psi(\tilde{q}) = \varphi(\tilde{A} + \tilde{B}i + \tilde{C}j + \tilde{D}k) = \begin{pmatrix} \tilde{A} & -\tilde{B} & -\tilde{C} & -\tilde{D} \\ \tilde{B} & \tilde{A} & -\tilde{D} & \tilde{C} \\ \tilde{C} & \tilde{D} & \tilde{A} & -\tilde{B} \\ \tilde{D} & -\tilde{C} & \tilde{B} & \tilde{A} \end{pmatrix}.$$

We note that ψ is bijective and preserves the operations - it follows immediately by direct calculation, thus it is a isomorphism of fields. Therefore $\mathbb{H}_F \simeq \mathbb{H}_{2F}$. \square

Remark. The isomorphism $\mathbb{H}_F \simeq \mathbb{H}_{2F}$ allows us to stat that each fuzzy quaternion number of \mathbb{H}_F admits a matrix representation under the form of the elements of \mathbb{H}_{2F} .

Remark. Any element of \mathbb{H}_{2F} can be written

$$Q_1 = \begin{pmatrix} \widetilde{A} & -\widetilde{B} & -\widetilde{C} & -\widetilde{D} \\ \widetilde{B} & \widetilde{A} & -\widetilde{D} & \widetilde{C} \\ \widetilde{C} & \widetilde{D} & \widetilde{A} & -\widetilde{B} \\ \widetilde{D} & -\widetilde{C} & \widetilde{B} & \widetilde{A} \end{pmatrix} = \widetilde{A} \begin{pmatrix} \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \end{pmatrix} + \widetilde{B} \begin{pmatrix} \widetilde{0} & -\widetilde{1} & \widetilde{0} & \widetilde{0} \\ \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & -\widetilde{1} \\ \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \end{pmatrix} +$$

$$+\widetilde{C} \begin{pmatrix} \widetilde{0} & \widetilde{0} & -\widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \\ \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & -\widetilde{1} & \widetilde{0} & \widetilde{0} \end{pmatrix} + \widetilde{D} \begin{pmatrix} \widetilde{0} & \widetilde{0} & \widetilde{0} & -\widetilde{1} \\ \widetilde{0} & \widetilde{0} & -\widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \end{pmatrix}.$$

Remark. Similarly, a fuzzy quaternion number $\widetilde{q} = \widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k$ can be represented as

$$Q_2 = \begin{pmatrix} \widetilde{A} & \widetilde{B} & \widetilde{C} & \widetilde{D} \\ -\widetilde{B} & \widetilde{A} & -\widetilde{D} & \widetilde{C} \\ -\widetilde{C} & \widetilde{D} & \widetilde{A} & -\widetilde{B} \\ -\widetilde{D} & -\widetilde{C} & \widetilde{B} & \widetilde{A} \end{pmatrix} = \widetilde{A} \begin{pmatrix} \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \end{pmatrix} + \widetilde{B} \begin{pmatrix} \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ -\widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & -\widetilde{1} \\ \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \end{pmatrix} +$$

$$+\widetilde{C} \begin{pmatrix} \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \\ -\widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & -\widetilde{1} & \widetilde{0} & \widetilde{0} \end{pmatrix} + \widetilde{D} \begin{pmatrix} \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \\ \widetilde{0} & \widetilde{0} & -\widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ -\widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \end{pmatrix}.$$

Remark. The 48 possible matrix representation of fuzzy quaternion numbers are in fact the matrices whose transpose is its negation (skew-symmetric matrices, i.e. $-Q = Q^T$).

Proposition 4.3. *In this representations, the conjugate of a fuzzy quaternion number corresponds to the transpose of the matrix Q_n , $n = 1, 48$*

$$\overline{\widetilde{q}} = \widetilde{A} - \widetilde{B}i - \widetilde{C}j - \widetilde{D}k = (Q_n)^T.$$

For example, for the representation Q_1 of a fuzzy quaternion number, we have

$$\overline{\widetilde{q}} = (Q_1)^T = \begin{pmatrix} \widetilde{A} & \widetilde{B} & \widetilde{C} & \widetilde{D} \\ -\widetilde{B} & \widetilde{A} & \widetilde{D} & -\widetilde{C} \\ -\widetilde{C} & -\widetilde{D} & \widetilde{A} & \widetilde{B} \\ -\widetilde{D} & \widetilde{C} & -\widetilde{B} & \widetilde{A} \end{pmatrix}.$$

Or, for second representation Q_2 ,

$$\bar{q} = (Q_2)^T = \begin{pmatrix} \bar{A} & -\bar{B} & -\bar{C} & -\bar{D} \\ \bar{B} & \bar{A} & \bar{D} & -\bar{C} \\ \bar{C} & -\bar{D} & \bar{A} & \bar{B} \\ \bar{D} & \bar{C} & -\bar{B} & \bar{A} \end{pmatrix}$$

Remark. If $\bar{C} = \bar{D} = \bar{0}$ then $\bar{q} = \bar{A} + \bar{B}i$ and it results the representation of fuzzy complex numbers as diagonal matrices with two 2×2 blocks.

For example, for the representation Q_1 when $\bar{C} = \bar{D} = \bar{0}$ it results

$$Q_1 = \begin{pmatrix} \bar{A} & -\bar{B} & \bar{0} & \bar{0} \\ \bar{B} & \bar{A} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{A} & -\bar{B} \\ \bar{0} & \bar{0} & \bar{B} & \bar{A} \end{pmatrix}$$

and for Q_2 we obtain

$$Q_2 = \begin{pmatrix} \bar{A} & \bar{B} & \bar{0} & \bar{0} \\ -\bar{B} & \bar{A} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{A} & -\bar{B} \\ \bar{0} & \bar{0} & \bar{B} & \bar{A} \end{pmatrix}.$$

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Fuzzy M -Open Sets

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Abstract

In this paper, we introduce the relatively new notion of fuzzy M -open subset which is strictly weaker than fuzzy open. We prove that the collection of all fuzzy M -open subsets of a fuzzy space forms a fuzzy topology that is finer than the original one. Several characterizations and properties of this class are also given as well as connections to other well-known "fuzzy generalized open" subsets.

Keywords: Fuzzy M -open, Fuzzy countable set, Fuzzy anti locally countable space.
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1. Introduction

Fuzzy topological spaces were first introduced by (Chakraborty & Ahsanullah, 1992; Chang, 1968). Let (X, \mathfrak{T}) be a fuzzy topological space (simply, Fts). If λ is a fuzzy set (simply, F-set), then the closure of λ , the interior of λ and the derived set of λ will be denoted by $Cl_{\mathfrak{T}}(\lambda)$, $Int_{\mathfrak{T}}(\lambda)$ and $d_{\mathfrak{T}}(\lambda)$, respectively. If no ambiguity appears, we use $\bar{\lambda}$, $\overset{o}{\lambda}$ and λ' instead, respectively. A F-set λ is called F -semi-open (simply, FSO) (Mahmoud *et al.*, 2004) if there exists a fuzzy open (simply, F-open) set μ such that $\mu \leq \lambda \leq Cl_{\mathfrak{T}}(\mu)$. Clearly λ is a FSO-set if and only if $\lambda \leq Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(\lambda))$. A complement of a FSO-set is called F -semi-closed (simply, FSC). The fuzzy semi-interior of λ is the union of all fuzzy semi-open subsets contained in λ and is denoted by $sInt(\lambda)$. λ is called fuzzy preopen (simply, FPO) if $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$. λ is called fuzzy α -open if $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(\lambda)))$ and fuzzy β -open if $\lambda \leq Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda)))$. Finally, λ is called fuzzy regular-open (simply, FRO) if $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$. Complements of FRO-sets are called fuzzy regular-closed (simply, FRC). The collection of all FSO (resp., FPO, FRO, FRC, $F\alpha$ -open and $F\beta$ -open) subsets of

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X is denoted by $SO(X, \mathfrak{T})$ (resp., $FPO(X, \mathfrak{T})$, $FRO(X, \mathfrak{T})$, $FRC(X, \mathfrak{T})$, $F\alpha(X, \mathfrak{T})$ and $F\beta(X, \mathfrak{T})$). We remark that $F\alpha(X, \mathfrak{T})$ is a topological space and $F\alpha(X, \mathfrak{T}) = FSO(X, \mathfrak{T}) \wedge FPO(X, \mathfrak{T})$. A fuzzy space (X, \mathfrak{T}) is called *locally countable* (*P-space*, *anti locally countable*, respectively) if each $\lambda \in X$ has a countable neighborhood (countable intersections of fuzzy open subsets are fuzzy open, non-empty fuzzy open subsets are uncountable, respectively). For more on the preceding notions, the reader is referred to (Al-Hawary, 2017, 2008; Chakraborty & Ahsanullah, 1992; Chang, 1968; Chaudhuri & Das, 1993; Mahmoud *et al.*, 2004; Wong, 1974).

In this paper, we introduce the relatively new notions of FMO, which is weaker than the class of fuzzy open subsets. In section 2, we also show that the collection of all FMO subsets of a space (X, \mathfrak{T}) forms a fuzzy topology that is finer than \mathfrak{T} and we investigate the connection of FMO notion to other classes of "fuzzy generalized open" subsets as well as several characterizations of FMO and fuzzy M -closed notions via the operations of interior and closure. In section 3, several interesting properties and constructions of FMO subsets are discussed in the case of anti locally countable spaces.

2. Fuzzy M -open set

We begin this section by introducing the notion of FMO and fuzzy M -closed subsets.

Definition 2.1. A fuzzy subset λ of a space (X, \mathfrak{T}) is called FMO (simply, FMO) if for every $\nu \leq \lambda$, there exists an open fuzzy subset $\omega \in X$ such that $\nu \leq \omega$ and such that $\omega \setminus sInt(\lambda)$ is countable. The complement of a FMO subset is called fuzzy M -closed (simply, FMC).

Clearly every FO-set is FMO, but the converse needs not be true.

Example 2.1. Let $X = \{a, b\}$ and $\mathfrak{T} = \{0, 1, \chi_{\{a\}}\}$. Set $\lambda = \chi_{\{b\}}$. Then λ is FMO but not FO.

Next, we show that the collection of all FMO subsets of a space (X, \mathfrak{T}) forms a fuzzy topology \mathfrak{T}_M that contains \mathfrak{T} .

Theorem 2.1. *If (X, \mathfrak{T}) is a fuzzy space, then (X, \mathfrak{T}_M) is a space such that $\mathfrak{T} \leq \mathfrak{T}_M$.*

Proof. We only need to show (X, \mathfrak{T}_M) is a space. Clearly since 0 and 1 are FO-sets, they are FMO. If $\lambda, \psi \in \mathfrak{T}_M$ and $\nu \leq \lambda \wedge \psi$, then there exist FO-sets ω, υ in X both containing λ such that $\omega \setminus sInt(\lambda)$ and $\upsilon \setminus sInt(\psi)$ are countable. Now $\lambda \leq \omega \wedge \upsilon$ and for every $\delta \leq (\omega \wedge \upsilon) \setminus sInt(\lambda \wedge \psi) = (\omega \wedge \upsilon) \setminus (sInt(\lambda) \wedge sInt(\psi))$ either $\delta \leq \omega \setminus sInt(\lambda)$ or $\delta \leq \upsilon \setminus sInt(\psi)$. Thus $(\omega \wedge \upsilon) \setminus sInt(\lambda \wedge \psi) \leq \omega \setminus sInt(\lambda)$ or $(\omega \wedge \upsilon) \setminus sInt(\lambda \wedge \psi) \leq \upsilon \setminus sInt(\psi)$ and thus $(\omega \wedge \upsilon) \setminus sInt(\lambda \wedge \psi)$ is countable. Therefore, $\lambda \wedge \psi \in \mathfrak{T}_M$. If $\{\lambda_\alpha : \alpha \in \Delta\}$ is a collection of FMO subsets of X , then for every $\lambda \leq \vee\{\lambda_\alpha : \alpha \in \Delta\}$, $\nu \leq \lambda_\beta$ for some $\beta \in \Delta$. Hence there exists a FO-set ω of X containing λ such that

$\omega \setminus sInt(\lambda)$ is countable. Now as $\omega \setminus sInt(\bigvee\{\lambda_\alpha : \alpha \in \nabla\}) \leq \omega \setminus \bigvee\{sInt(\lambda_\alpha) : \alpha \in \nabla\} \leq \omega \setminus sInt(\lambda)$, $\omega \setminus sInt(\bigvee\{\lambda_\alpha : \alpha \in \nabla\})$ is countable and hence $\bigvee\{\lambda_\alpha : \alpha \in \nabla\} \in \mathfrak{T}_M$. \square

Corollary 2.1. *If (X, \mathfrak{T}) is a p -space, then $\mathfrak{T} = \mathfrak{T}_M$.*

Next we show that FMO notion is independent of both FPO and FSO notions.

Example 2.2. Consider \mathbb{R} with the standard fuzzy topology. Then $\chi_{\mathbb{Q}}$ is FPO but not FMO. Also $\chi_{[0,1]}$ is FSO but not FMO.

Example 2.3. In Example 2.1, $\chi_{\{b\}}$ is FMO but neither FPO nor FO.

Next we characterize \mathfrak{T}_M when X is a locally countable fuzzy space.

Theorem 2.2. *If (X, \mathfrak{T}) is a locally countable fuzzy space, then \mathfrak{T}_M is the discrete fuzzy topology.*

Proof. Let λ be a fF-set in X and $v \leq \lambda$. Then there exists a countable neighborhood ω of λ and hence there exists a FO-set η containing λ such that $\eta \leq \omega$. Clearly $\eta \setminus sInt(\lambda) \leq \omega \setminus sInt(\lambda) \leq \omega$ and thus $\eta \setminus sInt(\lambda)$ is countable. Therefore λ is FMO and so \mathfrak{T}_M is the discrete fuzzy topology. \square

Corollary 2.2. *If (X, \mathfrak{T}) is a countable fuzzy space, then \mathfrak{T}_M is the discrete fuzzy topology.*

The following result, in which a new characterization of FMO subsets is given, will be a basic tool throughout the rest of the paper.

Lemma 2.1. *A subset λ of a fuzzy space X is FMO if and only if for every $v \leq \lambda$, there exists a FO- subset ω containing λ and a countable subset π such that $\omega - \pi \leq sInt(\lambda)$.*

Proof. Let $\lambda \in \mathfrak{T}_M$ and $v \leq \lambda$, then there exists a FO- subset ω containing λ such that $\omega \setminus sInt(\lambda)$ is countable. Let $\pi = \omega \setminus sInt(\lambda) = \omega \wedge (X \setminus sInt(\lambda))$. Then $\omega - \pi \leq sInt(\lambda)$. \square

Conversely, let $v \leq \lambda$. Then there exists a FO- subset ω containing λ and a countable subset π such that $\omega - \pi \leq sInt(\lambda)$. Thus $\omega \setminus sInt(\lambda) = \pi$ is countable.

The next result follows easily from the definition and the fact that the intersection of fuzzy M -closed sets is again fuzzy M -closed.

Lemma 2.2. *A subset λ of a fuzzy space X is fuzzy M -closed if and only if $Cl_M(\lambda) = \lambda$.*

We next study restriction and deletion operations.

Theorem 2.3. *If λ is FMO subset of X , then $\mathfrak{T}_M|_\lambda \subseteq (\mathfrak{T}|_\lambda)_M$.*

Proof. Let $\rho \in \mathfrak{T}_M|_\lambda$. Then $\rho = \nu \wedge \lambda$ for some FMO subset ν . For every $\lambda \leq \rho$, there exist $\delta_\nu, \delta_\lambda \in \mathfrak{T}$ containing λ and countable sets γ_ν and γ_λ such that $\delta_\nu - \gamma_\nu \leq sInt(\nu)$ and $\delta_\lambda - \gamma_\lambda \leq sInt(\lambda)$. Therefore, $\nu \leq \lambda \wedge (\delta_\nu \wedge \delta_\lambda) \in \mathfrak{T}_\lambda$, $\gamma_\nu \vee \gamma_\lambda$ is countable and

$$\begin{aligned} \lambda \wedge (\delta_\nu \wedge \delta_\lambda) - (\gamma_\nu \vee \gamma_\lambda) &\leq (\delta_\nu \wedge \delta_\lambda) \wedge (1 - \gamma_\nu) \wedge (1 - \gamma_\lambda) \\ &= (\delta_\nu - \gamma_\nu) \wedge (\delta_\lambda - \gamma_\lambda) \\ &\leq sInt(\nu) \wedge sInt(\lambda) \wedge \lambda \\ &= sInt(\nu \wedge \lambda) \wedge \lambda \\ &= sInt(\rho) \wedge \lambda \\ &\leq sInt_\lambda(\rho). \end{aligned}$$

Therefore, $\rho \in (\mathfrak{T}|_\lambda)_M$. □

Corollary 2.3. *If λ is a FO subset of X , then $\mathfrak{T}_M|_\lambda \leq (\mathfrak{T}|_\lambda)_M$.*

In the next example, we show that if λ in the preceding Theorem is not FMO, then the result needs not be true.

Example 2.4. Consider \mathbb{R} with the standard fuzzy topology and let $\lambda = \chi_{\mathbb{R} \setminus \mathbb{Q}}$. Then λ is not FMO and so not FO. As $\chi_{(0,1)}$ is FMO, then $\theta = \chi_{(0,1)} \wedge \lambda \in \mathfrak{T}_M|_\lambda$ while if $\theta \in (\mathfrak{T}|_\lambda)_M$ then for every $\lambda \leq \theta$, there exists $\omega \in \mathfrak{T}|_\lambda$ and a countable $\delta \leq \lambda$ such that $\omega - \delta \leq sInt(\theta) = 0$. Thus $\omega \leq \delta$ and hence ω is countable which is a contradiction.

In the next example, we show that $(\mathfrak{T}|_\lambda)_M$ needs not be a subset of $\mathfrak{T}_M|_\lambda$.

Example 2.5. Consider \mathbb{R} with the standard fuzzy topology, $\lambda = \chi_{\mathbb{Q}}$ and $\mu = \chi_{(0,2)}$. If $\mu \in \mathfrak{T}_M|_\lambda$, then $\mu = \delta \wedge \lambda$ for some $\delta \in \mathfrak{T}_M$ which is impossible as $\chi_{\sqrt{2}} \leq \delta - \lambda$. On the other hand to show $\mu \in (\mathfrak{T}|_\lambda)_M$, let $\nu \leq \mu$. If $\nu \leq \lambda$, pick $q_1, q_2 \leq \lambda$ such that $0 < q_1 < \nu < q_2 < 2$ and let $\omega = \chi_{(q_1, q_2)} \wedge \lambda$. Then $\lambda \leq \omega - 0 \leq \mu = sInt(\mu)$. Thus in both cases $\mu \in (\mathfrak{T}|_\lambda)_M$.

Theorem 2.4. *Let (X, \mathfrak{T}) be a fuzzy space and λ is FMC-set. Then $Cl_{\mathfrak{T}}(\lambda) \leq \gamma \vee \vartheta$ for some closed subset γ and a countable subset ϑ .*

Proof. Let λ be FMC-set. Then $1 - \lambda$ is FMO and hence for every $\lambda \leq 1 - \lambda$, there exists a FO-set ω containing λ and a countable set ϑ such that $\omega - \vartheta \leq sInt(1 - \lambda) \leq 1 - Cl_{\mathfrak{T}}(\lambda)$. Thus

$$Cl_{\mathfrak{T}}(\lambda) \leq 1 - (\omega - \vartheta) \leq 1 - (\omega \wedge (X - \vartheta)) \leq 1 \wedge ((1 - \omega) \vee \vartheta) \leq (1 - \omega) \vee \vartheta.$$

Letting $\gamma = 1 - \omega$. Then γ is closed such that $Cl_{\mathfrak{T}}(\lambda) \leq \gamma \vee \vartheta$. \square

3. Anti-locally countable fuzzy spaces

In this section, several interesting properties and constructions of FMO subsets are discussed in case of anti locally countable fuzzy spaces.

Theorem 3.1. *A fuzzy space (X, \mathfrak{T}) is anti locally countable if and only if (X, \mathfrak{T}_M) is anti locally countable.*

Proof. Let $\lambda \in \mathfrak{T}_M$ and $\nu \leq \lambda$. Then by Lemma 2.1, there exists a FO- subset ω containing λ and a countable μ such that $\omega - \mu \leq sInt(\lambda)$. Hence $sInt(\lambda)$ is uncountable and so is λ . The converse follows from the fact that every FO-set is FMO. \square

Corollary 3.1. *If (X, \mathfrak{T}) is anti locally countable fuzzy space and λ is FMO, then $Cl_{\mathfrak{T}}(\lambda) = Cl_{\mathfrak{T}_M}(\lambda)$.*

Proof. Clearly $Cl_{\mathfrak{T}_M}(\lambda) \leq Cl_{\mathfrak{T}}(\lambda)$. On the other hand, let $\lambda \leq Cl_{\mathfrak{T}}(\lambda)$ and μ be an FMO subset containing λ . Then by Lemma 2.1, there exists a FO- subset ν containing λ and a countable set η such that $\nu - \eta \leq sInt(\mu)$. Thus $(\nu - \eta) \wedge \lambda \leq sInt(\mu) \wedge \lambda$ and so $(\nu \wedge \lambda) - \eta \leq sInt(\mu) \wedge \lambda$. As $\lambda \in \nu$ and $\lambda \in Cl_{\mathfrak{T}}(\lambda)$, $\nu \wedge \lambda \neq 0$ and then as ν and λ are FMO, $\nu \wedge \lambda$ is FMO and as X is anti locally countable, $\nu \wedge \lambda$ is uncountable and so is $(\nu \wedge \lambda) - \eta$. Thus $\nu \wedge \lambda$ is uncountable as it contains the uncountable set $sInt(\mu) \wedge \lambda$. Therefore, $\mu \wedge \lambda \neq 0$ which means that $\lambda \in Cl_{\mathfrak{T}_M}(\lambda)$. \square

By a similar argument, we can easily prove the following result:

Corollary 3.2. *If (X, \mathfrak{T}) is anti locally countable and λ is FMC, then $Int_{\mathfrak{T}}(\lambda) = Int_{\mathfrak{T}_M}(\lambda)$.*

Theorem 3.2. *Let (X, \mathfrak{T}) be an anti locally countable fuzzy space. Then $F\alpha(X, \mathfrak{T}) \subseteq F\alpha(X, \mathfrak{T}_M)$.*

Proof. If $\lambda \in F\alpha(X, \mathfrak{T})$, then $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(\lambda)))$ and by Corollary 3.1, $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}_M}(Int_{\mathfrak{T}}(\lambda)))$. Now by Corollary 3.2 and as $Cl_{\mathfrak{T}_M}(Int_{\mathfrak{T}}(\lambda))$ is FMC, $\lambda \leq Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}_M}(Int_{\mathfrak{T}}(\lambda)))$ and by Corollary 3.2 again, $\lambda \leq Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}_M}(Int_{\mathfrak{T}_M}(\lambda)))$ which means $\lambda \in F\alpha(X, \mathfrak{T}_M)$. \square

The converse of the preceding result needs not be true as shown next.

Example 3.1. Consider \mathbb{R} with the standard fuzzy topology and let $\lambda = \chi_{\mathbb{R} \setminus \mathbb{Q}}$. Then $\lambda \in F\alpha(\mathbb{R}, \mathfrak{T}_M)$ but $\lambda \notin F\alpha(\mathbb{R}, \mathfrak{T})$.

Similarly, one can show that in an anti locally countable fuzzy space, $F\beta(X, \mathfrak{T}_M) \leq F\beta(X, \mathfrak{T})$.

Theorem 3.3. *Let (X, \mathfrak{T}) be an anti locally countable fuzzy space. Then $d_{\mathfrak{T}}(\mu) = d_{\mathfrak{T}_M}(\mu)$ for every subset F -set μ .*

Proof. If $\lambda \leq d_{\mathfrak{T}}(\mu)$ and ν is any FMO subset containing λ , then there exists a FO- subset ω containing λ and a countable γ such that $\omega - \gamma \leq sInt(\nu) \leq \nu$. Thus $(\omega - \gamma) \wedge (\lambda - \{\lambda\}) \leq sInt(\nu) \wedge (\mu - \lambda) \leq \nu \wedge (\mu - \lambda)$ and as $\lambda \in d_{\mathfrak{T}}(\mu)$ and ν^o is open containing λ , we have $\nu^o \wedge (\mu - \lambda) \neq 0$ and so $\nu \wedge (\mu - \lambda) \neq 0$. Therefore $\lambda \in d_{\mathfrak{T}_M}(\mu)$. \square

The converse is obvious as every FO subset is FMO.

Theorem 3.4. *Let (X, \mathfrak{T}) be an anti locally countable fuzzy space. Then $FRO(X, \mathfrak{T}) = FRO(X, \mathfrak{T}_M)$.*

Proof. If $\lambda \in FRO(X, \mathfrak{T})$, then $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$ and by Corollary 3.1, $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}_M}(\lambda))$. Now by Corollary 3.2 and as $Cl_{\mathfrak{T}_M}(\lambda)$ is FMC, $\lambda = Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}_M}(\lambda))$ which means $\lambda \in FRO(X, \mathfrak{T}_M)$. Conversely, if $\lambda \in FRO(X, \mathfrak{T}_M)$, then $\lambda = Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}_M}(\lambda))$. Then as λ is FMO, by Corollary 3.1, $\lambda = Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}}(\lambda))$ and as $Cl_{\mathfrak{T}}(\lambda)$ is FMC being a FC-set, then by $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$ which means $\lambda \in FRO(X, \mathfrak{T})$. \square

The converse of the preceding result need not be true as shown next.

Example 3.2. Let $X = \{a, b, c, d, e\}$ and $\mathfrak{T} = \{0, 1, \chi_{\{a\}}, \chi_{\{a,b\}}, \chi_{\{a,b,c\}}, \chi_{\{a,b,c,d\}}\}$. Then (X, \mathfrak{T}) is an anti locally countable fuzzy space such that $FRO(X, \mathfrak{T}) = \{0, 1\}$ while $FRO(X, \mathfrak{T}_M) = \mathfrak{T}$.

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Decomposition of Continuity in a Fuzzy Sequential Topological Space

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Abstract

A new class of fuzzy sequential sets called fs-preopen sets is introduced and characterized. An fs-precontinuous mapping and strongly fs-precontinuous mapping are defined and studied. A necessary and sufficient condition under which an fs-preopen set is fs-open, has been established. Apart from that, different kinds of sets, namely, fs- α -open set, fs- δ -set, fs- \mathcal{A} -set, locally fs-closed set, fs- S -preopen set have been studied. The interrelationships between them are investigated. Using them and the associated continuities, various decompositions of fs-continuity have been established.

Keywords: Fuzzy sequential topological spaces, fs-preopen, fs- α -open, fs- δ -set, fs- \mathcal{A} -set, locally fs-closed, fs- S -preopen sets.

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1. Introduction

In the last few decades, there has been interests in the study of generalized open sets and generalized continuity in topological spaces. Various authors studied different kinds of generalized open sets in topological spaces. In the fuzzy setting, fuzzy semi-open sets and fuzzy semicontinuity were introduced and studied by K. K. Azad ([Azad, 1981](#)), fuzzy pre-open sets ([Mashhour et al., 1982](#)) by A. S. Mashhour, M. E. Abd El-Monsef and S. N. EI-Deeb. Again, fs-semiopen sets and fs-semicontinuity have been studied in ([Tamang & Sarkar, 2016](#)).

The purpose of this work is to study different generalized open sets and the associated continuous functions in a fuzzy sequential topological space.

Apart from the introduction in Section 1, Section 2 is devoted to the study of preopen sets and precontinuity in a fuzzy sequential topological space. An important result from this Section is a

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necessary and sufficient condition for a preopen set to be open. Section 3 deals with the introduction and study of some more generalized open sets and their respective continuities. Finally the Section has been concluded with some decompositions of continuity.

Throughout the paper X will denote a non empty set and I the unit interval $[0, 1]$. Sequences of fuzzy sets in X called fuzzy sequential sets (fs-sets) will be denoted by the symbols $A_f(s), B_f(s), C_f(s)$, etc. An fs-set $X_f^l(s)$ is a sequence of fuzzy sets $\{X_f^n\}_n$ where $l \in I$ and $X_f^n(x) = l$, for all $x \in X, n \in \mathbb{N}$,

A family $\delta(s)$ of fuzzy sequential sets on a set X satisfying the properties

- (i) $X_f^r(s) \in \delta(s)$ for $r = 0$ and 1 ,
- (ii) $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$ and
- (iii) for any family $\{A_{fj}(s) \in \delta(s), j \in J\}$, $\bigvee_{j \in J} A_{fj}(s) \in \delta(s)$

is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called fuzzy sequential topological space (FSTS) (Singha et al., 2014). The members of $\delta(s)$ are called open fuzzy sequential (fs-open) sets in X . Complement of an open fuzzy sequential set in X is called closed fuzzy sequential (fs-closed) set in X . In an FSTS, the closure, interior, continuous functions, semiopen sets etc. are defined in the usual manner (See (Singha et al., 2014), (Tamang & Sarkar, 2016), (Tamang et al., 2016)).

2. FS-preopen sets and FS-precontinuity

Definition 2.1. (i) An fs-set $A_f(s)$ in an FSTS, is said to be an fs-preopen set if $A_f(s) \leq \overline{(A_f(s))^o}$.
 (ii) An fs-set $\overline{A_f(s)}$ in an FSTS, is said to be an fs-preclosed set if its complement is fs-preopen or equivalently if $\overline{\overline{A_f(s)}} \leq A_f(s)$.

If $A_f(s)$ is both fs-preopen and fs-preclosed, then it is called an fs-preclopen set.

Definition 2.2. An fs-set $A_f(s)$ is called fs-dense in an FSTS $(X, \delta(s))$, if $\overline{A_f(s)} = X_f^1(s)$.

Fundamental properties of fs-preopen (fs-preclosed) sets are:

- Every fs-open (fs-closed) set is fs-preopen (fs-preclosed).
- Arbitrary union (intersection) of fs-preopen (fs-preclosed) sets is fs-preopen (fs-preclosed).

Example 2.1 shows that an fs-preopen (fs-preclosed) set may not be fs-open (fs-closed), the intersection (union) of any two fs-preopen (fs-preclosed) sets need not be an fs-preopen (fs-preclosed) set. Unlike in a general topological space, the intersection of an fs-preopen set with an fs-open set may fail to be an fs-preopen set.

Example 2.1. Consider the fs-sets $A_f(s), B_f(s)$ and $C_f(s)$ in $X = [0, 1]$, defined as follows:

$$\begin{aligned}
 A_f^1(x) &= 0, \text{ if } 0 \leq x \leq \frac{1}{2} \\
 &= \frac{1}{4}, \text{ if } \frac{1}{2} < x \leq 1 \\
 \text{and } A_f^n &= \overline{1} \text{ for all } n \neq 1.
 \end{aligned}$$

$$\begin{aligned}
 B_f^1(x) &= \frac{1}{2}, \text{ if } 0 \leq x \leq \frac{1}{2} \\
 &= 0, \text{ if } \frac{1}{2} < x \leq 1 \\
 \text{and } B_f^n &= \bar{0} \text{ for all } n \neq 1.
 \end{aligned}$$

$$\begin{aligned}
 C_f^1(x) &= \frac{3}{4}, \text{ if } 0 \leq x \leq \frac{1}{2} \\
 &= 1, \text{ if } \frac{1}{2} < x \leq 1 \\
 \text{and } C_f^n &= \bar{0} \text{ for all } n \neq 1.
 \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Now,

- (i) $A_f(s)$ and $C_f(s)$ are fs-preopen sets but their intersection is not fs-preopen.
- (ii) $C_f(s)$ is fs-preopen but is not fs-open.

Theorem 2.1. *Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is fs-preopen if and only if there exists an fs-open set $O_f(s)$ in X such that $A_f(s) \leq \overline{O_f(s)} \leq A_f(s)$.*

Proof. Straightforward. □

Corollary 2.1. *Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is fs-preclosed if and only if there exists an fs-closed set $C_f(s)$ in X such that $\overset{o}{A_f(s)} \leq C_f(s) \leq A_f(s)$.*

Proof. Straightforward. □

Theorem 2.2. *An fs-set is fs-clopen (both fs-closed and fs-open) if and only if it is fs-closed and fs-preopen.*

Proof. Proof is omitted. □

Theorem 2.3. *In an FSTS, every fs-set is fs-preopen if and only if every fs-open set is fs-closed.*

Proof. Suppose every fs-set in an FSTS $(X, \delta(s))$, is fs-preopen and let $A_f(s)$ be an fs-open set. Then, $A_f^c(s) = \overline{A_f^c(s)}$ is fs-preopen and hence $\overline{A_f^c(s)} \leq \overline{\overline{A_f^c(s)}}^o = \overline{A_f^c(s)}^o = (A_f^c(s))^o$. Thus, $A_f^c(s)$ is fs-open and hence $A_f(s)$ is fs-closed.

Conversely, suppose every fs-open set is fs-closed and let $A_f(s)$ be any fs-set. By the assumption, $\overline{A_f(s)} = \overline{A_f(s)}^o$ and hence $A_f(s)$ is fs-preopen. □

Theorem 2.4. (a) *Closure of an fs-preopen set is fs-regular closed.*

(b) *Interior of an fs-preclosed set is fs-regular open.*

Proof. We prove only (a). Let $A_f(s)$ be an fs-preopen set in X . Since $\overline{A_f(s)}^o \leq \overline{A_f(s)}$, we have $\overline{\overline{A_f(s)}^o} \leq \overline{A_f(s)} = \overline{A_f(s)}$. Now $A_f(s)$ being fs-preopen, $A_f(s) \leq \overline{A_f(s)}^o$ and hence $\overline{A_f(s)} \leq \overline{\overline{A_f(s)}^o}$. Thus, $\overline{A_f(s)}$ is fs-regular closed. □

The set of all fs-preopen sets in X , is denoted by $FSPO(X)$.

Theorem 2.5. In an FSTS $(X, \delta(s))$, (i) $\delta(s) \subseteq FSPO(X)$, (ii) If $V_f(s) \in FSPO(X)$ and $U_f(s) \leq V_f(s) \leq \overline{U_f(s)}$, then $U_f(s) \in FSPO(X)$.

Proof. (i) Follows from definition.

(ii) Let $V_f(s) \in FSPO(X)$, that is, $V_f(s) \leq \overline{(\overline{V_f(s)})^o}$. We have,

$$U_f(s) \leq V_f(s) \leq \overline{U_f(s)}$$

Therefore, $U_f(s) \leq V_f(s) \leq \overline{(\overline{V_f(s)})^o} \leq \overline{(\overline{U_f(s)})^o}$. Hence the result. □

Definition 2.3. An fs-set $A_f(s)$ in an FSTS, is called an fs-preneighbourhood of an fs-point $P_f(s) = (p_{fx}^M, r)$, if there exists an fs-preopen set $B_f(s)$ such that $P_f(s) \leq B_f(s) \leq A_f(s)$.

Theorem 2.6. For an fs-set $A_f(s)$ in an FSTS $(X, \delta(s))$, the following are equivalent:

- (i) $A_f(s)$ is fs-preopen.
- (ii) There exists an fs-regular open set $B_f(s)$ containing $A_f(s)$ such that $\overline{A_f(s)} = \overline{B_f(s)}$.
- (iii) ${}_scl(A_f(s)) = \overline{(A_f(s))^o}$.
- (iv) The semi-closure of $A_f(s)$ is fs-regular open.
- (v) $A_f(s)$ is an fs-preneighbourhood of each of its fs-points.

Proof. (i) \Rightarrow (ii) Let $A_f(s)$ be fs-preopen. This implies

$$\begin{aligned} A_f(s) &\leq \overline{(A_f(s))^o} \leq \overline{A_f(s)} \\ \Rightarrow \overline{A_f(s)} &\leq \overline{\overline{(A_f(s))^o}} \leq \overline{A_f(s)} \\ \Rightarrow \overline{\overline{(A_f(s))^o}} &= \overline{A_f(s)} \\ \Rightarrow \overline{A_f(s)} &= \overline{B_f(s)} \end{aligned}$$

where $B_f(s) = \overline{(A_f(s))^o}$ is an fs-regular open set containing $A_f(s)$.

(ii) \Rightarrow (iii) Let $\overline{A_f(s)} = \overline{B_f(s)}$, where $B_f(s)$ is an fs-regular open set containing $A_f(s)$. Then,

$$A_f(s) \leq B_f(s) = \overline{(B_f(s))^o} = \overline{(A_f(s))^o}$$

Also, $\overline{(A_f(s))^o}$ is fs-semiclosed. Let $C_f(s)$ be an fs-semiclosed set containing $A_f(s)$. Thus,

$$\overline{(A_f(s))^o} \leq \overline{(C_f(s))^o} \leq C_f(s).$$

Hence ${}_scl(A_f(s)) = \overline{(A_f(s))^o}$.

(iii) \Rightarrow (iv) Obvious.

(iv) \Rightarrow (i) Suppose ${}_scl(A_f(s))$ is fs-regular open. Now,

$$\begin{aligned} A_f(s) &\leq {}_scl(A_f(s)) \\ \Rightarrow \overline{(A_f(s))^o} &\leq \overline{({}_scl(A_f(s)))^o} = {}_scl(A_f(s)) \leq \overline{(A_f(s))^o} \\ \Rightarrow A_f(s) &\leq {}_scl(A_f(s)) = \overline{(A_f(s))^o} \end{aligned}$$

Thus, $A_f(s)$ is fs-preopen.

(i) \Rightarrow (v) and (v) \Rightarrow (i) are obvious. □

Corollary 2.2. An fs-set is fs-regular open if and only if it is fs-semiclosed and fs-preopen.

Proof. Proof is omitted. □

Theorem 2.7. In an FSTS $(X, \delta(s))$, the following are equivalent:

- (i) $\overline{A_f(s)} \in \delta(s)$ for all $A_f(s) \in \delta(s)$.
- (ii) Every fs-regular closed set in X is fs-preopen.
- (iii) Every fs-semiopen set in X is fs-preopen.
- (iv) The closure of every fs-preopen set in X is fs-open.
- (v) The closure of every fs-preopen set in X is fs-preopen.

Proof. (i) \Rightarrow (ii) Let $A_f(s)$ be an fs-regular closed set, that is, $\overline{\overset{o}{A_f(s)}} = A_f(s)$. By (i), $A_f(s) \in \delta(s)$ and hence $A_f(s)$ is fs-preopen.

(ii) \Rightarrow (iii) Let $A_f(s)$ be an fs-semiopen set, that is, $A_f(s) \leq \overline{\overset{o}{A_f(s)}}$. By (ii), $\overline{\overset{o}{A_f(s)}}$ is fs-preopen.

Also, we have, $A_f(s) \leq \overline{\overset{o}{A_f(s)}} \leq \overline{A_f(s)}$. Thus, $A_f(s)$ is fs-preopen.

(iii) \Rightarrow (iv) Let $A_f(s)$ be an fs-preopen set, that is, $A_f(s) \leq \overline{(A_f(s))^o}$. This implies, $\overline{A_f(s)} \leq \overline{(A_f(s))^o}$. Thus, $\overline{A_f(s)}$ being fs-semiopen, is fs-preopen and the result follows.

(iv) \Rightarrow (v) Obvious.

(v) \Rightarrow (i) Let $A_f(s) \in \delta(s)$. Then, $A_f(s)$ is fs-preopen and hence $\overline{A_f(s)}$ is fs-preopen. Therefore, $\overline{A_f(s)} \leq \overline{(A_f(s))^o} \leq \overline{A_f(s)}$ and hence $\overline{A_f(s)} \in \delta(s)$. □

Theorem 2.8. In an FSTS $(X, \delta(s))$, the following are equivalent:

- (i) Every non-zero fs-open set is fs-dense.
- (ii) For every non-zero fs-preopen set $A_f(s)$, we have ${}_scl(A_f(s)) = X_f^1(s)$.
- (iii) Every non-zero fs-preopen set is fs-dense.

Proof. (i) \Rightarrow (ii) Let $A_f(s)$ be a non-zero fs-preopen set. By Theorem 2.6 (iii), ${}_scl(A_f(s)) = \overline{(A_f(s))^o}$. Also, there exists an fs-open set $O_f(s)$ such that $A_f(s) \leq O_f(s) \leq \overline{A_f(s)}$. By (i), $\overline{O_f(s)} = X_f^1(s)$. Therefore, $\overline{A_f(s)} = X_f^1(s)$ and hence ${}_scl(A_f(s)) = X_f^1(s)$.

(ii) \Rightarrow (iii) Easy to prove.

(iii) \Rightarrow (i) Since every fs-open set is fs-preopen, the proof is straightforward. □

Definition 2.4. Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. We define fs-preclosure ${}_pcl(A_f(s))$ and fs-preinterior ${}_pint(A_f(s))$ of $A_f(s)$ by

$$\begin{aligned} {}_pcl(A_f(s)) &= \wedge \{B_f(s); A_f(s) \leq B_f(s) \text{ and } B_f^c(s) \in FSPO(X)\} \\ {}_pint(A_f(s)) &= \vee \{C_f(s); C_f(s) \leq A_f(s) \text{ and } C_f(s) \in FSPO(X)\} \end{aligned}$$

Clearly, ${}_pcl(A_f(s))$ is the smallest fs-preclosed set containing $A_f(s)$ and ${}_pint(A_f(s))$ is the largest fs-preopen set contained in $A_f(s)$. Further,

- (i) $A_f(s) \leq {}_pcl(A_f(s)) \leq \overline{A_f(s)}$ and $\overset{o}{A_f(s)} \leq {}_pint(A_f(s)) \leq A_f(s)$.
- (ii) $A_f(s)$ is fs-preopen if and only if $A_f(s) = {}_pint(A_f(s))$
- (iii) $A_f(s)$ is fs-preclosed if and only if $A_f(s) = {}_pcl(A_f(s))$
- (iv) $A_f(s) \leq B_f(s) \Rightarrow {}_pint(A_f(s)) \leq {}_pint(B_f(s))$ and ${}_pcl(A_f(s)) \leq {}_pcl(B_f(s))$.

Definition 2.5. A mapping $g : (X, \delta(s)) \rightarrow (Y, \delta'(s))$ is said to be

(i) fs-precontinuous if $g^{-1}(B_f(s))$ is fs-preopen in X , for every $B_f(s) \in \delta'(s)$.

(ii) fs-preopen if $g(A_f(s))$ is fs-preopen in Y , for every $A_f(s) \in \delta(s)$.

(iii) fs-preclosed if $g(A_f(s))$ is fs-preclosed in Y , for every fs-closed set $A_f(s)$ in X .

It is easy to check that an fs-continuous (fs-open, fs-closed) function is fs-precontinuous (fs-preopen, fs-preclosed). That the converse may not be true, is shown by Example 2.2.

Example 2.2. Consider the fs-sets $A_f(s), B_f(s), C_f(s)$ in a set X , defined as follows:

$$A_f(s) = \left\{ \frac{1}{4}, \bar{1}, \bar{1}, \dots \right\}$$

$$B_f(s) = \left\{ \frac{1}{2}, \bar{0}, \bar{0}, \dots \right\}$$

$$C_f(s) = \left\{ \frac{3}{8}, \bar{1}, \bar{1}, \dots \right\}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Let $\delta'(s) = \{C_f(s), X_f^0(s), X_f^1(s)\}$ and define $g : (X, \delta(s)) \rightarrow (X, \delta'(s))$ by $g(x) = x$ for all $x \in X$. The function g is fs-precontinuous but not fs-continuous.

Again, the map $h : (X, \delta'(s)) \rightarrow (X, \delta(s))$ defined by $h(x) = x$ for all $x \in X$, is both fs-preopen and fs-preclosed but neither fs-open nor fs-closed.

Theorem 2.9. Let $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ be a map. Then the following conditions are equivalent:

(i) g is fs-precontinuous.

(ii) the inverse image of an fs-closed set in Y under g , is fs-preclosed in X .

(iii) For any fs-set $A_f(s)$ in X , $g(pcl(A_f(s))) \leq \overline{g(A_f(s))}$.

Proof. (i) \Rightarrow (ii) Suppose g be an fs-precontinuous map and $B_f(s)$ be an fs-closed set in Y . Then,

$$B_f^c(s) \text{ is fs-open in } Y$$

$$\Rightarrow (g^{-1}(B_f(s)))^c = g^{-1}(B_f^c(s)) \text{ is fs-preopen in } X$$

$$\Rightarrow g^{-1}(B_f(s)) \text{ is fs-preclosed in } X.$$

(ii) \Rightarrow (iii) Let $A_f(s)$ be an fs-set in X . Then, $g^{-1}(\overline{g(A_f(s))})$ is fs-preclosed in X and hence $g^{-1}(\overline{g(A_f(s))}) = pcl(g^{-1}(\overline{g(A_f(s))}))$. Again,

$$A_f(s) \leq g^{-1}(g(A_f(s)))$$

$$\Rightarrow pcl(A_f(s)) \leq pcl(g^{-1}(g(A_f(s)))) \leq g^{-1}(\overline{g(A_f(s))})$$

$$\Rightarrow g(pcl(A_f(s))) \leq g(g^{-1}(\overline{g(A_f(s))})) \leq \overline{g(A_f(s))}.$$

(iii) \Rightarrow (i) Let $B_f(s)$ be an fs-open set in Y . Then for the fs-closed set $B_f^c(s)$, we have

$$g(pcl(g^{-1}(B_f^c(s)))) \leq \overline{g(g^{-1}(B_f^c(s)))} \leq \overline{B_f^c(s)} = B_f^c(s)$$

Thus, ${}_p cl(g^{-1}(B_f^c(s))) \leq g^{-1}(B_f^c(s))$. Therefore, ${}_p cl(g^{-1}(B_f^c(s))) = g^{-1}(B_f^c(s))$ and hence $(g^{-1}(B_f(s)))^c = g^{-1}(B_f^c(s))$ is fs-preclosed in X . \square

In Theorem 2.8 (Tamang & Sarkar, 2016), it has been proved that the inverse image of an fs-semiopen set is fs-semiopen, under an fs-semicontinuous open map. The next Theorem shows that the result is true even if we take an fs-semicontinuous preopen map.

Theorem 2.10. *Suppose $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ be an fs-semicontinuous preopen mapping. Then the inverse image of every fs-semiopen set in Y under g , is fs-semiopen in X .*

Proof. Let $B_f(s)$ be an fs-semiopen set in Y . Then there exists an fs-open set $O_f(s)$ in Y such that

$$\begin{aligned} O_f(s) \leq B_f(s) \leq \overline{O_f(s)} \\ \Rightarrow g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq g^{-1}(\overline{O_f(s)}) \end{aligned}$$

We claim that $g^{-1}(\overline{O_f(s)}) \leq \overline{g^{-1}(O_f(s))}$. Let $P_f(s) \in \overline{g^{-1}(O_f(s))}$. This implies $g(P_f(s)) \in \overline{O_f(s)}$. Consider a weak open Q-nbd $U_f(s)$ of $P_f(s)$, then $g(U_f(s))$ is a weak Q-nbd of $g(P_f(s))$. Therefore,

$$\begin{aligned} & \overline{g(U_f(s))} q_w O_f(s) \\ \Rightarrow & W_f(s) q_w O_f(s) \text{ where } W_f(s) = \overline{g(U_f(s))} \\ \Rightarrow & W_f^n(y) + O_f^n(y) > 1 \text{ for some } y \in Y \\ \Rightarrow & O_f(s) \text{ is a weak open Q-nbd of the fs-point } (p_{fy}^n, W_f^n(y)) \\ \Rightarrow & g(U_f(s)) q_w O_f(s) \\ \Rightarrow & U_f(s) q_w g^{-1}(O_f(s)) \\ \Rightarrow & P_f(s) \in \overline{g^{-1}(O_f(s))}. \end{aligned}$$

Thus $g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq \overline{g^{-1}(O_f(s))}$. Since $g^{-1}(O_f(s))$ is fs-semiopen, $g^{-1}(B_f(s))$ is fs-semiopen. \square

Corollary 2.3. *Suppose $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ be an fs-semicontinuous preopen mapping. Then the inverse image of every fs-semiclosed set in Y under g , is fs-semiclosed in X .*

Proof. The proof is omitted. \square

Corollary 2.4. *Suppose $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ be an fs-semicontinuous preopen map and $h : (Y, \eta(s)) \rightarrow (Z, \delta'(s))$ be fs-semicontinuous. Then hog is fs-semicontinuous.*

Proof. The proof is omitted. \square

Theorem 2.11. *Suppose $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ be an fs-precontinuous preopen mapping. Then the inverse image of every fs-preopen set in Y under g , is fs-preopen in X .*

Proof. Let $B_f(s)$ be an fs-preopen set in Y . Then there exists an fs-open set $O_f(s)$ in Y such that

$$\begin{aligned} B_f(s) &\leq O_f(s) \leq \overline{B_f(s)} \\ \Rightarrow g^{-1}(B_f(s)) &\leq g^{-1}(O_f(s)) \leq g^{-1}(\overline{B_f(s)}). \end{aligned}$$

As in Theorem 2.10, we have $g^{-1}(\overline{B_f(s)}) \leq \overline{g^{-1}(B_f(s))}$. Thus $g^{-1}(B_f(s)) \leq g^{-1}(O_f(s)) \leq \overline{g^{-1}(B_f(s))}$, where $g^{-1}(O_f(s))$ is fs-preopen. Hence $g^{-1}(B_f(s))$ is fs-preopen. \square

Corollary 2.5. *Suppose $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ be an fs-precontinuous preopen mapping. Then the inverse image of every fs-preclosed set in Y under g , is fs-preclosed in X .*

Proof. The proof is omitted. \square

Corollary 2.6. *Suppose $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ be an fs-precontinuous preopen map and $h : (Y, \eta(s)) \rightarrow (Z, \delta'(s))$ be an fs-precontinuous map. Then hog is fs-precontinuous.*

Proof. The proof is omitted. \square

Theorem 2.12. *Suppose $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ be an fs-continuous open map. Then the g -image of an fs-preopen set in X , is fs-preopen in Y .*

Proof. Let $A_f(s)$ be an fs-preopen set in X . Then there exists an fs-open set $O_f(s)$ in X such that $A_f(s) \leq O_f(s) \leq \overline{A_f(s)}$. This implies $g(A_f(s)) \leq g(O_f(s)) \leq \overline{g(A_f(s))}$. Since $g(O_f(s))$ is fs-open in Y , $g(A_f(s))$ is fs-preopen. \square

Corollary 2.7. *Pre-openness in an FSTS, is a topological property.*

Proof. Proof follows from Theorem 2.12. \square

Theorem 2.13. *Let $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ and $h : (Y, \eta(s)) \rightarrow (Z, \delta'(s))$ be two mappings, such that hog is fs-preclosed. Then g is fs-preclosed if h is an injective fs-precontinuous preopen mapping.*

Proof. Let $A_f(s)$ be an fs-closed set in X . Then, $hog(A_f(s))$ is fs-preclosed in Z and hence $g(A_f(s)) = h^{-1}(hog(A_f(s)))$ is fs-preclosed in Y . \square

Theorem 2.14. *If $g : (X, \delta(s)) \rightarrow (Y, \delta'(s))$ be fs-precontinuous and $h : (Y, \delta'(s)) \rightarrow (Z, \eta(s))$ be fs-continuous, then $hog : (X, \delta(s)) \rightarrow (Z, \eta(s))$ is fs-precontinuous.*

Proof. Omitted. \square

Previously, we showed that the intersection of any two fs-preopen sets may not be fs-preopen and an fs-preopen set may not be fs-open. Now, we investigate and establish conditions, under which the intersection of any two fs-preopen sets is fs-preopen and conditions, under which an fs-preopen set is fs-open.

Theorem 2.15. *The intersection of any two fs-preopen sets is fs-preopen if the closure is preserved under finite intersection.*

Proof. Proof is simple and hence omitted. □

Theorem 2.16. *In an FSTS $(X, \delta(s))$, if every fs-set is either fs-open or fs-closed, then every fs-preopen set in X is fs-open.*

Proof. Let $A_f(s)$ be an fs-preopen set in X . If $A_f(s)$ is not fs-open, then it is fs-closed and hence $\overline{A_f(s)} = A_f(s)$. Therefore, $A_f(s) \leq \overline{(A_f(s))^o} = \overset{o}{A_f(s)}$ and hence the theorem. □

For a fuzzy sequential topological space $(X, \delta(s))$, $\delta^*(s)$ will denote the fuzzy sequential topology on X , obtained by taking $FSPO(X)$ as a subbase.

Definition 2.6. A mapping $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ is called strongly fs-precontinuous if the inverse image of each fs-preopen set in Y is fs-open in X .

By the definition of a strong fs-precontinuous mapping, the following two results are obvious.

Theorem 2.17. (i) *A map $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ is strongly fs-precontinuous if and only if $g : (X, \delta(s)) \rightarrow (Y, \eta^*(s))$ is fs-continuous.*

(ii) *If $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ is strongly fs-precontinuous, then it is fs-continuous.*

Remark. Converse of (ii) of Theorem 2.17 may not be true, as is shown by the following Example.

Example 2.3. Consider the fs-sets $A_f(s), B_f(s), C_f(s)$ in a set X , defined as follows:

$$\begin{aligned}
 A_f(s) &= \left\{ \overline{\frac{1}{4}}, \overline{1}, \overline{1}, \dots \right\} \\
 B_f(s) &= \left\{ \overline{\frac{1}{2}}, \overline{0}, \overline{0}, \dots \right\} \\
 C_f(s) &= \left\{ \overline{\frac{3}{8}}, \overline{1}, \overline{1}, \dots \right\}
 \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Consider the identity map $id : (X, \delta(s)) \rightarrow (X, \delta(s))$. Then, id is fs-continuous but not strongly fs-precontinuous, as the inverse image of fs-preopen set $C_f(s)$ is not fs-open.

We conclude the section with a necessary and sufficient condition for an fs-preopen set to be fs-open.

Theorem 2.18. *In an FSTS $(Y, \eta(s))$, the following are equivalent:*

(i) *Every fs-preopen set in Y is fs-open.*

(ii) *Every fs-continuous function $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ is strongly fs-precontinuous, where $(X, \delta(s))$ is any FSTS.*

Proof. (i) \Rightarrow (ii) is straightforward.

(ii) \Rightarrow (i) The identity map $g : (Y, \eta(s)) \rightarrow (Y, \eta(s))$ is fs-continuous and hence is strongly fs-precontinuous. Let $B_f(s)$ be an fs-preopen set in Y , then $B_f(s) = g^{-1}(B_f(s))$ is fs-open in Y . □

3. Decomposition of Continuity

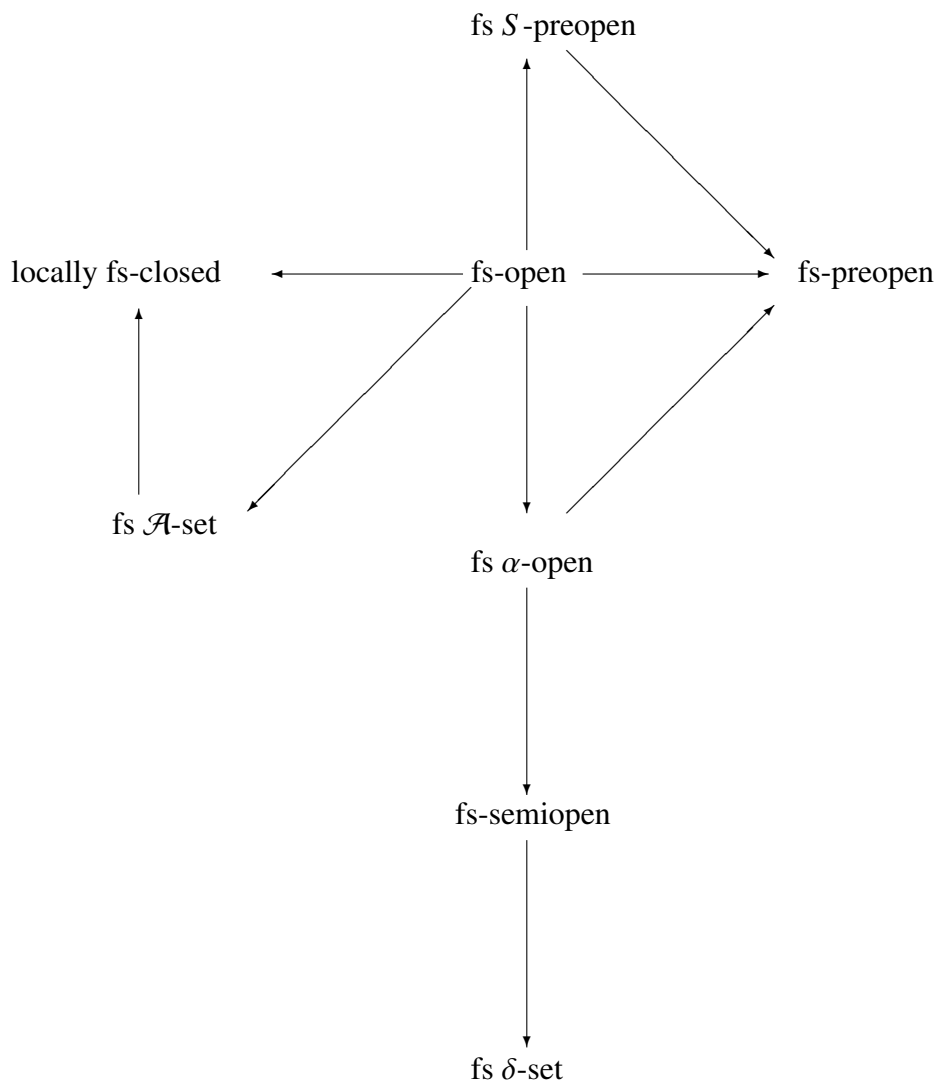
In (Tamang & Sarkar, 2016) and in the last section, nearly open sets like fs-semiopen, fs-preopen and fs-regular open sets in a fuzzy sequential topological space have been studied. Here, we study some more of such sets and the respective continuities and the section is concluded establishing a few decompositions of fs-continuity. In this section, for our convenience, we denote the closure and interior by cl and int respectively.

Definition 3.1. Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is called

- (i) fs α -open if $A_f(s) \leq int\ cl\ int A_f(s)$;
- (ii) locally fs-closed if $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $V_f(s)$ is fs-closed;
- (iii) an fs \mathcal{A} -set if $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $V_f(s)$ is fs-regular closed;
- (iv) an fs δ -set if $int\ cl A_f(s) \leq cl\ int A_f(s)$;
- (v) fs S-preopen if $A_f(s)$ is fs-preopen and $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $int V_f(s)$ is fs-regular open.

We denote the collection of all fs α -open sets, fs-semiopen sets, fs-preopen sets, fs \mathcal{A} -sets, fs S-preopen sets, locally fs-closed sets and fs δ -sets in an FSTS $(X, \delta(s))$, by $\alpha(X)$, $FSSO(X)$, $FSPO(X)$, $\mathcal{A}(X)$, $FSSPO(X)$, $FSLC(X)$ and $\delta(X)$ respectively.

The relationships among different fs-sets defined above, are given by the following diagram:



The implications in the above diagram are not reversible. To show this, here we give examples. In (Tamang & Sarkar, 2016) and in Section 2 respectively, it is already shown that an fs-semiopen and an fs-preopen set may not be fs-open.

Example 3.1. Example to show that an fs α -open set may not be fs-open.

Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X , defined as follows:

$$\begin{aligned}
 A_f(s) &= \left\{ \overline{\frac{1}{4}}, \overline{1}, \overline{1}, \dots \right\} \\
 B_f(s) &= \left\{ \overline{\frac{1}{2}}, \overline{0}, \overline{0}, \dots \right\} \\
 C_f(s) &= \left\{ \overline{\frac{3}{8}}, \overline{1}, \overline{1}, \dots \right\}
 \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS, where $C_f(s)$ is fs α -open but is not fs-open.

Example 3.2. Example to show that a locally fs-closed set may not be fs-open.

Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X , defined as follows:

$$\begin{aligned}
 A_f(s) &= \left\{ \overline{\frac{1}{4}}, \overline{\frac{3}{4}}, \overline{\frac{1}{4}}, \overline{\frac{3}{4}}, \dots \right\} \\
 B_f(s) &= \left\{ \overline{\frac{1}{4}}, \overline{\frac{1}{4}}, \overline{\frac{1}{4}}, \overline{\frac{1}{4}}, \dots \right\}
 \end{aligned}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS, where $B_f(s)$ is locally fs-closed but not fs-open.

Example 3.3. Example to show that an fs \mathcal{A} -set may not be fs-open.

Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$, $D_f(s)$ in a set X , defined as follows:

$$\begin{aligned}
 A_f(s) &= \left\{ \overline{1}, \overline{\frac{1}{2}}, \overline{1}, \overline{\frac{1}{2}}, \dots \right\} \\
 B_f(s) &= \left\{ \overline{\frac{1}{2}}, \overline{0}, \overline{\frac{1}{2}}, \overline{0}, \dots \right\} \\
 C_f(s) &= \left\{ \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \dots \right\} \\
 D_f(s) &= \left\{ \overline{\frac{1}{2}}, \overline{1}, \overline{\frac{1}{2}}, \overline{1}, \dots \right\}
 \end{aligned}$$

Consider the fuzzy sequential topological space $(X, \delta(s))$, where $\delta(s) = \{A_f(s), B_f(s), X_f^0(s), X_f^1(s)\}$. Here, $C_f(s) = A_f(s) \wedge D_f(s)$, where $A_f(s)$ is fs-open and $D_f(s)$ is fs-regular closed. Hence $C_f(s)$ is an fs \mathcal{A} -set but is not fs-open.

Example 3.4. Example to show that a locally fs-closed set may not be an fs \mathcal{A} -set.

Consider the FSTS $(X, \delta(s))$, given in Example 3.2. Here, $B_f(s)$ is a locally fs-closed set but not an fs \mathcal{A} -set.

Example 3.5. Example to show that an fs-semiopen set may not be an fs \mathcal{A} -set.

Consider the FSTS $(X, \delta(s))$, given in Example 3.1.

The fs-set $C_f(s)$ is an fs-semiopen set but not an fs \mathcal{A} -set.

Example 3.6. Example to show that an fs-semiopen set may not be fs α -open.

In the FSTS, given in Example 3.3, the fs-set $C_f(s)$ is fs-semiopen but not fs α -open.

Example 3.7. Example to show that an fs-preopen set may not be fs α -open.

Consider the fs-sets $A_f(s), B_f(s)$ in a set X , defined as follows:

$$A_f(s) = \left\{ \frac{\overline{1}}{4}, \frac{\overline{1}}{4}, \frac{\overline{1}}{4}, \dots \right\}$$

$$B_f(s) = \left\{ \frac{\overline{1}}{5}, \frac{\overline{1}}{5}, \frac{\overline{1}}{5}, \dots \right\}$$

Then $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ is a fuzzy sequential topology on X . In this FSTS, $B_f(s)$ is fs-preopen but not fs α -open.

Example 3.8. Example to show that an fs δ -set may not be fs-semiopen.

In the FSTS, given in Example 3.2, the fs-set $B_f(s)$ is an fs δ -set but is not fs-semiopen.

Example 3.9. Example to show that an fs-preopen set may not be an fs S -preopen set.

Consider the FSTS, given in Example 3.1, the fs-set $C_f(s)$ is fs-preopen but not fs S -preopen.

Example 3.10. Example to show that an fs S -preopen set may not be an fs-open set.

Consider the FSTS, given in Example 3.7, the fs-set $B_f(s)$ is fs S -preopen but not fs-open.

Definition 3.2. A mapping $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ is called

- (i) fs α -continuous if $g^{-1}(B_f(s))$ is fs α -open in X , for every fs-open set $B_f(s)$ in Y .
- (ii) fs lc -continuous if $g^{-1}(B_f(s))$ is locally fs-closed in X , for every fs-open set $B_f(s)$ in Y .
- (iii) fs \mathcal{A} -continuous if $g^{-1}(B_f(s))$ is an fs \mathcal{A} -set in X , for every fs-open set $B_f(s)$ in Y .
- (iv) fs δ -continuous if $g^{-1}(B_f(s))$ is an fs δ -set in X , for every fs-open set $B_f(s)$ in Y .
- (v) fs S -precontinuous if $g^{-1}(B_f(s))$ is fs S -preopen in X , for every fs-open set $B_f(s)$ in Y .

Theorem 3.1. An fs-set in an FSTS, is fs α -open if and only if it is fs-semiopen and fs-preopen.

Proof. Let $A_f(s)$ be an fs α -open set, that is, $A_f(s) \leq \text{int } cl \text{ int } A_f(s)$. Clearly, $A_f(s)$ is fs-semiopen. Also,

$$\begin{aligned} \text{int } A_f(s) \leq cl A_f(s) &\Rightarrow cl \text{ int } A_f(s) \leq cl A_f(s) \\ &\Rightarrow \text{int } cl \text{ int } A_f(s) \leq \text{int } cl A_f(s) \\ &\Rightarrow A_f(s) \leq \text{int } cl A_f(s) \end{aligned}$$

Thus, $A_f(s)$ is fs-preopen.

Conversely, suppose $A_f(s)$ be fs-semiopen and fs-preopen, that is, $A_f(s) \leq cl\ int A_f(s)$, $A_f(s) \leq int\ cl A_f(s)$. Then,

$$\begin{aligned} int\ cl A_f(s) &\leq cl A_f(s) \leq cl\ int A_f(s) \\ \Rightarrow A_f(s) &\leq int\ cl\ int A_f(s) \end{aligned}$$

Hence, $A_f(s)$ is fs α -open. □

Corollary 3.1. A mapping $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ is fs α -continuous if and only if it is fs-semicontinuous and fs-precontinuous.

Definition 3.3. Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. Then α fs-closure ${}_{\alpha}cl A_f(s)$ and α fs-interior ${}_{\alpha}int A_f(s)$ of $A_f(s)$ are defined by

$$\begin{aligned} {}_{\alpha}cl A_f(s) &= \bigwedge \{V_f(s); A_f(s) \leq V_f(s) \text{ and } V_f^c(s) \in \alpha(X)\} \\ {}_{\alpha}int A_f(s) &= \bigvee \{U_f(s); U_f(s) \leq A_f(s) \text{ and } U_f(s) \in \alpha(X)\} \end{aligned}$$

Complement of an fs α -open set is called an fs α -closed set. Hence, it is clear that ${}_{\alpha}cl(A_f(s))$ is the smallest fs α -closed set containing $A_f(s)$ and ${}_{\alpha}int(A_f(s))$ is the largest fs α -open set contained in $A_f(s)$. Further,

- (i) $A_f(s) \leq {}_{\alpha}cl(A_f(s)) \leq \overline{A_f(s)}$ and $\overset{\circ}{A_f(s)} \leq {}_{\alpha}int(A_f(s)) \leq A_f(s)$.
- (ii) $A_f(s)$ is fs α -open if and only if $A_f(s) = {}_{\alpha}int(A_f(s))$
- (iii) $A_f(s)$ is fs α -closed if and only if $A_f(s) = {}_{\alpha}cl(A_f(s))$
- (iv) $A_f(s) \leq B_f(s) \Rightarrow {}_{\alpha}int(A_f(s)) \leq {}_{\alpha}int(B_f(s))$ and ${}_{\alpha}cl(A_f(s)) \leq {}_{\alpha}cl(B_f(s))$.

Theorem 3.2. Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. Then,

- (i) ${}_{\alpha}int A_f(s) = A_f(s) \wedge int\ cl\ int A_f(s)$.
- (ii) if $A_f(s)$ is both fs-preopen and fs-preclosed, then $A_f(s) = int\ cl A_f(s) \wedge A_f(s)$ and thus $A_f(s)$ is fs S-preopen;
- (iii) if $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $int V_f(s)$ is fs-regular open, then ${}_{\alpha}int A_f(s) = int A_f(s)$;
- (iv) if $A_f(s)$ is an fs δ -set, then ${}_{\alpha}int A_f(s) = {}_p int A_f(s)$.

Proof. (i) Easy to prove.

(ii) Given $A_f(s) \leq int\ cl A_f(s)$ and $cl\ int A_f(s) \leq A_f(s)$. Then,

$$A_f(s) = int\ cl A_f(s) \wedge A_f(s).$$

Since $int A_f(s) = int\ cl\ int A_f(s)$, hence $A_f(s)$ is fs S-preopen.

(iii) We have $int\ cl\ int A_f(s) \leq int\ cl\ int V_f(s) = int V_f(s)$. Therefore,

$$\begin{aligned} {}_{\alpha}int A_f(s) &= A_f(s) \wedge int\ cl\ int A_f(s) \\ &\leq A_f(s) \wedge int V_f(s) \\ &= int A_f(s) \end{aligned}$$

Also, $intA_f(s) \leq \alpha intA_f(s)$. Hence $intA_f(s) = \alpha intA_f(s)$.

(iv) Given $int clA_f(s) \leq cl intA_f(s)$. Since $\alpha intA_f(s)$ is an fs-preopen set contained in $A_f(s)$, we have

$$\alpha intA_f(s) \leq \rho intA_f(s)$$

Now,

$$\rho intA_f(s) \leq int clA_f(s) \leq int cl intA_f(s)$$

Thus, $\rho intA_f(s) \leq A_f(s) \wedge int cl intA_f(s) = \alpha intA_f(s)$. Hence the result. □

Lemma 3.1. *An fs-set $A_f(s)$ is locally fs-closed if and only if $A_f(s) = U_f(s) \wedge cl(A_f(s))$, where $U_f(s)$ is an fs-open set.*

Proof. Omitted. □

Theorem 3.3. *Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. Then $A_f(s)$ is an fs \mathcal{A} -set if it is fs-semiopen and locally fs-closed.*

Proof. Suppose $A_f(s)$ be fs-semiopen and locally fs-closed. Then, $A_f(s) \leq cl intA_f(s)$ and $A_f(s) = U_f(s) \wedge clA_f(s)$, where $U_f(s)$ is fs-open. Since $clA_f(s) = cl intA_f(s)$ is fs-regular closed, the result follows. □

Corollary 3.2. *A mapping $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ is fs \mathcal{A} -continuous if it is fs-semicontinuous and fs lc-continuous.*

Remark. Unlike in a general topological space, the converse of Theorem 3.3 may not be true and it has been shown by the next Example.

Example 3.11. Let $X = \{x, y\}$. Consider the fs-sets $A_f(s), B_f(s), C_f(s), D_f(s)$ and $E_f(s)$ in X , where

$$\begin{aligned} A_f^1 &= \overline{0.3}, A_f^n(x) = 1 \text{ and } A_f^n(y) = 0 \text{ for all } n \neq 1; \\ B_f^1(x) &= 0.4, B_f^1(y) = 0.7, B_f^n(x) = 0 \text{ and } B_f^n(y) = 1 \text{ for all } \\ &n \neq 1; \\ C_f^1 &= \overline{0.7} \text{ and } C_f^n = \overline{1} \text{ for all } n \neq 1; \\ D_f^1(x) &= 0.6, D_f^1(y) = 0.3, D_f^n(x) = 1 \text{ and } D_f^n(y) = 0 \text{ for all } \\ &n \neq 1; \\ E_f^1(x) &= 0.4, E_f^1(y) = 0.3 \text{ and } E_f^n = \overline{0} \text{ for all } n \neq 1. \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), C_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. In the FSTS $(X, \delta(s))$, $D_f(s)$ being an fs-regular closed set, the fs-set $E_f(s) = B_f(s) \wedge D_f(s)$ is an fs \mathcal{A} -set but not fs-semiopen.

Theorem 3.4. Let $(X, \delta(s))$ be an FSTS and $A_f(s)$ be an fs-set in X . Then the following statements are equivalent:

- (i) $A_f(s)$ is an fs-open set.
- (ii) $A_f(s)$ is fs α -open and locally fs-closed.
- (iii) $A_f(s)$ is fs-preopen and locally fs-closed.
- (iv) $A_f(s)$ is fs-preopen and an fs \mathcal{A} -set.
- (v) $A_f(s)$ is fs S-preopen and an fs δ -set.

Proof. (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are obvious.

(iii) \Rightarrow (iv) Let $A_f(s)$ be fs-preopen and locally fs-closed. Then,

$$A_f(s) \leq \text{int } clA_f(s) \quad \text{and} \quad A_f(s) = U_f(s) \wedge clA_f(s),$$

where $U_f(s)$ is fs-open. $clA_f(s)$ being fs-regular closed, the result follows.

(iv) \Rightarrow (i) Let $A_f(s)$ be an fs-preopen and an fs \mathcal{A} -set. Then,

$$A_f(s) \leq \text{int } clA_f(s) \quad \text{and} \quad A_f(s) = U_f(s) \wedge clA_f(s),$$

where $U_f(s)$ is fs-open. Since $\text{int}A_f(s) = U_f(s) \wedge \text{int } clA_f(s)$, $A_f(s)$ is fs-open.

(i) \Rightarrow (v) Obvious.

(v) \Rightarrow (i) Let $A_f(s)$ be an fs S-preopen and an fs δ -set. Using Theorem 3.2, (iii) and (iv),

$$\text{int}A_f(s) = {}_{\alpha}\text{int}A_f(s) = {}_p\text{int}A_f(s) = A_f(s).$$

Hence, $A_f(s)$ is fs-open. □

By Theorems 3.1, 3.3 and 3.4, we have the following relationships among the different classes of fs-sets of an FSTS $(X, \delta(s))$:

- (i) $\alpha(X) = FSPO(X) \cap FSSO(X)$.
- (ii) $\mathcal{A}(X) \supseteq FSSO(X) \cap FS LC(X)$.
- (iii) $\delta(s) = \alpha(X) \cap FS LC(X)$.
- (iv) $\delta(s) = FSPO(X) \cap FS LC(X)$.
- (v) $\delta(s) = FSPO(X) \cap \mathcal{A}(X)$.
- (vi) $\delta(s) = FSSPO(X) \cap \delta(X)$.

Theorem 3.5. In an FSTS $(X, \delta(s))$, the following are equivalent:

- (i) $clA_f(s) \in \delta(s)$ for every $A_f(s) \in \delta(s)$.
- (v) $\mathcal{A}(X) = \delta(s)$.

Proof. (i) \Rightarrow (ii) It is obvious that $\delta(s) \subseteq \mathcal{A}(X)$. For the reverse inclusion, let $A_f(s) \in \mathcal{A}(X)$, then

$$A_f(s) = U_f(s) \wedge V_f(s),$$

where $U_f(s)$ is fs-open and $V_f(s)$ is fs-regular closed. By (i), $V_f(s) \in \delta(s)$ and hence $A_f(s) \in \delta(s)$.

(ii) \Rightarrow (i) Suppose $\mathcal{A}(X) = \delta(s)$. Let $A_f(s) \in \delta(s)$, then $clA_f(s)$ is fs-regular closed and hence belongs to $\mathcal{A}(X) = \delta(s)$. □

We conclude the chapter by stating the following decompositions of fs-continuity:

Theorem 3.6. *Let $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ be a map. Then g is fs-continuous if and only if*

- (i) g is fs α -continuous and fs lc-continuous.*
- (ii) g is fs-precontinuous and fs lc-continuous.*
- (iii) g is fs-precontinuous and fs \mathcal{A} -continuous.*
- (iv) g is fs S -continuous and fs δ -continuous.*

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Structural Machines as a Mathematical Model of Biological and Chemical Computers

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Abstract

To rigorously describe and study the living morphological computers, we develop a formal model of algorithms and computations called a structural machine providing theoretical tools for exploration of possibilities of biological computations and extension of their applications. We study properties of structural machines demonstrating how they can model the most popular in computer science model of computation — Turing machine. We also prove that structural machines can solve some problems more efficiently than Turing machines. We also show how structural machines model a programmable amorphous biological computer called a Physarum machine, as well as an inductive Turing machine.

Keywords: model of computation, algorithm, structural machine, Turing machine, Physarum machine, bio-inspired computing, computing automaton, slime mould, unconventional computing, computational efficiency.
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1. Introduction

To better understand and further develop information processing, researchers created different mathematical models of algorithms and computation building foundations of theoretical computer science. After development and proliferation of digital computers, Turing machine became the most popular of those models impersonating the paradigm of algorithmic computational devices according to the Church-Turing Thesis. In spite of this, creation of new mathematical models has continued. There were three reasons for this situation in theoretical computer science. First, many researchers tried to build a model more powerful than Turing machine. Regardless of many attempts, the first mathematical model of algorithms and computation, for which it was proved

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that it was really more powerful than Turing machine, was inductive Turing machine constructed in (Burgin, 1983). Note that although Turing machines with an oracle are more powerful than conventional Turing machines, a Turing machine with an oracle is not a constructive theoretical device, while inductive Turing machines are virtually as constructive as conventional Turing machines. Second, by its construction, Turing machine is a very simple edifice well suited for theoretical research but reflecting only some essential features of digital computers while ignoring other features. At the same time, exactly these properties of a Turing machine made it very inefficient as a computing device. Therefore, researchers thrived to build more efficient than Turing machines mathematical models, representing more key features of ever-developing computers. Examples of such models are Kolmogorov algorithms (Kolmogorov, 1953), Random Access Machines (Shepherdson & Sturgis, 1963), Random Access Machines with the Stored Program (Elgot & Robinson, 1964) and some others. The third incentive for the development of new models has been necessity to address various kinds of computations that the conventional Turing machine was not able to properly model. This intent brought forth many novel models such as:

- cellular automata (Von Neumann *et al.*, 1966), Turing machines with many heads and tapes, vector machines and systolic arrays (Kung & Leiserson, 1978) for modeling and exploration of parallel computations;
- Petri nets (Petri, 1962) and grid automata (Burgin, 2003; Burgin & Eberbach, 2009a) for modeling and exploration of concurrent computations;
- formal grammars (Chomsky, 1956) for modeling and exploration of linguistic transformations; and some others.

Discovery of unconventional types of computation and research in building unconventional computers such as biological, chemical, or non-linear physical computers (Adamatzky, 2001; Adamatzky *et al.*, 2005; Calude & Dinneen, 2015; Adamatzky, 2016) demand innovative mathematical models of computation that reflect peculiarity of unconventional computations (Stepney *et al.*, 2005; Cooper, 2013; Kendon *et al.*, 2015) and original models of algorithms that represent control in such computations. This is the exact goal of the development and exploration of structural machines in this paper. The paper is organized as follows. In the second section, we describe structural machines and study their properties. In the third section, we study relations between structural machines and other popular computational models. In particular, we show how structural machines model Turing machines (Theorem 1) and inductive Turing machines (Theorem 3) demonstrating (Theorem 2) that structural machines are more efficient than Turing machines. In the fourth section, we briefly describe Physarum machines, which perform biological computations, and show how structural machines model Physarum machines. In the fifth section, we discuss the obtained results and suggest future directions for research.

2. Structural machines

There are structural machines of different orders. Here we study structural machines of the first order, which work with first-order structures, and structural machines of the second order, which work with first-order and second-order structures.

Definition 2.1. From (Burgin, 2012). A first-order structure is a triad of the form $\mathbf{A} = (A, r, \mathcal{R})$ In this expression, we have:

- the set A , which is called the substance of the structure \mathbf{A} and consists of elements of the structure \mathbf{A} , which are called structure elements of the structure \mathbf{A}
- the set \mathcal{R} , which is called the arrangement of the structure \mathbf{A} and consists of relations in the structure \mathbf{A} , which are called structure relations of the structure \mathbf{A}
- the incidence relation r , which connects groups of elements from A with relations from \mathcal{R} .

For instance, if \mathbf{R} is an n -ary relation from \mathcal{R} and $a_1, a_2, a_3, \dots, a_n$ are elements from A , then the expression $r(\mathbf{R}; a_1, a_2, a_3, \dots, a_n)$ means that the elements $a_1, a_2, a_3, \dots, a_n$ belong to the relation \mathbf{R} with the name R , i.e., r connects the elements $a_1, a_2, a_3, \dots, a_n$ with the relation \mathbf{R} .

Describing structures, it is significant to distinguish relations and their names because when a structural machine functions, relations, as a rule, are changing, while their names can remain the same. For instance, when a structure \mathbf{A} on a set A has a ternary relation \mathbf{R} with the name R and in the process of computation, the machine additionally connects three elements from A by a link assigning it to the relation \mathbf{R} . As a result, \mathbf{R} becomes larger but preserves the same name R .

However, it is also possible, that a structural machine changes names of the structure relations, for example by splitting one relation into two new relations of the same arity.

Lists, queues, words, texts, graphs, directed graphs, labeled graphs, mathematical and chemical formulas, tapes of Turing machines and Kolmogorov complexes are particular cases of structures that have only unary and binary relations. Note that labels, names, types and properties are unary relations.

In the case when the set \mathcal{R} of relations consists of one binary and several unary relations, then the first order structure is a labeled (named) graph. When \mathcal{R} contains only binary and unary relations, then the first order structure is a labeled (named) multigraph.

Example 2.1. Let us consider the first order structure $\mathbf{A} = (A, r, \mathcal{R})$ where $A = \{a, b, c, d, e, f, g, h\}$ and \mathcal{R} consists of one binary relation P and one ternary relation Q , the graphical representation of which is given in Figure 1.

In the case of a Physarum machine, we can interpret elements of the relation Q in Figure 1 as two clusters of nutrients, colonized by a single slime mould. There are higher degrees of connections between growth zones and branches of the protoplasmic network inside clusters but there are only few links bringing these two clusters together.

Another possibility is when a first order structure $A = (A, r, \mathcal{R})$, in which the set \mathcal{R} consists of a single binary relation R can faithfully represent the structure of a living slime mould established by blobs of slime mould and active zones. Namely, elements from the set A represent blobs of slime mould and active zones, while elements from the relation R represent connecting tubes.

However, a slime mould often has a more sophisticated structure. The network of blobs, active zones and protoplasmic tubes is not uniform but forms clusters. These clusters are also connected by thick protoplasmic tubes, which represent the incidence relation that connects groups of elements from A . To model a slime mould with clusters, we need second-order structures.

Definition 2.2. From (Burgin, 2012). A second-order structure is a triad of the form

$$\mathbf{A} = (A, r, \mathcal{R})$$

Here

- the set A , which is called the substance of the structure \mathbf{A} and consists of elements of the structure \mathbf{A} , which are called structure elements of the structure \mathbf{A}
- the set \mathcal{R} , which is called the arrangement of the structure \mathbf{A} and consists of relations in the structure \mathbf{A} , which are called structure relations of the structure \mathbf{A}
- r is the incidence relation that connects groups of elements from A with relations from \mathcal{R}
- $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$
- \mathcal{R}_1 is the set of relations in the set A
- \mathcal{R}_2 is the set of relations in the set \mathcal{R}_1 , i.e., elements from \mathcal{R}_2 are relations between relations from \mathcal{R}_1

Second-order structures are used to represent data processed by structural machines of the second order.

Relations from the set \mathcal{R} determine the intrinsic structure of the structure $\mathbf{A} = (A, r, \mathcal{R})$. However, efficient operation with, utilization and modeling of first-order and higher-order structures demands additional (extrinsic) structures (Burgin, 2012). One of these extrinsic structures is pre-topology (ech, 1966) determined by neighborhoods in the set A .

Definition 2.3. If \mathbf{R} is an n -ary relation from \mathcal{R} , then:

1. the substantial R -neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is a set of the form

$$O_{RS}a = \{a\} \cup \{d; \exists i, k \exists a_2, \dots, a_n \in A ((1 \leq k \leq n) \wedge (a_1, a_2, \dots, a_i = a, \dots, a_k = d, \dots, a_n) \in \mathbf{R})\}$$

2. the link R -neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is a set of the form

$$O_{RL}a = \{(a_1, a_2, \dots, a_n) \in \mathbf{R}; a_1, a_2, \dots, a_n \in O_R a\}$$

3. the full R -neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set

$$O_{RF}a = O_{RS}a \cup O_{RL}a$$

Informally, the substantial R -neighborhood of a structure element a consists of all structure elements connected to a by the relation \mathbf{R} . The link R -neighborhood of a structure element a consists of all elements from the relation \mathbf{R} that contain a . The full R -neighborhood of a structure element a is the union of the substantial R -neighborhood and link R -neighborhood of a .

Examples:

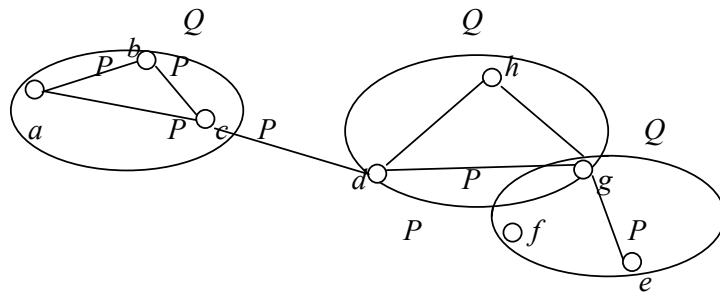


Figure 1. The graphical representation of a first order structure.

1. Taking the structure presented in Fig. 1, we see that $\{a, b, c, f\}$ is the substantial P -neighborhood of the structure element a determined by the relation P from the structure in Fig. 1.
2. Taking the structure presented in Fig. 1, we see that $\{a, b, c\}$ is the substantial Q -neighborhood of the structure element a determined by the relation Q from the structure in Fig. 1.
3. Taking the structure presented in Fig. 1, we see that $\{e, g\}$ is the substantial P -neighborhood of the structure element e determined by the relation P from the structure in Fig. 1.
4. Taking the structure presented in Fig. 1, we see that $\{f, g, e\}$ is the substantial Q -neighborhood of the structure element f determined by the relation Q from the structure in Fig. 1.
5. Taking the structure presented in Fig. 1, we see that $\{(a, b), (a, c)\}$ is the link P -neighborhood of the structure element a determined by the relation Q from the structure in Fig. 1.
6. Taking the structure presented in Fig. 1, we see that $\{(a, b, c)\}$ is the link Q -neighborhood of the structure element a and of the structure element b where both neighborhoods are determined by the relation Q from the structure in Fig. 1.

Definition 3 implies the following result.

Lemma 2.1. *Each structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ has the uniquely defined substantial R -neighborhood.*

Relations between relations from \mathcal{R} imply relations between neighborhoods.

Proposition 2.1. *For any structure element a , the inclusion $R \subseteq Q$ of two relations $R, Q \in \mathcal{R}$ implies inclusions of neighborhoods*

$$O_{RS}a \subseteq O_{QS}a, O_{RL}a \subseteq O_{QL}a \text{ and } O_{RF}a \subseteq O_{QF}a .$$

Neighborhoods of elements allow us to build neighborhoods of sets of elements.

Definition 2.4. If R is an n -ary relation from \mathcal{R} and $Z \subseteq A$, then:

(α) the substantial R -neighborhood of the set Z in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set

$$O_{RS}Z = \cup_{a \in Z} O_{RS}a$$

(β) the link R -neighborhood of the set Z in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is a set of the form

$$O_{RL}Z = \cup_{a \in Z} O_{RL}a$$

(γ) the full R -neighborhood of the set Z in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set

$$O_{RF}Z = O_{RS}Z \cup O_{RL}Z$$

Lemma 1 implies the following result.

Lemma 2.2. *The substantial (link or full) R -neighborhood of any set Z of structure elements is uniquely defined.*

Proposition 1 implies the following result.

Proposition 2.2. *For any set Z of structure elements, the inclusion $R \subseteq Q$ of two relations $R, Q \in \mathcal{R}$ implies inclusions of neighborhoods $O_{RS}Z \subseteq O_{QS}Z, O_{RL}Z \subseteq O_{QL}Z$ and $O_{RF}Z \subseteq O_{QF}Z$.*

Substantial neighborhoods of sets of structure elements determine pretopology in the set A of all structure elements. We remind that a pretopological space is defined as a set X with a preclosure operator (ech closure operator) cl . Let 2^X be the power set of X . A preclosure operator on a set X is a mapping $cl : 2^X \rightarrow 2^X$ that satisfies the following axioms (ech, 1966):

- $cl(\emptyset) = \emptyset$
- $Z \subseteq cl(Z)$ for any $Z \subseteq X$
- $cl(Z \cup Y) \subseteq cl(Z) \cup cl(Y)$ for any $Z, Y \subseteq X$
- $Y \subseteq Z$ implies $cl(Y) \subseteq cl(Z)$ for any $Z, Y \subseteq X$

Properties of substantial R -neighborhoods allow us to prove the following result.

Proposition 2.3. *Substantial R -neighborhoods define a pretopology in the set A .*

Proof. Let us define the closure $cl(Z)$ of a set $Z \subseteq A$ equal to its substantial R -neighborhood $O_{RS}Z$ and check the axioms of pretopological spaces.

Axioms 1 and 2 are true by definition as $cl(\emptyset) = \emptyset$ and $Z \subseteq cl(Z)$ for any $Z \subseteq A$. In addition,
 $cl(Z \cup Y) = O_{RS}(Z \cup Y) = \cup_{a \in Z \cup Y} O_{RS}a = (\cup_{a \in Z} O_{RS}a) \cup (\cup_{a \in Y} O_{RS}a) = O_{RS}Z \cup O_{RS}Y = cl(Z) \cup cl(Y)$

This gives us Axiom 3. Axiom 4 is implied by Proposition 2. □

Definition 2.5. • The substantial \mathcal{R} -neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set

$$O_{RS}a = \cup_{R \in \mathcal{R}} O_{RS}a$$

- The link \mathcal{R} -neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is a set of the form

$$O_{\mathcal{R}\mathbf{L}}a = \cup_{\mathbf{R} \in \mathcal{R}} O_{\mathbf{R}\mathbf{S}}a$$

- The full \mathcal{R} -neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set

$$O_{\mathcal{R}\mathbf{F}}a = \cup_{\mathbf{R} \in \mathcal{R}} O_{\mathbf{R}\mathbf{F}}a$$

Definition 5 implies the following result.

Lemma 2.3. *The substantial (link or full) \mathcal{R} -neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is uniquely defined.*

Properties of union and Definitions 4 and 5 imply the following result.

Lemma 2.4. $O_{\mathcal{R}\mathbf{F}}a = O_{\mathbf{R}\mathbf{S}}a \cup O_{\mathcal{R}\mathbf{L}}a$.

Proposition 3 and Definition 4 imply the following result.

Corollary 2.1. *Substantial \mathcal{R} -neighborhoods define a pretopology in the set A .*

Neighborhoods of elements allow us to build neighborhoods of sets of elements.

Definition 2.6. If $Z \subseteq A$, then:

α : the substantial \mathcal{R} -neighborhood of the set Z in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set

$$O_{\mathcal{R}\mathbf{S}}Z = \cup_{a \in Z} O_{\mathcal{R}\mathbf{S}}a$$

β : the link \mathcal{R} -neighborhood of the set Z in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is a set of the form

$$O_{\mathcal{R}\mathbf{L}}Z = \cup_{a \in Z} O_{\mathcal{R}\mathbf{L}}a$$

γ : the full \mathbf{R} -neighborhood of the set Z in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set

$$O_{\mathcal{R}\mathbf{F}}Z = O_{\mathcal{R}\mathbf{S}}Z \cup O_{\mathcal{R}\mathbf{L}}Z$$

Lemma 3 implies the following result.

Lemma 2.5. *The substantial (link or full) \mathcal{R} -neighborhood of any set Z of structure elements is uniquely defined.*

Let us study properties of neighborhoods in structures.

Definition 2.7. If R is an n -ary relation from \mathcal{R} , then the substantial R -neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is symmetric if for any elements $a_1, a_2, a_3, \dots, a_n$ from A and any permutation $i_1, i_2, i_3, \dots, i_n$ of the numbers $1, 2, 3, \dots, n$, we have

$$(a_1, a_2, a_3, \dots, a_n) \in R$$

if and only if

$$(a_{i_1}, a_{i_2}, a_{i_3}, \dots, a_{i_n}) \in R$$

Lemma 2.6. *If the substantial R -neighborhood $O_{RS}a$ of a structure element a is symmetric, a structure element b belongs to the R -neighborhood $O_{RS}a$ if and only if a belongs to the substantial R -neighborhood $O_{RS}b$ of b .*

Definition 2.8. If \mathbf{R} is an n -ary relation from \mathcal{R} , then the R -neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is transitive if for any elements $a_1, a_2, a_3, \dots, a_n, d_1, d_2, d_3, \dots, d_n$ from A and any numbers i and j from the set $\{1, 2, 3, \dots, n\}$, we have

If $(a_1, a_2, a_3, \dots, a_n) \in \mathbf{R}$ and $(d_1, \dots, a_i, \dots, d_n) \in \mathbf{R}$, then there are elements $b_1, b_2, b_3, \dots, b_n$ from A such that $(b_1, \dots, a_i, \dots, d_j, \dots, b_n) \in \mathbf{R}$

Proposition 2.4. *If a relation $\mathbf{R} \in \mathcal{R}$ is symmetric and transitive, then substantial R -neighborhoods of two structure elements either coincide or do not intersect.*

Corollary 2.2. *If all relations in \mathcal{R} are symmetric and transitive, then substantial \mathcal{R} -neighborhoods of two structure elements either coincide or do not intersect.*

Corollary 2.3. *If \mathcal{R} consists of one symmetric binary relation R , i.e., \mathbf{A} is a graph and R is also transitive, then any substantial \mathcal{R} -neighborhood (R -neighborhood) is a complete graph*

Proposition 2.5. *If a relation $\mathbf{R} \in \mathcal{R}$ is symmetric and transitive, then the system of substantial R -neighborhoods forms a base of a topology in A .*

Corollary 2.4. *If all relations in \mathcal{R} are symmetric and transitive, then the system of substantial \mathcal{R} -neighborhoods forms a base of a topology in A .*

Definition 2.9. The type $T(A)$ of a first-order structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set $\{(R, \alpha(R)); R \in \mathcal{R}\}$ of pairs $(R, \alpha(R))$; where $\alpha(R)$ is the arity of the relation \mathbf{R} with the name R .

f

We assume that two first-order structures $\mathbf{A} = (A, r, \mathcal{R})$ and $\mathbf{B} = (B, p, \mathcal{P})$ have the same type if there is a one-to-one mapping $f : T(A) \rightarrow T(B)$ such that if $f(R, \alpha(R)) = (P, \alpha(P))$, then $\alpha(R) = \alpha(P)$.

For instance, all binary relations have the same type.

A structural machine M works with structures of a given type and has three components:

- The control device C_M regulates the state of the machine M
- The processor P_M performs transformation of the processed structures and its actions (operations) depend on the state of the machine M and the state of the processed structures
- The functional space $S p_M$ consists of three components:
 - The input space In_M , which contains the input structure.
 - The output space Out_M , which contains the output structure.
 - The processing space PS_M , in which the input structure(s) is transformed into the output structure(s).

We assume that all structures — the input structure, the output structure and the processed structures — have the same type.

Computation of a structural machine M determines the trajectory of computation, which is a tree in general case and a sequence in the deterministic case and a single processor unit.

There are two forms functional spaces Sp_M and USp_M :

- Sp_M is the set of all structures that can be processed by the structural machine M and is called a categorical functional space
- USp_M is a structure for which all structures that can be processed by the structural machine M are substructures and is called a universal functional space

There are three basic types of control devices:

- A central control device controls all processors of the structural machine
- A cluster control device controls a cluster of processors in the structural machine
- An individual control device controls a single processor in the structural machine

There are two basic types of processors:

- A localized processor is a single abstract device
- A distributed processor, which is also called a total processor, consists of a system of unit processors or processor units

In turn, there are three basic types of distributed processors:

- A homogeneous distributed processor consists of a system of identical unit processors, i.e., all these unit processors are copies of one processor
- An almost homogeneous distributed processor consists of a system in which almost all unit processors are identical
- A heterogeneous distributed processor consists of a system of different unit processors

As a result, we have three structural types of processors.

There are also three basic classes of distributed processors:

- In a synchronous distributed processor, all unit processors perform each operation at the same time
- In an almost synchronous distributed processor, almost all unit processors perform each operation at the same time
- In an asynchronous distributed processor, unit processors function independently

A cellular automaton is a synchronous distributed processor, while a Petri net is an asynchronous distributed processor.

It is natural to suppose that each unit processor performs only local operations. In a general case, each unit processor moves from one structure element to another, performing operations in their neighborhoods (Fig. 2). This makes it possible to consider a localized processor as a special type of a distributed processor with one unit processor.

A standard example of a localized processor is the head of a Turing machine with one head or a finite automaton. One head of TM correspond to one growth zone of the slime mould.

A standard example of a homogeneous distributed processor is the system of all heads of a Turing machine with several heads. It is also possible to perceive a Physarum machine as multi-processor structural machine.

Examples of heterogeneous distributed processors are processing devices in evolutionary automata such as evolutionary finite automata, evolutionary Turing machines or evolutionary inductive Turing machines (Burgin & Eberbach, 2009b).

Note that not all heterogeneous distributed processors are the same and to discern their properties it is possible to use measures of homogeneity constructed in (Burgin & Bratalskii, 1986).

There are three types of localized processors:

- A processor localized to one structure element (e.g., a node)
- A processor localized to an R -neighborhood of one structure element (e.g., a node) where \mathbf{R} is a relation from \mathcal{R}
- A processor localized to an \mathcal{R} -neighborhood of one structure element (e.g., a node)

In what follows, localization of a processor is formalized by the concept of the processor topos. There are three sorts of distributed processors:

- A constant distributed processor has a fixed number of localized unit processors.
- A variable distributed processor can change the number of localized unit processors.
- A growing distributed processor can increase the number of localized unit processors.

Growing distributed processors are special kinds of variable distributed processors.

There are three types of variable (growing) distributed processors:

- In a bounded variable (growing) distributed processor, the quantity of localized unit processors is always between two numbers, e.g., between 1 and 10, (is bounded by some number).
- An unbounded variable distributed processor can use any finite number of localized unit processors in its functioning.
- An infinite distributed processor can use any (even infinite) number of localized unit processors.

Cellular automata give examples of structural machines with infinite distributed processors. One-dimensional cellular automata work with such structures as words. Two-dimensional cellular automata work with such structures as two-dimensional arrays.

Now let us consider characteristics of unit processors in structural machines. Each unit processor p of a structural machine M has its topos, observation zone and operation zone.

Definition 2.10. The topos T_p of a unit processor p is the part of the structure occupied by this processor. When we take into account time of processing, the topos T_p of the processor p is denoted by $T_p(t)$.

This definition allows defining topos for total processors of structural machines.

Definition 2.11. The topos T_A of a total (distributed) processor A is the union of the topos of all its unit processors. When we take into account time of processing, the topos T_A of the processor A is denoted by $T_A(t)$.

Note that in the general case, the topos of a processor can be infinite although the constructive conditions on algorithms prohibit infinite topos. Cellular automata give an example of a processor with an infinite topos.

Axiom T. Topoi of different unit processors do not intersect.

Localization of unit processors implies restrictions on their topos. Namely, the topos of a unit processor localized to one structure element consists of this structure element, the topos of a unit processor localized to an R -neighborhood of one structure element is a part of this neighborhood, and the topos of a unit processor localized to an \mathcal{R} -neighborhood of one structure element is a part of that neighborhood.

Definition 2.12. The observation zone Ob_p of a unit processor p is the part of the structure $S p_M$ observed by this processor from its topos. When we take into account time of processing, the observation zone Ob_p of the processor p is denoted by observation zone $Ob_p(t)$.

As in the case of topos, we define observation zones for total processors.

Definition 2.13. The observation zone Ob_A of a total (distributed) processor A is the union of the observation zones of all its unit processors. When we take into account time of processing, the observation zone Ob_A of the processor A is denoted by observation zone $Ob_A(t)$.

It is natural to suppose that observation zones of processors impact their functioning.

Axiom Z. Operations performed by unit processors depend only on their observation zone.

Definition 2.14. The operation zone Op_p of a unit processor p is the part of the structure $S p_M$ that can be changed by this processor from its topos. When we take into account time of processing, the operation zone Op_p of the processor p is denoted by observation zone $Op_p(t)$.

For instance, the head of a Turing machine with one linear tape is unit processor. The topos of the head is one cell in the tape. Its observation zone is the same cell and the symbol written in it. Its operation zone is the symbol written in the cell occupied by the head.

Usually, these parts of the functional space PS_M satisfy the following conditions:

$$T_p \subseteq Op_p \subseteq Ob_p$$

and

$$T_p(t) \subseteq Op_p(t) \subseteq Ob_p(t)$$

Informally, it means that the topos of a processor is inside its operation zone, while it is possible to perform operations only inside the observation zone.

These conditions are true, for example, for Turing machines, but they are not satisfied for pushdown automata (Hopcroft *et al.*, 2001).

Often we have $Op_p = Ob_p$ and T_p consists of a single node (element from A).

As in the case of observation zones, we define operation zones for total processors.

Definition 2.15. The operation zone Op_A of a total (distributed) processor A is the union of the operation zones of all its unit processors.. When we take into account time of processing, the operation zone Op_A of the processor A is denoted by operation zone $Op_A(t)$.

Functioning of processors in structural machines includes not only structure transformations but also transitions from one topos to another.

Definition 2.16. The transition zone Tr_p of a unit processor p consists of all topoi where p can move in one step from its present topos.

For instance, the transition zone Tr_h of the head h of a Turing machine with one linear tape consists of three adjacent cells with h is situated in the middle cell.

In some cases, it is useful to assume that the transition zone of a unit processor is included in its observation zone.

Definition 2.17. A processor unit p is called: ϕ topologically uniform if all its topoi are isomorphic ϕ operationally uniform if all its operation zones are isomorphic ϕ transitionally uniform if all its transition zones are isomorphic ϕ observationally uniform if all its observation zones are isomorphic ϕ topologically standardized if all its topoi have the same type, e.g., are R -neighborhoods ϕ operationally standardized if all its operation zones have the same type, e.g., are R -neighborhoods ϕ transitionally standardized if all its transition zones have the same type, e.g., are R -neighborhoods ϕ observationally standardized if all its observation zones have the same type, e.g., are R -neighborhoods

For instance, processor unit p is topologically uniform if all its topoi consist of a single node.

Lemma 2.7. Any topologically (operationally, transitionally or observationally) uniform processor unit p is topologically (correspondingly, operationally, transitionally or observationally) standardized.

Usually, processors of abstract and physical automata (machines) are topologically operationally, transitionally and observationally uniform. At the same time, processors in chemical and biological automata (machines) can be non-uniform. An example of a non-uniform processor

is a processor that can read from or write to up to five cells in the memory. Thus, when this processor writes to one cell, its topos consists of one cell, while when this processor writes to three cells, its topos consists of these three cells.

Topoi, observation zones and operation zones of unit processors allow us to define topoi, observation zones and operation zones of distributed processors.

There are different types of processor units. A processor unit can be:

- Controlled (by the central control device of the structural machine).
- Autonomous, when it has its own control device.
- Cooperative, when it has its own control device but the functioning of this processor unit depends on the states both of its own control device and of the central control device of the structural machine.

For instance, in a many-head Turing machine T , all heads are controlled processor units. The control device of T controls them. At the same time, all finite automata in a cellular automaton are autonomous processor units.

We remind that a finite state machine also called a finite state automaton is an abstract system that can be in a finite number of different finite states and functioning of which is described as changes of its states.

Proposition 2.6. *A structural machine M is a finite state machine if and only if:*

- *Its structural space $S p_M$ is finite, i.e., in the case of universal structural space, it is a finite structure, or in the case of categorical structural space, it consists of a finite number of finite structures.*
- *The number of unit processors is finite and each of them can be in a finite number of different finite states.*

For instance, a finite automaton is a finite state machine, while a Turing machine is not a finite state machine.

Definition 2.18. A temporally finite state machine is an abstract system that can be in a finite number of different finite states at any moment of time and functioning of which is described as changes of its states.

Proposition 2.7. *A structural machine M is a temporally finite state machine if and only if:*

- *At any moment of time, its structural space $S p_M$ is finite, i.e., in the case of universal structural space, it is a finite structure, or in the case of categorical structural space, it consists of a finite number of finite structures.*
- *At any moment of time, the number of unit processors is finite and each of them can be in a finite number of different finite states.*

- Any operation of each unit processor involves only a finite number of structure elements and relations

For instance, a Turing machine is a temporally finite state machine, while finite dimensional and general machines of (Blum et al., 1989) are not temporally finite state machines.

Definition 2.19. An operation of a processor is local or more exactly, unilocal if it is performed with one structural element (e.g., node) and some (all) of its relations (a pointed operation), e.g., deleting a structural element (e.g., a node) and all its binary connections (links or edges), adding a link to a structural element or changing a label of a structural element.

For instance, the head h of a Turing machine performs only local operations, while the head of a pushdown automaton can perform nonlocal operations. Processors of automata that perform operations of unrestricted formal grammars are mostly nonlocal.

Definition 2.20. An operation of a processor P is \mathcal{R} -local if it is performed with elements (e.g., nodes) from the \mathcal{R} -neighborhood of a definite element (e.g., node) and with some (all) of their relations (a singularly local operation).

An operation of a processor P is topologically \mathcal{R} -local if it is performed with elements (e.g., nodes) from the \mathcal{R} -neighborhood of a topos of P (e.g., node) and with some (all) of their relations (a topologically local operation).

Proposition 2.8. *If (1) a structural machine M operates with structures in which \mathcal{R} contains only one binary relation, (2) a topos of a topologically uniform processor P of M is one structural element, and (3) an operation O of P is topologically local, then O is singularly local.*

Definition 2.21. a) An operation of a processor is \mathcal{R} -local or totally local if it is performed with elements (e.g., nodes) from the \mathcal{R} -neighborhood of a definite element (e.g., node) and with some (all) of their relations. b) An operation of a processor P is topologically \mathcal{R} -local if it is performed with elements (e.g., nodes) from the \mathcal{R} -neighborhood of a topos of P (e.g., node) and with some (all) of their relations (a wholly local operation).

Proposition 2.9. *If a topos of a topologically uniform processor P is one structural element and an operation O of P is wholly local, then O is totally local.*

Definitions imply the following result.

Proposition 2.10. *If R belongs to \mathcal{R} , then any R -local operation is \mathcal{R} -local.*

Let us consider operations performed by processors of structural machines.

The first group of operations consists of the transition operations:

1. Moving the processor from one topos, e.g., a structure element, to another topos. This operation is local when both elements belong to some relation from \mathcal{R} .
2. Changing the operation zone of the processor
3. Changing the observation zone of the processor

The second group of operations consists of the substantial transforming operations:

1. Adding a structure element, e.g., a node.
2. Deleting (removing) a structure element, e.g., a node, from a neighborhood of the element where the processor is situated and all relations that include this element.
3. Deleting (removing) a link from a relation R that connects some structure elements with the element where the processor is situated.
4. Adding a link to a relation \mathbf{R} that connects some structure elements with the element where the processor is situated.
5. Deleting (removing) a relation \mathbf{R} from \mathcal{R} .
6. Adding a new relation to R .

The third group of operations consists of the symbolic transforming operations:

1. Renaming a node
2. Naming a node
3. Denaming a node, i.e., deleting the name of a node
4. Renaming a link
5. Naming a link
6. Denaming a link, i.e., deleting the name of a link

Example 2.2. Operation of deleting the element f from the first order structure $\mathbf{A} = (A, r, \mathcal{R})$ in Fig. 2 where (A) shows the structure before operation and (B) shows the structure after operation. Note that such operations often change relations in the processed structure. Besides, the processor (processor unit) moves from the place (position) f to the place (position) g (see Fig. 2). This operation is performed according to the instruction $(q, f, f) \rightarrow (q, g, \sim f)$, in which q is the state of the processor (processor unit), f is the place (position) of the processor (processor unit) before the operation, g is the place (position) of the processor (processor unit) after the operation and $\sim f$ means elimination of f .

Example 2.3. Operation of adding the relation P for elements g and f in the first order structure $\mathbf{A} = (A, r, \mathcal{R})$ where (A) shows the structure before operation and (B) shows the structure after operation (see Fig. 3). Besides, the processor (processor unit) moves from the place (position) g to the place (position) e . This operation is performed according to the instruction $(p, g, f) \rightarrow (p, e, P(g, f))$, in which p is the state of the processor (processor unit), g is the place (position) of the processor (processor unit) before the operation, e is the place (position) of the processor (processor unit) after the operation, f is an observed element and means addition of the pair (g, f) to the relation P .

Structural machines can simulate Turing machines, Kolmogorov algorithms (machines), storage modification machines and cellular automata. We prove some of these results in the next section.

Structural machines also can simulate processes generated by logical calculi (Schumann & Adamatzky, 2011), λ -calculus and formal grammars being able to perform operations used in

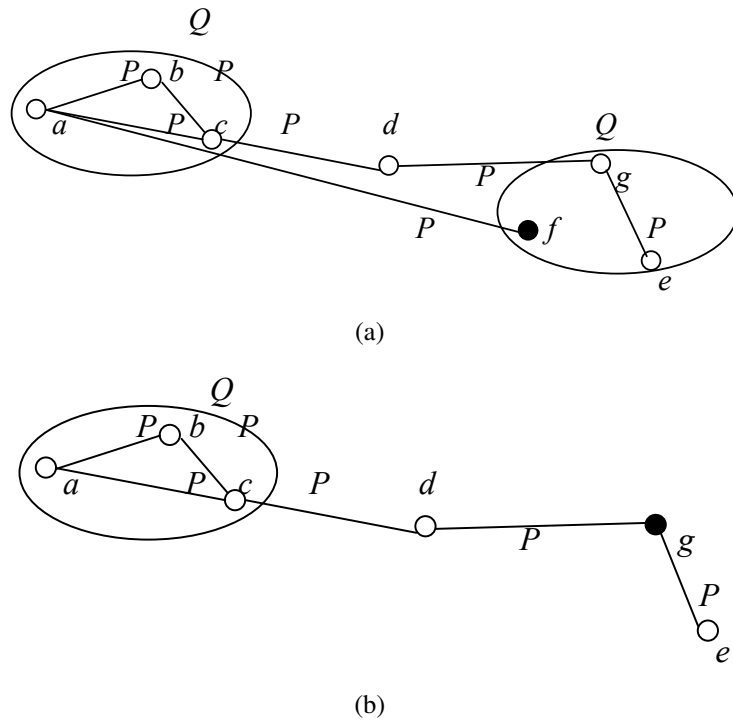


Figure 2. The graphical representation of an operation on first order structures.

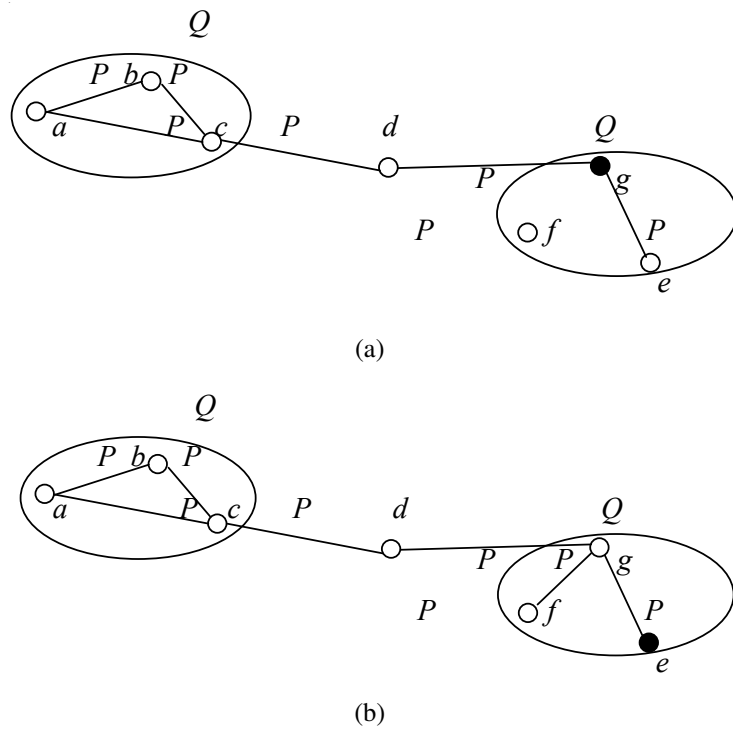


Figure 3. The graphical representation of an operation on first order structures.

various databases. Structural machines can also compute partial recursive functions and limit partial recursive functions.

It is possible to build structural machines that can work not only with discrete but also with continuous data because structures can be continuous and there are no restrictions on relations in processed structures. This possibility turns artificial neural networks, Shannons differential analyzer (Shannon, 1941), a finite dimensional and general machines of (Blum et al., 1989) and Type 2 Turing machines (Weihrauch, 2000) into special cases of structural machines.

This shows that it is practical to discern discrete structural machines, which work with discrete structures, have discrete systems of states and operations, and continuous structural machines. In continuous structural machines one two or all three of the following components can be continuous, i.e., continuous processed structures, continuous system of states and/or continuous operations.

Being able to construct various forms of structural machines and their flexibility show that it is natural to use structural machines for a theoretical study of natural computations performed by biological, chemical and physical systems. For instance, in Section 4, we show how to use structural machines as abstract automata for simulation of such biological automata as Physarum machines (Adamatzky, 2007) based on slime mold computations.

3. Structural machines, Turing machines and inductive Turing machines

In this section, we study relations between structural machines and other popular computational models.

Theorem 3.1. *A structural machine with a centralized processor can simulate any Turing machine with the linear time complexity.*

It is sufficient to simulate a Turing machine with one linear tape and one head (cf., for example, (Burgin, 2005; Hopcroft et al., 2001)).

When we want to model a Turing machine by a structural machine, it is easy to build a linear structure of elements with sequential elements connected by a binary relation and this structure will represent the linear tape of the Turing machine. However, the question is how the structural machine will discern left from right simulating the moves of the Turing machine. Here we give three solutions to this problem.

Let us consider a Turing machine $T = (A, Q, q_0, F, R)$. This formula shows that the Turing machine T is determined by the alphabet $\Sigma = \{a_1, a_2, \dots, a_m\}$, the set of states $Q = \{q_0, q_1, q_2, \dots, q_n\}$, the set of the final states $F = \{p_1, p_2, \dots, p_u\} \subseteq Q$, the start state q_0 and the system of rules R , each of which has the form

$$q_h a_i \rightarrow a_j q_k$$

$$q_h \Lambda \rightarrow a_j q_k$$

$$q_h a_i \rightarrow \Lambda q_k$$

$$q_h a_i \rightarrow q_k L$$

$$q_h a_i \rightarrow q_k cR$$

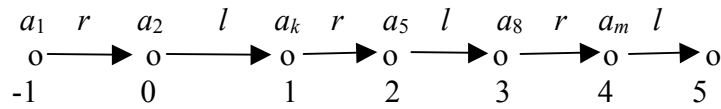


Figure 4. The graphical representation of processed first-order structures.

where $a_i, a_j \in A, q_h, q_k \in Q, R$ means that the head of T moves to the right, L means that the head of T moves to the left and Λ is the empty symbol, which denotes empty cell of the memory.

Here we give a linear solution to the problem of simulation, i.e., we use only one-dimensional first-order structures in the simulation.

Let us describe the structural machine $M_T = (C, P, \mathcal{R})$ with the control device C , processor P and processing space \mathcal{R} , which coincides with the input space and output space, such that M_T simulates the Turing machine T .

In it, C is a finite automaton with the states $Q^o = \{q_0^{odd}, q_1^{odd}, q_2^{odd}, \dots, q_n^{odd}, q_0^{even}, q_1^{even}, q_2^{even}, \dots, q_n^{even}\}$, the final states $F^o = \{p_1^{odd}, p_2^{odd}, \dots, p_u^{odd}, p_1^{even}, p_2^{even}, \dots, p_u^{even}\} \subseteq Q^o$, and the start state q_0^{odd} .

The processor P is similar to the head of a Turing machine but it observes not only the content of a cell but also its connections, and P can change not only the content of the observed cell but also its connections.

$\mathcal{R} = \{R, L, \Sigma\}$ where: R is a binary relation elements of which are denoted (named) by r ; L is a binary relation elements of which are denoted (named) by l ; Σ contains unary relations a where $a \in \Sigma$.

Relations r and l are also called connections or links. The unary relations a from Σ are used for naming the structure elements by symbols from Σ .

The machine M_T processes first-order structures that have the form $A = (N, z, \mathcal{R}_a)$ where $N = \{-h, -h + 1, \dots, -1, 0, 1, 2, 3, \dots, k; -h < k + 1\}$, the processing space \mathcal{R}_a contains two binary relations $R_a \subseteq R$ and $L_a \subseteq L$ in which links r connect elements $2t$ and $2t + 1$ for all integer numbers t with $-h \leq 2t \leq k - 1$, while links l connect elements $2t - 1$ and $2t$ for all integer numbers t with $-h + 1 \leq 2t \leq k$; and m unary relations a_1, a_2, \dots, a_m and z is the assignment of relations from \mathcal{R}_a (names from the alphabet Σ) to elements from N . Note that the number $-h$ can be positive, e.g., when $h = -3$.

An arbitrary input structure has the form $A_0 = (N_0 = \{1, 2, 3, \dots, m\}, z_0, \mathcal{R}_0)$ where \mathcal{R}_0 contains two binary relations $R_0 \subseteq R$ and $L_0 \subseteq L$ in which links r connect elements $2t$ and $2t + 1$ for all t with $1 \leq 2t \leq x - 1$, while links l connect elements $2t - 1$ and $2t$ for all t with $1 \leq 2t \leq x$; and m unary relations a_1, a_2, \dots, a_m and z_0 is the assignment of relations from \mathcal{R}_0 (names from the alphabet Σ) to the elements from N_0 , of the relation r to the pairs of elements $2t$ and $2t + 1$ for all t with $1 \leq 2t \leq x - 1$ and of the relation l to the pairs of elements $2t - 1$ and $2t$ for all t with $1 \leq 2t \leq x$.

Expressions in the rules for the machine M_T have the following meanings:

$a_i(r, l)$ in the left part of a rule means that the processor observes three relations a_i, r and l at the named element where the processor is situated.

$a_i(l)$ in the left part of a rule means that the processor observes only two relations a_i and l at the named element where the processor is situated.

$a_i(r)$ in the left part of a rule means that the processor observes only two relations a_i and r at the named element where the processor is situated.

$o(r, l)$ in the left part of a rule means that the processor observes two relations (connections) r and l at the element where the processor is situated.

$o(l)$ in the left part of a rule means that the processor observes only one relation (connection) l at the element where the processor is situated.

$o(r)$ in the left part of a rule means that the processor observes only one relation (connection) r at the element where the processor is situated.

The symbol o means a structure element to which no unary relation a_i is assigned, that is, an empty or not named structure element.

The symbol a_i in a rule means a structure element to which the unary relation (name) a_i is assigned.

We build rules of the in the machine M_T following way:

Renaming rules

Two rules $q_h^{odd} a_i(r, l) \rightarrow a_j q_k^{odd}$ and $q_h^{even} a_i(r, l) \rightarrow a_j q_k^{even}$ are assigned to the rule $q_h a_i \rightarrow a_j q_k$

Two rules $q_h^{odd} o(r, l) \rightarrow a_j q_k^{odd}$ and $q_h^{even} o(r, l) \rightarrow a_j q_k^{even}$ are assigned to the rule $q_h \Lambda \rightarrow a_j q_k$

Two rules $q_h^{odd} a_j(r, l) \rightarrow o q_k^{odd}$ and $q_h^{even} a_j(r, l) \rightarrow o q_k^{even}$ are assigned to the rule $q_h a_j \rightarrow \Lambda q_k$

Transition rules

Two rules $q_h^{odd} a_i(r, l) \rightarrow q_k^{even} r$ and $q_h^{even} a_i(r, l) \rightarrow q_k^{odd} l$ are assigned to the rule $q_h a_i \rightarrow q_k L$

Two rules $q_h^{odd} a_i(r, l) \rightarrow q_k^{even} l$ and $q_h^{even} a_i(r, l) \rightarrow q_k^{odd} r$ are assigned to the rule $q_h a_i \rightarrow q_k R$

Two rules $q_h^{odd}(r, l) \rightarrow q_k^{even} r$ and $q_h^{even} o(r, l) \rightarrow a_j q_k^{odd} l$ are assigned to the rule $q_h \Lambda \rightarrow q_k L$

Two rules $q_h^{odd}(r, l) \rightarrow q_k^{even} l$ and $q_h^{even} o(r, l) \rightarrow q_k^{odd} r$ are assigned to the rule $q_h \Lambda \rightarrow q_k R$

Construction rules

$q_h^{odd} a_j(l) \rightarrow q_h^{odd} a_j(l)[o]$ [creation of a new structural element near the named (full) end-element where the processor is situated],

$q_h^{odd} a_j(r) \rightarrow q_h^{odd} a_j(r)[o]$ [creation of a new structural element near the named (full) end-element where the processor is situated],

$q_h^{even} a_j(l) \rightarrow q_h^{even} [o]$ [creation of a new structural element near the named (full) end-element where the processor is situated],

$q_h^{even} a_j(r) \rightarrow q_h^{even} [o]$ [creation of a new structural element near the named (full) end-element where the processor is situated],

$q_h^{odd} a_j(l) \rightarrow q_h^{odd} a_j(r, l)$ [connecting a new structural element to the nearby (full) named end-element where the processor is situated],

$q_h^{odd} a_j(r) \rightarrow q_h^{odd} a_j(r, l)$ [connecting a new structural element to the nearby (full) named end-element where the processor is situated],

$q_h^{even} a_j(l) \rightarrow q_h^{even} a_j(r, l)$ [connecting a new structural element to the nearby (full) named end-element where the processor is situated],

$q_h^{even} a_j(r) \rightarrow q_h^{even} a_j(r, l)$ [connecting a new structural element to the nearby (full) named end-element where the processor is situated],

$q_h^{odd} o(l) \rightarrow q_h^{odd} [o]$ [creation of a new structural element near the empty (not named) end-element where the processor is situated],

$q_h^{odd}o(r) \rightarrow q_h^{odd}[o]$ [creation of a new structural element near the empty (not named) end-element where the processor is situated],

$q_h^{even}o(l) \rightarrow q_h^{even}[o]$ [creation of a new structural element near the empty (not named) end-element where the processor is situated],

$q_h^{even}o(r) \rightarrow q_h^{even}[o]$ [creation of a new structural element near the empty (not named) end-element where the processor is situated],

$q_h^{odd}o(l) \rightarrow q_h^{odd}o(r, l)$ [connecting a new structural element to the nearby empty (not named) end-element where the processor is situated],

$q_h^{even}o(r) \rightarrow q_h^{even}o(r, l)$ [connecting a new structural element to the nearby empty (not named) end-element where the processor is situated],

$q_h^{even}o(l) \rightarrow q_h^{even}o(r, l)$ [connecting a new structural element to the nearby empty (not named) end-element where the processor is situated],

$q_h^{even}o(l) \rightarrow q_h^{even}o(r, l)$ [connecting a new structural element to the nearby empty (not named) end-element where the processor is situated].

Construction operations add new structural elements and lacking relations, allowing processor to perform prescribed movements in all cases.

The structural machine M_T works following these rules. When it comes either to a state q_h^{even} or q_h^{odd} with q_h from F , then the machine M_T stops and the created structure with filled structure elements is the result of computation of the structural machine M_T .

When the processor of the structural machine M_T is at an odd structure element, then the state of the machine is some q_h^{odd} , and when the processor is at an even structure element, then the state of the machine is some q_h^{even} . When the head of the Turing machine T comes to the end of the tape, the processor of the structural machine M_T creates an empty structure element and the adequate connection to this element. The goal is to allow the processor of M_T to go (by this connection) to the empty structure element, simulating in such a way the move of the head of T to the empty cell.

As a result of these operations, the structural machine M_T simulates the Turing machine T , while the number of performed operations is $O(n)$ when T makes n operations.

Theorem is proved.

Being able to simulate Turing machines, structural machines can be more efficient than Turing machines. To show this, we take some alphabet Σ and consider the following algorithmic problem.

The Word Symmetry Problem. Given an arbitrary word w in the alphabet Σ , find if $w = uu^*$ for some word u where u^* is the inverse of u .

A word $w = uu^*$ is called a palindrome because written backwards it coincides with itself.

Theorem 3.2. *A structural machine with a centralized processor can solve the Word Symmetry Problem with the linear time complexity, i.e., with time $T(n) = O(n)$.*

Let us describe the structural machine $M = (C, P, \mathcal{R})$ with the control device C , processor P and processing space \mathcal{R} , which coincides with the input space and output space, such that M solves the Word Symmetry Problem with the linear time complexity.

In it, the control device C is a finite automaton with the states $Q = \{q_0, (q_0, a), (q_1, a), (q_2, a), q_f; a \in \Sigma\}$, the final states $F = \{q_f\} \subseteq Q$, and the start state q_0 .

The processor P is similar to the head of a Turing machine but it observes not only the content of a cell but also its connections, and P can change not only the content of the observed cell but

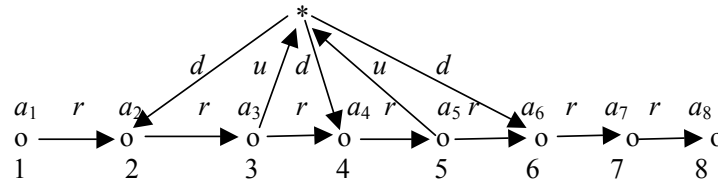


Figure 5. The graphical representation of processed first order structures.

also its connections.

$\mathcal{R} = \{R, L, U, D, \mathcal{E}\}$ where:

R is a binary relation elements of which are denoted (named) by r ;

U is a binary relation elements of which are denoted (named) by u ;

DB is a binary relation elements of which are denoted (named) by db ;

DE is a binary relation elements of which are denoted (named) by de ;

L is a binary relation elements of which are denoted (named) by l ;

Σ contains unary relations a where $a \in \Sigma$.

Relations r, u, d and l are also called connections or links. The unary relations a from Σ are used for naming the structure elements by symbols from Σ .

The machine M processes first-order structures that have the form $A = (N \cup \{*\}; z, \mathcal{R}a)$ where $N = \{1, 2, 3, \dots, k\}$, $\mathcal{R}a = \{ \text{our binary relations } D_a \subseteq D, U_a \subseteq U, R_a \subseteq R \text{ and } L_a \subseteq L \text{ in which links } r \text{ and } l \text{ can connect elements } t \text{ and } t + 1 \text{ for all } t = 1, 2, 3, \dots, k - 1, \text{ while links } u \text{ and } d \text{ can connect elements } t \text{ with the element } *\}$ and z is the assignment of relations from $\mathcal{R}a$ (names from the alphabet Σ) to elements from N .

An arbitrary input structure has the form $A_0 = (N_0 = \{1, 2, 3, \dots, h\}, z_0, \mathbf{R}_0)$ where $\mathbf{R}_0 = \{ \text{a binary relation } \mathbf{R}_0 \subseteq \mathbf{R} \text{ and } m \text{ unary relations } a_1, a_2, \dots, a_m \in \Sigma \}$ and links r from \mathbf{R}_0 connect elements t and $t + 1$ for all $t = 1, 2, 3, \dots, h - 1$.

The processor P of the machine M performs the following operation:

mv $[c(a, b)]$ denotes the transition of P from its topos (the cell where it is situated) with the name a to the cell with the name b by the link with the name c , which connects these two cells.

ch $[q \rightarrow p]$ denotes changing the state of M from q to the name p .

rn $[c(a, b) \rightarrow k(a, b)]$ denotes renaming of the link with the name c by giving it the new name k .

rn $[a \rightarrow b]$ denotes renaming of the cell where P is situated by changing its name a to the name b .

rn $[c(a, e); e \rightarrow b]$ denotes renaming of the cell that has the name e and is connected by a link with the name c to the cell where P is situated by changing its name e to the name b .

bd $[*]$ denotes building a new cell.

nm $[* \rightarrow b]$ denotes naming a new cell.

bd $[c(a, *)]$ denotes building a link with the name c from the cell where P is situated with the name a to a new cell.

bd $[c(a, b)]$ denotes building a link with the name c from the cell where P is situated with the name a to a cell with the name b .

Note that all these operations are local and \mathcal{R} -local. There are five types of neighborhoods

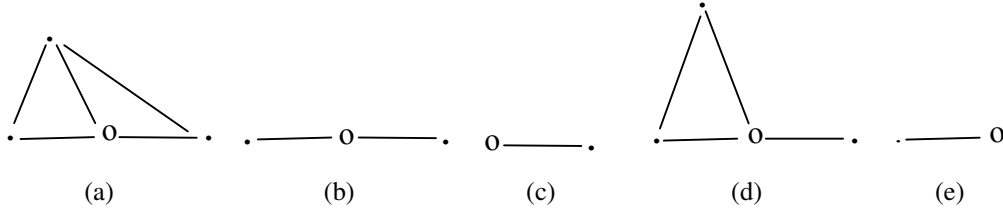


Figure 6. Types of neighborhoods of elements in processed first-order structures.

(Fig. 6) and all operations are performed only with parts (structure elements and links) of these neighborhoods.

Now we describe the rules of the machine M . In the rules, the left side describes the observation/control zone, while the right side shows the performed operation. Symbols a_j denote arbitrary elements from the alphabet Σ . We build the rules of the machine M in following way:

$$\begin{aligned}
& [a_1; q_0] \rightarrow \mathbf{rn}[a \rightarrow N]; \mathbf{ch}[q_0 \rightarrow q_f] \\
& [a_1; r(a_1, a_2); q_0] \rightarrow \mathbf{ch}[q_0 \rightarrow (q_0, a_1)]; \mathbf{mathbf{fmv}}[r(a_1, a_2)] \\
& [a_2; r(a_1, a_2); (q_0, a_1)] \rightarrow \mathbf{rn}[a_2 \rightarrow N]; \mathbf{ch}[(q_0, a_1) \rightarrow q_f] \text{ for all } a_2 \neq a_1 \quad [a_1; r(a_1, a_2); (q_0, a_1)] \rightarrow \\
& \mathbf{rn}[a_2 \rightarrow T]; \mathbf{ch}[(q_0, a_1) \rightarrow q_f] \\
& [a_2; r(a_1, a_2), r(a_2, a_3); (q_0, a_1)] \rightarrow \mathbf{bd}[\star]; \mathbf{nm}[\star \rightarrow \alpha]; \mathbf{bd}[db(a_2, \alpha)]; \mathbf{ch}[(q_0, a_1) \rightarrow (q_1, a_1)]; \\
& \mathbf{rn}[r(a_1, a_2) \rightarrow l(a_1, a_2)]; \mathbf{mv}[r(a_2, a_3)] \\
& [a_3; r(a_2, a_3), r(a_3, a_4), db(a_2, \alpha); (q_1, a_1)] \rightarrow \mathbf{bd}[u(a_3, \alpha)]; \mathbf{mv}[r(a_3, a_4)]; \mathbf{ch}[(q_1, a_1) \rightarrow (q_0, a_1)] \\
& [a_4; r(a_3, a_4), r(a_4, a_5), u(a_3, \alpha); (q_0, a_1)] \rightarrow \mathbf{bd}[u(a_4, \alpha)]; \mathbf{mv}[r(a_4, a_5)]; \mathbf{ch}[(q_0, a_1) \rightarrow (q_1, a_1)] \\
& [a_k; r(a_{k-1}, a_k), u(a_{k-1}, \alpha); (q_0, a_1)] \rightarrow \mathbf{rn}[a_k \rightarrow N]; \mathbf{ch}[(q_0, a_1) \rightarrow q_f] \text{ for all } a_k \neq a_1 \\
& [a_k; r(a_{k-1}, a_k), u(a_{k-1}, \alpha); (q_1, a_1)] \rightarrow \mathbf{rn}[a_k \rightarrow N]; \mathbf{ch}[(q_1, a_1) \rightarrow q_f] \\
& [a_k = a_1; r(a_{k-1}, a_k), u(a_{k-1}, \alpha); (q_0, a_1)] \rightarrow \mathbf{ch}[(q_0, a_1) \rightarrow q_1]; \mathbf{mv}[r(a_{k-1}, a_k)] \text{ (the case } a_k = a_1) \\
& [a_{k-1}; r(a_{k-1}, a_k), u(a_{k-1}, \alpha); (q_0, a_1)] \rightarrow \mathbf{ch}[(q_0, a_1) \rightarrow (q_2, a_{k-1})]; \mathbf{rn}[r(a_{k-1}, a_k) \rightarrow l(a_{k-1}, a_k)]; \\
& \mathbf{rn}[r(a_{k-1}, a_{k-2}) \rightarrow l(a_{k-1}, a_{k-2})]; \mathbf{rn}[u(a_{k-2}, \alpha) \rightarrow de(a_{k-2}, \alpha)]; \mathbf{mv}[u(a_{k-1}, \alpha)] \\
& [\alpha; de(a_{k-2}, \alpha), db(a_2, \alpha); (q_2, a_{k-1})] \rightarrow \mathbf{mv}[db(a_2, \alpha)] \\
& [a_2; r(a_2, a_3), db(a_2, \alpha), l(a_1, a_2); (q_2, a_{k-1})] \rightarrow \mathbf{rn}[a_k \rightarrow N]; \mathbf{ch}[q_0 \rightarrow q_f] \text{ for all } a_{k-1} \neq a_2 \\
& [a_2; r(a_2, a_3), db(a_2, \alpha), l(a_1, a_2); (q_2, a_2)] \rightarrow \mathbf{rn}[db(a_2, \alpha) \rightarrow u(a_2, \alpha)]; \mathbf{mv}[r(a_2, a_3)] \text{ (the case } a_{k-1} = \\
& a_2) \\
& [a_3; r(a_2, a_3), r(a_3, a_4), u(a_3, \alpha); (q_2, a_2)] \rightarrow \mathbf{ch}[(q_2, a_2) \rightarrow (q_2, a_3)]; \mathbf{rn}[r(a_2, a_3) \rightarrow l(a_2, a_3)]; \mathbf{rn}[r(a_3, a_4) \rightarrow \\
& l(a_3, a_4)]; \mathbf{rn}[u(a_4, \alpha) \rightarrow db(a_4, \alpha)]; \mathbf{mv}[u(a_3, \alpha)] \\
& [\alpha; de(a_{k-2}, \alpha), db(a_4, \alpha); (q_2, a_3)] \rightarrow \mathbf{mv}[de(a_{k-2}, \alpha)] \quad [a_{k-2}; r(a_{k-2}, a_{k-3}), de(a_{k-2}, \alpha), l(a_{k-1}, a_{k-2}); (q_2, a_3)] \rightarrow \\
& \mathbf{rn}[a_{k-2} \rightarrow N]; \mathbf{ch}[(q_3, a_3) \rightarrow q_f] \text{ for all } a_{k-2} \neq a_3 \\
& [a_{k-2} = a_3; r(a_{k-2}, a_3), de(a_2, \alpha), l(a_{k-1}, a_{k-2}); (q_2, a_3)] \rightarrow \mathbf{rn}[de(a_2, \alpha) \rightarrow u(a_2, \alpha)]; \mathbf{mv}[r(a_{k-2}, a_{k-3})] \\
& \text{(the case } a_{k-2} = a_3) \\
& [a_{k-2}; r(a_{k-2}, a_{k-3}), de(a_{k-2}, \alpha), l(a_{k-1}, a_{k-2}); (q_2, a_3)] \rightarrow \mathbf{rn}[a_{k-2} \rightarrow T]; \mathbf{ch}[(q_3, a_3) \rightarrow q_f]
\end{aligned}$$

The result of computations is defined in the following way:

- When the machine M comes to the final state and the name of the processor topos is N , then the result is negative, i.e., the input word is not symmetric.

- When the machine M comes to the final state and the name of the processor topos is T , then the result is positive, i.e., the input word is symmetric.

Using these rules the machine M checks if the input word is symmetric or not. In the process of computation, the processor P comes to each structure element (cell) not more than two times and in each cell, the processor P performs not more than five operations. Assuming, as it is done in the theory of algorithms and computation, that each operation takes one unit of time, we see that the machine M can solve the Word Symmetry Problem with the time complexity $T(n) = 10n$. Theorem is proved. At the same time, by Barzdins theorem, any deterministic Turing machine can solve the Word Symmetry Problem only with time complexity $O(n^2)$ (cf. (Trahtenbrot, 1974)). It means that computational complexity of structural machines is essentially less than computational complexity of Turing machines for some problems.

Another problem with time complexity $O(n^2)$ for Turing machines and with time complexity $O(n)$ for structural machines is inversion of a given word.

Theorem 3.3. *A structural machine with a centralized processor can simulate any simple inductive Turing machine.*

Proof. It is demonstrated how structural machine with a centralized processor can simulate any Turing machine with one tape. In the theory of algorithms and computation, it is proved that a Turing machine with one tape can simulate a Turing machine with any number of tapes. At the same time, a simple inductive Turing machine has exactly the same structure and operations (instructions) as a Turing machine with three tapes. Consequently, working in the inductive mode, a structural machine with a centralized processor can simulate any simple inductive Turing machine. \square

It is also possible to simulate membrane computations (Păun & Rozenberg, 2002) with structural machines, while some classes of structural machines work as neural Turing machines (Graves et al., 2014). However, these results are presented in another work of the authors.

4. Modeling slime mold computations and Physarum machines by structural machines

Physarum polycephalum belongs to the species of order *Physarales*, subclass *Myxogastromycetidae*, class *Myxomycetes*, division *Myxozetelida*. It is commonly known as a true, acellular or multi-headed slime mould, see introduction in (Stephenson et al., 1994). Plasmodium is a ‘vegetative’ phase, a single cell with a myriad of diploid nuclei. The plasmodium is visible to the naked eye. The plasmodium looks like an amorphous yellowish mass with networks of protoplasmic tubes.

The plasmodium behaves and moves as a giant amoeba forming a network of biochemical oscillators (Matsumoto et al., 1986; Nakagaki et al., 2000). The plasmodium’s behavior is determined by external stimuli and excitation waves travelling and interacting inside the plasmodium. The plasmodium can be considered as a reaction-diffusion (Adamatzky, 2007) encapsulated in an elastic growing membrane.

When plasmodium is placed on an appropriate substrate, the plasmodium propagates, searches for sources of nutrients and follows gradients of chemo-attractants, humidity and illumination



Figure 7. Physarum forms a network of protoplasmic tubes on virtually any surface.



Figure 8. A spanning tree approximated by live Physarum. Planar data points are represented by oat flakes (dark blobs), edges of the tree by protoplasmic tubes; few duplicated tubes are considered as a single edge.

forming veins of protoplasm, or protoplasmic tubes (Fig. 7). The veins can branch, and eventually the plasmodium spans the sources of nutrients with a dynamic proximity graph, resembling, but not perfectly matching graphs from the family of k -skeletons (Kirkpatrick et al., 1985).

Due to its unique features and relative ease of experimentation with, the plasmodium has become a test biological substrate for implementation of various computational tasks. The problems solved by the plasmodium include maze-solving, spanning trees and proximity graphs (Fig. 8,

calculation of efficient networks, construction of logical gates, sub-division of spatial configurations of data points, and robot control (see overview in (Adamatzky, 2016)). A computation in the plasmodium is implemented by interacting biochemical and excitation waves, redistribution of electrical charges on plasmodium's membrane and spatiotemporal dynamics of mechanical waves. Plasmodium of *P. polycephalum* performs complex computation by three general mechanisms: morphological adaptation of its body plan and transport network, wave propagation of information through its protoplasmic transport network, and competition and entrainment of oscillations in partial bodies – relatively small fragments of plasmodium connected via protoplasmic tubes. All three mechanisms are closely associated with one another (for example, morphological adaptation is dependent on local oscillatory activity and protoplasmic flux). In (Adamatzky, 2007), it is demonstrated how to simulate Kolmogorov algorithms (Kolmogorov, 1953; Kolmogorov & Uspensky, 1958) with living slime mould in experimental laboratory conditions.

To model a Physarum machine by a structural machine, we have to interpret components of a slime mould as components of a structural machine and behaviour of the slime mould as computations of the structural machine.

A Physarum machine PM is realized by a many-headed slime mould, which is a single cell with a myriad of diploid nuclei. It is possible to treat this cell as a primitive object SM with a set of inner states. Examples of such states are “to be alive or “not to be alive. In a context of physical measurements it would be more correct to use.

In this context, we represent the object SM by the control device C_M of the structural machine M , which models the Physarum machine. The control device C_M can be assigned to be an active growing zone.

A many-headed slime mould has several active growth zones exploring concurrently the physical space around the slime mould (Fig. 9). Thus, it is natural to treat a Physarum machine as a structural machine with a distributed processor P and to interpret each active growth zone as the operation zone of a unit processor p .

As it was already demonstrated, a first-order structure $\mathbf{A} = (A, r, \mathcal{R})$, in which the set \mathcal{R} consists of a single binary relation R naturally represents the structure of a living slime mould established by blobs of slime mould and active zones where structural elements (e.g., nodes) from the set A represent blobs of slime mould and active zones, while elements from the relation R (e.g., edges) represent connecting tubes. A Physarum machine has two types of nodes: stationary nodes presented by sources of nutrient (oat flakes), and dynamic nodes, which are sites where two or more protoplasmic tubes originate (Adamatzky, 2007).

However, a slime mould often has a more sophisticated structure. Despite being a single cell, the slime mould can colonize substantial areas, up to hundreds of centimeters. The network of blobs, active zones and protoplasmic tubes is not uniform but forms clusters. These clusters are also connected by thick protoplasmic tubes, which represent the incidence relation that connects groups of elements from A . Therefore, we use second-order structures to model a slime mould with clusters. Thus, taking a second-order structure $\mathbf{A} = (A, r, \mathcal{R})$, in which the set \mathcal{R} consists of a binary relation R , a system of binary relations $C_1, C_2, C_3, \dots, C_n$, and a binary relation Q , we represent the structure of a living slime with clusters in the following way:

- elements from the set A represent blobs of slime mould and active zones,

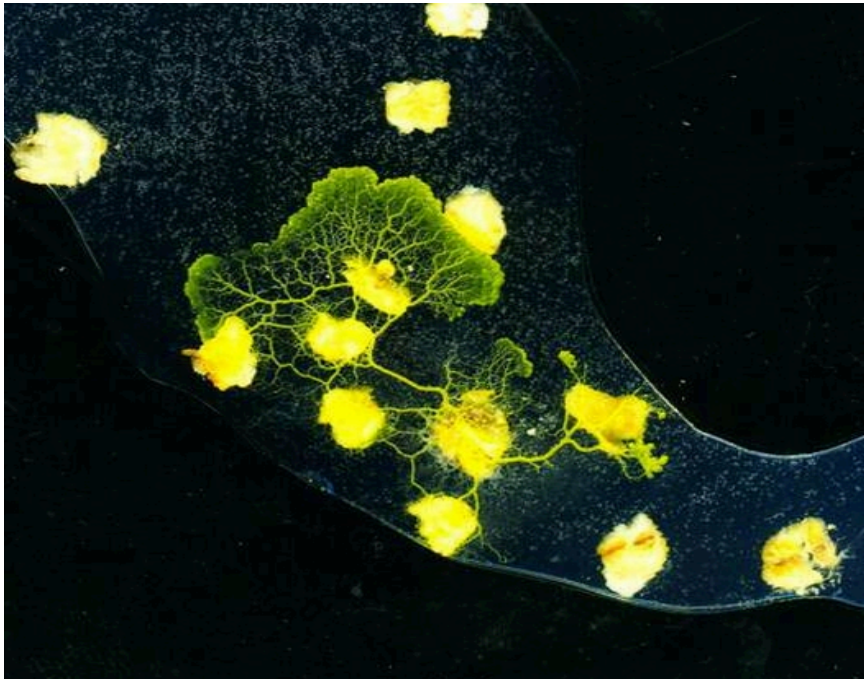


Figure 9. Physarum propagates in several directions simultaneously attracted by sources of nutrients.

- elements from the relation R represent tubes connecting blobs of the slime mould and active zones,
- each relation C_i represents one cluster of the slime mould, namely, if the cluster with the number i consists of blobs and active zones $a_1, a_2, a_3, \dots, a_m$, then $C_i = (a_1, a_2, a_3, \dots, a_m) \subseteq A^m$
- elements from the relation Q represent tubes connecting clusters.

This allows us to consider the sensorial space of the slime mould as the input space In_M of the machine M because the slime mould sees the world as a configuration of gradient fields.

The output space Out , which contains the output structure. The output space is the morphology of the slime mould, i.e., the configuration of growth zones, blobs occupying nutrients, and network protoplasmic tubes connecting them, is moulded by the output space Out_M of the machine M .

In a similar way, the cyto-skeletal network inside the slime mould body forms the processing space of the Physarum machine and is naturally modeled by the processing space PS_M of the structural machine M .

In slime mould, oscillatory patterns control the behaviors of the cell. In structural machines, oscillatory patterns are represented by the names of the nodes (structural elements) and links between these elements.

5. Conclusion

We determined and studied structural machines demonstrating that they can simulate Turing machines (Theorem 1) and inductive Turing machines (Theorem 3). We also proved that structural machines are more efficient than Turing machines (Theorem 2). In addition, we explained how structural machines describe and model behavior biological computers such as Physarum machine, which is based on slime mould *P. polycephalum*. This shows the big computational potential of slime mould as a biological computer. Further work can go in two directions: implementation of practical algorithms on structural machines and development of structural machines models for ultra-cellular computing based on cytoskeleton. The development of practical algorithms is necessary to allow the structural machines to 'enter the real world' and not just remain one of the many formal accomplishments of theoretical computer science. Physarum machines can solve dozens of problems from computational geometry, graph optimization and control. They also can be used as organic electronic elements (Erokhin et al., 2012; Gale et al., 2015; Adamatzky, 2014; Whiting et al., 2015). The structural machine might form a platform for developing Physarum programming languages, compilers and interface between human operators and the slime mould (Schumann & Adamatzky, 2011; Schumann & Pancerz, 2014, 2015). The development of structural machine models of ultra-cellular computing is necessary because the behavior of the slime mould, as of most other cells, is governed by actin and tubuline networks inside the cells. Here we mention actin because it is a dominating cytoskeleton protein in *P. polycephalum*. Actin is a filament-forming protein forming a communication and information processing cytoskeletal network of eukaryotic cells. Actin filaments play a key role in developing synaptic structure, memory and learning of animals and humans. This is why it is important to develop abstractions of the information processing on the actin filaments. While designing experimental laboratory prototypes of computing devices from living slime mould *P. polycephalum* (Adamatzky, 2016), we found that actin networks might play a key role in distributed sensing, decentralized information processing and parallel decision making in a living cell (Adamatzky & Mayne, 2015; Mayne et al., 2015). The actin-automata exhibit a wide a range of mobile and stationary patterns, which were later used to design computational models of quantum (Siccardi & Adamatzky, 2016) and Boolean (Siccardi et al., 2016) gates implementable on actin fibre, as well as realization of universal computation with cyclic tag systems (Martínez et al., 2017). The previously proposed model of an actin filament in a form of a finite-state machine, or automaton network, (Adamatzky & Mayne, 2015) constitutes a very special case of studied in this paper structural machines, which provide much more powerful tools for exploration of possibilities of biologically based computation. Detailed formalization of the information processing capabilities of the actin networks, including their polymerization and growths, and interaction with other intra-cellular proteins would immensely advance nano-computing and theoretical computer science making an imperative impact on development of future and emergent computing architectures. Structural machines provide means for simulation of membrane computations (Păun, 2000; Păun & Rozenberg, 2002), while some classes of structural machines work as neural networks or neural Turing machines (Graves et al., 2014). Exposition of these results is given in another work of the authors. It is necessary to remark that due to their flexibility, definite classes of structural machines allow much better modeling of quantum computations than conventional models. Now there are various theoretical models of quantum computation: quantum

Turing machines (Deutsch, 1985) and quantum circuits (Feynman, 1986) correspond to discrete computing, while modular functors describe topological quantum computation (Freedman *et al.*, 2003), to mention but a few. It is interesting that while some theoretical models are recursive algorithms, which are not more powerful than Turing machines (Deutsch, 1985), other theoretical models are more powerful than Turing machines (Kieu, 2003). Structural machines provide a theoretical framework for unification of different models of quantum computation. This opportunity brings us to the open problem of modeling of quantum computation by structural machines.

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New Subclasses of Analytic and Bi-Univalent Functions Involving a New Integral Operator Defined by Polylogarithm Function

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Abstract

In the present investigation, we introduce two new subclasses of the function class σ of bi-univalent functions in the open unit disc. Also we find coefficient estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the function class and several related classes are also considered and connections to earlier known results are made.

Keywords: Analytic functions, univalent functions, bi-univalent functions, coefficient bounds.
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1. Introduction

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1.1)$$

Further, the class of all functions in A which are univalent in U is denoted by the symbol S .

The Koebe one-quarter theorem (Duren, 1983) states that the image of U under every function $f \in S$ contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \quad (z \in U)$$

and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}\right),$$

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where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U .

Let Σ denote the class of bi-univalent functions defined in the unit disk U . For a brief history and interesting examples in the class Σ , see (Srivastava *et al.*, 2010). The concept of bi-univalent function class was firstly studied by Lewin (Lewin, 1967) and obtained that the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie (Brannan & Clunie, 1980) conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Netanyahu (Netanyahu, 1969) showed that $\max |a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$.

Brannan and Taha (Brannan & Taha, 1986) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\delta^*(\alpha)$ and $K(\alpha)$ of starlike and convex function of order α ($0 < \alpha \leq 1$) respectively. The classes $\delta_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α , corresponding to the function classes $\delta^*(\alpha)$ and $K(\alpha)$, were also introduced similarly. For each of the function classes $\delta_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$, non-sharp estimates on the initial coefficients were found by them. In recent years, bounds for various subclasses of bi-univalent functions were investigated by many authors ((Frasin & Aouf, 2011), (Srivastava *et al.*, 2010), (Xu *et al.*, 2012b)). For each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N} \setminus \{1, 2\}$, the problem of determining coefficient estimate is still an open problem. In the year 2010, the following subclasses of the bi-univalent function class Σ was introduced by Srivastava *et al.* (Srivastava *et al.*, 2010) and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ was obtained.

Definition 1.1. (Srivastava *et al.*, 2010) A function $f(z)$ given by the TaylorMaclaurin series expansion (1.1) is said to be in the class $\mathcal{H}_\sigma^\alpha$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \left| \arg(f'(z)) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, z \in U)$$

and

$$\left| \arg(g'(w)) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, w \in U)$$

where the function g is given by

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

Theorem 1.1. (Srivastava *et al.*, 2010) Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma^\alpha$ ($0 < \alpha \leq 1$). Then

$$|a_2| \leq \alpha \sqrt{\frac{2}{\alpha + 2}} \quad \text{and} \quad |a_3| \leq \frac{\alpha(3\alpha + 2)}{3}.$$

Definition 1.2. (Srivastava *et al.*, 2010) A function $f(z)$ given by (1.1) is said to be in the class H_Σ^β ($0 \leq \beta < 1$) if the following conditions are satisfied:

$$f \in \Sigma, \quad \left| \operatorname{Re}(f'(z)) \right| > \beta \quad (0 \leq \beta < 1, z \in U)$$

and

$$\left| \operatorname{Re} \left(g'(w) \right) \right| > \beta \quad (0 \leq \beta < 1, w \in U)$$

where the function g is given by $f^{-1}(w) = g(w)$.

Theorem 1.2. (Srivastava et al., 2010) Let the function $f(z)$ given by (1.1) be in the class H_{Σ}^{β} ($0 \leq \beta < 1$). Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}} \quad \text{and} \quad |a_3| \leq \frac{(1-\beta)(5-3\beta)}{3}.$$

Here, in our present sequel to some of the aforementioned works (especially [15]), the following subclass of the analytic function class A is introduced. Also, by using the method of (Srivastava et al., 2010), (Frasin & Aouf, 2011), (Xu et al., 2012b) and (Xu et al., 2012a) different from that used by other authors, we obtain bounds for the coefficients $|a_2|$ and $|a_3|$ for the subclasses of bi-univalent functions considered Porwal and Darus and get more accurate estimates than that given in (Porwal & Darus, 2013). For the functions $f \in A$ given by (1.1) and $g \in A$, $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, their

Hadamard product or convolution (Duren, 1983) is defined by the power series

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

For $f(z) \in A$, Al-Shaqsi (AL-Shaqsi, 2014) defined the following integral operator:

$$\begin{aligned} L_c^{\delta} f(z) &= (1+c)^{\delta} \Phi_{\delta}(c; z) * f(z) \\ &= -\frac{(1+c)^{\delta}}{\Gamma(\delta)} \int_0^1 t^{c-1} \log\left(\frac{1}{t}\right)^{\delta-1} f(zt) dt \end{aligned} \quad (1.2)$$

$(c > 0, \delta > 1, z \in U)$

where Γ stands for the usual gamma function, $\Phi_{\delta}(c; z)$ is the well known generalization of the Riemann-zeta and polylogarithm functions, or the δ th polylogarithm function, given by

$$\Phi_{\delta}(c; z) = \sum_{k=1}^{\infty} \frac{z^k}{(k+c)^{\delta}}$$

where any term without $k+c = 0$ (see (Lerch, 1887) and (Bateman, 1953)(sections 1.10 and 1.12)). Also, $\Phi_{-1}(0; z) = \frac{z}{(1-z)^2}$ is Koebe function. One can find more details about polylogarithms in the theory of univalent functions in the study of Ponnusamy and Sabapathy (Ponnusamy & Sabapathy, 1996).

We also state that the operator $\mathcal{L}_c^{\delta} f(z)$ given by the relation (1.2) can be expressed by the series expansions as follows:

$$\mathcal{L}_c^{\delta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+c}{k+c} \right)^{\delta} a_k z^k.$$

First of all, we present the following lemma to prove our main result

Lemma 1.1. (Pommerenke & Jensen, 1975) If $h \in \mathbb{P}$ then $|c_k| \leq 2$ for each k , where \mathbb{P} is the family of all functions h analytic in E for which $\operatorname{Re}(h(z)) > 0$, then

$$h(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \dots$$

2. Coefficient Estimates for the class $\mathcal{B}_\sigma^\delta(\beta, \lambda, c)$

Definition 2.1. The class $\mathcal{B}_\sigma^\delta(\beta, \lambda, c)$ of the functions $f(z)$ determined by the equality (1.1) consists of those functions $f(z)$ that satisfy the following conditions:

$f \in \sigma$,

$$\operatorname{Re} \left(\frac{(1-\lambda)\mathcal{L}_c^\delta f(z) + \lambda\mathcal{L}_c^{\delta-1} f(z)}{z} \right) > \beta \quad (2.1)$$

where $0 \leq \beta < 1, \lambda \geq 1, c > 0, \operatorname{Re}\delta > 1, z \in U$ and

$$\operatorname{Re} \left(\frac{(1-\lambda)\mathcal{L}_c^\delta g(w) + \lambda\mathcal{L}_c^{\delta-1} g(w)}{w} \right) > \beta. \quad (2.2)$$

where $\mathcal{L}_c^{\delta-1}$ stands for polylogarithm function introduced and studied by Al-Shaqsi and the function g is given by $g(w) = f^{-1}(w)$.

Remark. If we let $c = 0$ and $\delta = -n$, for $n \in \mathbb{N} \cup \{0\}$, then we obtain

$$\mathcal{B}_\sigma^\delta(\beta, \lambda, c) = H_\Sigma(n, \beta, \lambda)$$

studied by Porwal and Darus (Porwal & Darus, 2013). This class contains the function $f \in \Sigma$ satisfying

$$\operatorname{Re} \left(\frac{(1-\lambda)\mathcal{D}^n f(z) + \lambda\mathcal{D}^{n+1} f(z)}{z} \right) > \beta$$

and

$$\operatorname{Re} \left(\frac{(1-\lambda)\mathcal{D}^n g(w) + \lambda\mathcal{D}^{n+1} g(w)}{w} \right) > \beta.$$

where \mathcal{D}^n stands for Salagean derivative introduced by Sălăgean (Salagean, 1983).

The class $\mathcal{B}_\sigma^{-n}(\beta, \lambda, 0)$ includes many earlier classes, which are mentioned below:

1. If we let $n = 0$, then we have

$$\mathcal{B}_\sigma^{-n}(\beta, \lambda, 0) = H_\Sigma^\lambda(\beta)$$

studied by Frasin and Aouf (Frasin & Aouf, 2011). This class contains the functions $f \in \Sigma$ satisfying

$$\operatorname{Re} \left(\frac{(1-\lambda)f(z)}{z} + \lambda f'(z) \right) > \beta$$

and

$$\operatorname{Re} \left(\frac{(1-\lambda)g(w)}{w} + \lambda g'(w) \right) > \beta.$$

2. If we let $n = 0$ and $\lambda = 1$, then we have

$$\mathcal{B}_\sigma^{-n}(\beta, 1, 0) = H_\Sigma(\beta)$$

studied by Srivastava et al. (Srivastava et al., 2010). This class contains the functions $f \in \Sigma$ satisfying

$$\operatorname{Re}(f'(z)) > \beta$$

and

$$\operatorname{Re}(g'(w)) > \beta.$$

The next theorem gives the estimate on coefficient of the function in the class $\mathcal{B}_\sigma^\delta(\beta, \lambda, c)$ given in Definition 2.1.

Theorem 2.1. Let the function $f(z)$ given by equation (1.1) be in the class $\mathcal{B}_\sigma^\delta(\beta, \lambda, c)$. Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^\delta}} \quad (2.3)$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{\left(1 + \frac{\lambda}{1+c}\right)^2 \left(\frac{1+c}{2+c}\right)^{2\delta}} + \frac{2(1-\beta)}{\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^\delta} \quad (2.4)$$

where $0 \leq \beta < 1$ and $\lambda \geq 1$.

Proof. Let $f \in \mathcal{B}_\sigma^\delta(\beta, \lambda, c)$, $\lambda \geq 1$ and $0 \leq \beta < 1$. Using argument inequalities in (2.1) and (2.2), we can state their forms as follows:

$$\frac{(1-\lambda)\mathcal{L}_c^\delta f(z) + \lambda\mathcal{L}_c^{\delta-1} f(z)}{z} = \beta + (1-\beta)p(z) \quad (z \in U) \quad (2.5)$$

and

$$\frac{(1-\lambda)\mathcal{L}_c^\delta g(w) + \lambda\mathcal{L}_c^{\delta-1} g(w)}{w} = \beta + (1-\beta)q(w) \quad (w \in U) \quad (2.6)$$

where $p(z)$ and $q(w)$ given by the equalities

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots \quad (2.7)$$

and

$$q(w) = 1 + q_1w + q_2w^2 + q_3w^3 + \dots \quad (2.8)$$

satisfy the inequalities $\operatorname{Re}(p(z)) > 0$ and $\operatorname{Re}(q(w)) > 0$ respectively. Equating coefficients (2.5) and (2.6) yields

$$\left(1 + \frac{\lambda}{1+c}\right)\left(\frac{1+c}{2+c}\right)^\delta a_2 = (1-\beta)p_1, \quad (2.9)$$

$$\left(1 + \frac{2\lambda}{1+c}\right) \left(\frac{1+c}{3+c}\right)^\delta a_3 = (1-\beta)p_2, \quad (2.10)$$

and

$$-\left(1 + \frac{\lambda}{1+c}\right) \left(\frac{1+c}{2+c}\right)^\delta a_2 = (1-\beta)q_1 \quad (2.11)$$

$$\left(1 + \frac{2\lambda}{1+c}\right) \left(\frac{1+c}{3+c}\right)^\delta (2a_2^2 - a_3) = (1-\beta)q_2. \quad (2.12)$$

From (2.9) and (2.11), we have

$$p_1 = -q_1 \quad (2.13)$$

and

$$2\left(1 + \frac{\lambda}{1+c}\right)^2 \left(\frac{1+c}{2+c}\right)^{2\delta} a_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \quad (2.14)$$

Also, adding (2.10) to (2.12), we get

$$2\left(1 + \frac{2\lambda}{1+c}\right) \left(\frac{1+c}{3+c}\right)^\delta a_2^2 = (1-\beta)(p_2 + q_2). \quad (2.15)$$

Applying Lemma 1.1 for equality (2.15), we have

$$|a_2|^2 \leq \frac{(1-\beta)(|p_2| + |q_2|)}{2\left(1 + \frac{2\lambda}{1+c}\right) \left(\frac{1+c}{3+c}\right)^\delta} \leq \frac{2(1-\beta)}{\left(1 + \frac{2\lambda}{1+c}\right) \left(\frac{1+c}{3+c}\right)^\delta}$$

This gives the bound on $|a_2|$ as asserted in (2.3).

Next, to find the bound on $|a_3|$, by subtracting (2.12) from (2.10), we get

$$2\left(1 + \frac{2\lambda}{1+c}\right) \left(\frac{1+c}{3+c}\right)^\delta (a_3 - a_2^2) = (1-\beta)(p_2 - q_2) \quad (2.16)$$

which, upon substitution of value of a_2^2 from (2.14) yields

$$a_3 = \frac{(1-\beta)^2(p_1^2 + q_1^2)}{2\left(1 + \frac{\lambda}{1+c}\right)^2 \left(\frac{1+c}{2+c}\right)^{2\delta}} + \frac{(1-\beta)(p_2 - q_2)}{2\left(1 + \frac{2\lambda}{1+c}\right) \left(\frac{1+c}{3+c}\right)^\delta}.$$

Applying the lemma 1 for the coefficients p_1, q_1, p_2 and q_2 , we readily get

$$|a_3| \leq \frac{4(1-\beta)^2}{\left(1 + \frac{\lambda}{1+c}\right)^2 \left(\frac{1+c}{2+c}\right)^{2\delta}} + \frac{2(1-\beta)}{\left(1 + \frac{2\lambda}{1+c}\right) \left(\frac{1+c}{3+c}\right)^\delta}.$$

□

Remark. Choosing $c = 0$ in Theorem 2.1, we have the following corollaries:

1. If we let $\delta = -n$, ($n \in \mathbb{N} \cup \{0\}$), then we obtain the following:

Corollary 2.1. (Porwal & Darus, 2013) Let the function $f(z)$ given by (1.1) be in the class $H_{\Sigma}(n, \beta, \lambda)$,

$0 \leq \beta < 1, \lambda \geq 1, n \in \mathbb{N}_0$. Then,

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{(1-\lambda)3^n + \lambda 3^{n+1}}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{[(1-\lambda)2^n + \lambda 2^{n+1}]^2} + \frac{2(1-\beta)}{(1-\lambda)3^n + \lambda 3^{n+1}}.$$

2. Especially, choosing $n = 0$ in Corollary 2.1, we have the following result:

Corollary 2.2. (Frasin & Aouf, 2011) Let the function $f(z)$ given by (1.1) be in the class $H_{\Sigma}^{\lambda}(\beta)$, $0 \leq \beta < 1$. Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{1+2\lambda}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(1+\lambda)^2} + \frac{2(1-\beta)}{(1+2\lambda)}.$$

Remark. The estimates for $|a_2|$ and $|a_3|$ of Corollary 2.2 and Corollary 2.3 show that Theorem 2.1 coincides with the the estimates obtained by Frasin and Aouf (Frasin & Aouf, 2011).

3. If we choose $n = 0$ and $\lambda = 1$, then we obtain the following corollary:

Corollary 2.3. (Srivastava et al., 2010) Let the function $f(z)$ given by (1.1) be in the class $H_{\Sigma}(\beta)$, $0 \leq \beta < 1$. Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}}$$

and

$$|a_3| \leq \frac{(1-\beta)(5-3\beta)}{3}.$$

3. Coefficient Estimates for the class $\mathcal{H}_{\sigma}^{\delta}(\alpha, \lambda, c)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_{\sigma}^{\delta}(\alpha, \lambda, c)$ if the following conditions are satisfied:

$$f \in \sigma, \quad \left| \arg \left(\frac{(1-\lambda)\mathcal{L}_c^{\delta}f(z) + \lambda\mathcal{L}_c^{\delta-1}f(z)}{z} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, \lambda \geq 1, z \in U) \quad (3.1)$$

and

$$\left| \arg \left(\frac{(1-\lambda)\mathcal{L}_c^{\delta}g(w) + \lambda\mathcal{L}_c^{\delta-1}g(w)}{w} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, \lambda \geq 1, w \in U) \quad (3.2)$$

where $\mathcal{L}_c^{\delta-1}$ stands for polylogarithm function and the function (by Al-Shaqsi) $g(w) = f^{-1}(w)$.

Theorem 3.1. Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\sigma^\delta(\alpha, \lambda, c)$. Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^\delta \alpha - (\alpha - 1)\left(1 + \frac{\lambda}{1+c}\right)^2 \left(\frac{1+c}{2+c}\right)^{2\delta}}} \quad (3.3)$$

and

$$|a_3| \leq \frac{4\alpha^2}{\left(1 + \frac{\lambda}{1+c}\right)^2 \left(\frac{1+c}{2+c}\right)^{2\delta}} + \frac{2\alpha}{\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^\delta}, \text{ where } 0 \leq \beta < 1 \text{ and } \lambda \geq 1. \quad (3.4)$$

Proof. Let $f \in \mathcal{H}_\sigma^\delta(\alpha, \lambda, c)$, $\lambda \geq 1$ and $0 < \alpha \leq 1$. We can write the argument inequalities in (3.1) and (3.2) as follows:

$$\frac{(1 - \lambda)\mathcal{L}_c^\delta f(z) + \lambda\mathcal{L}_c^{\delta-1} f(z)}{z} = [p(z)]^\alpha, \quad z \in U \quad (3.5)$$

$$\frac{(1 - \lambda)\mathcal{L}_c^\delta g(w) + \lambda\mathcal{L}_c^{\delta-1} g(w)}{w} = [q(w)]^\alpha, \quad w \in U \quad (3.6)$$

where $p(z)$ and $q(w)$ are given by (2.7) and (2.8) and satisfy the inequalities $\operatorname{Re}(p(z)) > 0$ and $\operatorname{Re}(q(w)) > 0$ respectively. Now, equating the coefficients of (3.5) and (3.6), we have

$$\left(1 + \frac{\lambda}{1+c}\right)\left(\frac{1+c}{2+c}\right)^\delta a_2 = \alpha p_1, \quad (3.7)$$

$$\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^\delta a_3 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2, \quad (3.8)$$

$$-\left(1 + \frac{\lambda}{1+c}\right)\left(\frac{1+c}{2+c}\right)^\delta a_2 = \alpha q_1, \quad (3.9)$$

and

$$\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^\delta (2a_2^2 - a_3) = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2, \quad (3.10)$$

From (3.7) and (3.9), we get

$$p_1 = -q_1 \quad (3.11)$$

and

$$2\left(1 + \frac{\lambda}{1+c}\right)^2 \left(\frac{1+c}{2+c}\right)^{2\delta} a_2^2 = \alpha^2(p_1^2 + q_1^2) \quad (3.12)$$

Also from (3.8) and (3.10), we obtain

$$2\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^\delta a_2^2 = \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 + q_1^2). \quad (3.13)$$

By using the relation (3.12) in (3.13), we find that

$$2\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^\delta a_2^2 = \alpha(p_2 + q_2) + \left(1 + \frac{\lambda}{1+c}\right)^2 \left(\frac{1+c}{2+c}\right)^{2\delta} a_2^2.$$

Thus we get

$$a_2^2 = \frac{\alpha^2(p_2 + q_2)}{2\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^\delta \alpha - (\alpha - 1)\left(1 + \frac{\lambda}{1+c}\right)^2 \left(\frac{1+c}{2+c}\right)^{2\delta}}. \tag{3.14}$$

Then, applying Lemma 1.1 for the aforementioned equality, we get desired estimate on $|a_2|$ as asserted in (3.3). Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$2\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^\delta (a_3 - a_2^2) = \alpha(p_2 - q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 - q_1^2). \tag{3.15}$$

Also from (3.11), (3.12) and (3.15) we find that

$$a_3 = \frac{\alpha(p_2 - q_2)}{2\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^\delta} + \frac{\alpha^2(p_1^2 + q_1^2)}{2\left(1 + \frac{\lambda}{1+c}\right)^2 \left(\frac{1+c}{2+c}\right)^{2\delta}}. \tag{3.16}$$

By applying the Lemma 1 for the equality (3.16), we obtain desired estimate and this complats the proof of the theorem. \square

Remark. If we let $c = 0$ in Theorem 3.1 and

1. $\delta = -n$, we obtain the following corollary:

Corollary 3.1. (Porwal & Darus, 2013) *Let the function $f(z)$ given by (1.1) be in the class $B_\Sigma(n, \alpha, \lambda)$, $0 < \alpha \leq 1, \lambda \geq 1, n \in \mathbb{N}_0$. Then,*

$$|a_2| \leq \frac{2\alpha}{\sqrt{4^n(\lambda + 1)^2 + \alpha(2 \cdot 3^n(1 + 2\lambda) - 4^n(\lambda + 1)^2)}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{[(1 - \lambda)2^n + \lambda 2^{n+1}]^2} + \frac{2\alpha}{(1 - \lambda)3^n + \lambda 3^{n+1}}.$$

2. Choosing $\delta = 0$, we obtain the following corollary:

Corollary 3.2. (Frasin & Aouf, 2011). *Let the function $f(z)$ given by (1.1) be in the class $B_\Sigma(\lambda, \beta)$, $0 < \alpha \leq 1, \lambda \geq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda + 1)^2 + \alpha(1 + 2\lambda - \lambda^2)}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\lambda + 1)^2} + \frac{2\alpha}{2\lambda + 1}.$$

3. Also, if we choose $\lambda = 1$, we have the following corollary:

Corollary 3.3. (Srivastava et al., 2010). Let the function $f(z)$ given by (1.1) be in the class H_{Σ}^{α} , $0 < \alpha \leq 1$. Then

$$|a_2| \leq \alpha \sqrt{\frac{2}{2 + \alpha}}$$

and

$$|a_3| \leq \frac{\alpha(3\alpha + 2)}{3}.$$

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Statistical Convergence and C^* -operator Algebras

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Abstract

In this article, we define in terms of Berezin symbols, reproducing kernels and statistical radial convergence the notion of generalized Engliš algebra, which is a C^* -operator algebra on the Hardy space $H^2(\mathbb{D})$, and study its some properties.

Keywords: Berezin symbol, Engliš algebra, statistical convergence, Toeplitz operators, Hardy space.
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1. Introduction

This article is mainly motivated with the papers by Engliš (Engliš, 1995), Karaev (Karaev, 2002, 2004, 2010) and Pehlivan and Karaev (Pehlivan & Karaev, 2004), where the authors systematically applied the Berezin symbols method in summability theory; and conversely, the summability methods are used in investigation of some important problems for C^* -operator algebras on the Hardy space and also on the Bergman space. In particular, Pehlivan and Karaev investigated in (Pehlivan & Karaev, 2004) compactness of the weak limit of the sequence of compact operators on a Hilbert space by using the notion of so-called statistical convergence.

In this article, we use statistical radial limits for the study of some special C^* -operator algebras on the classical Hardy space $H^2(\mathbb{D})$ over the unit disc \mathbb{D} of the complex plane \mathbb{C} . These results generalize some results in the paper (Engliš, 1995).

The Hardy space $H^2 = H^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ satisfying

$$\|f\|_2^2 := \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt < +\infty.$$

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Alternately, H^2 consists of all functions in $L^2(\mathbb{T})$ whose negative Fourier coefficients vanish; here $\mathbb{T} = \partial\mathbb{D}$ is the unit circle in \mathbb{C} . The orthogonal projection from $L^2(\mathbb{T})$ onto H^2 will be denoted P_+ , and $P_- : I - P_+$. For $\varphi \in L^\infty(\mathbb{T})$ the Toeplitz operator T_φ and Hankel operator H_φ with symbol φ are defined by $T_\varphi f = P_+ \varphi f$ and $H_\varphi f = P_- \varphi f$ and are bounded linear operators from H^2 into H^2 (i.e., $T_\varphi \in \mathcal{B}(H^2)$) and $L^2(\mathbb{T}) \ominus H^2$, respectively. For $\lambda \in \mathbb{D}$, the reproducing kernel (Szegö kernel) of H^2 is the function $k_\lambda \in H^2$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle$$

for every $f \in H^2$. The normalized reproducing kernel \widehat{k}_λ is the function $\frac{k_\lambda}{\|k_\lambda\|}$. It is well-known (and can be easily shown) that $k_\lambda(z) := \frac{1}{1-\lambda z}$. The algebra of all bounded linear operators on the Hilbert space H is denoted by $\mathcal{B}(H)$. For $T \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H} = \mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space over some set Ω with the reproducing kernel $k_{\mathcal{H},\lambda} \in \mathcal{H}$, its Berezin symbol \widetilde{T} is the complex-valued function on Ω defined by

$$\widetilde{T}(\lambda) = \langle T \widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \rangle \quad (\lambda \in \Omega).$$

It is well-known that $\widetilde{T}_\varphi(\lambda) = \widetilde{\varphi}(\lambda)$, where $\widetilde{\varphi}$ denotes the harmonic extension of φ into the unit disc \mathbb{D} (see Engliš (Engliš, 1995), Zhu (Zhu, 1990) and Karaev (Karaev, 2002)).

Engliš determined in (Engliš, 1995) in terms of nontangential and radial convergences some C^* -operator algebras on the Hardy space $H^2(\mathbb{D})$, which is defined by the boundary behavior of $\|T \widehat{k}_\lambda\|$, $\|T^* \widehat{k}_\lambda\|$ and $|\widetilde{T}(\lambda)|$. Let \mathcal{T} be the C^* -algebra generated by $\{T_\varphi : \varphi \in L^\infty(\mathbb{T})\}$. The following celebrated result due to Douglas (Douglas, 1972) (see also in (Nikolski, 1986)).

Theorem D (Douglas). *There is a C^* -homomorphism*

$$\sigma : \mathcal{T} \rightarrow L^\infty(\mathbb{T})$$

of \mathcal{T} onto $L^\infty(\mathbb{T})$ which satisfies $\sigma(T_\varphi) = \varphi$ ($\forall \varphi \in L^\infty(\mathbb{T})$). The kernel of σ coincides with the commutator ideal of \mathcal{T} , i.e., the ideal in \mathcal{T} generated by all commutators

$$[R, S] := RS - SR \quad (R, S \in \mathcal{T}).$$

σ is sometimes called the symbol map.

Note that the major goal of the Engliš’s paper (Engliš, 1995) is to develop an alternative approach for proving results akin to the Douglas theorem. The symbol of an operator $T \in \mathcal{T}$ is then obtained in (Engliš, 1995) as the nontangential boundary value of a certain function on \mathbb{D} associated with T (called the Berezin symbol (transform), \widetilde{T} , of T to be defined above). Following by (Engliš, 1995), remark that Engliš’s method also works for some operator algebras larger than the Toeplitz algebra, thereby yielding a number of interesting generalizations of the classical Toeplitz symbol calculus. This method is also applicable to the Bergman space, where the resulting symbol calculus is related to the one obtained by Berger and Coburn in (Berger & Coburn, 1986, 1987), Gürdal and Şöhret in (Gurdal & Sohret, 2011) and Zhu in (Zhu, 1987).

In the present article, we replace nontangential and radial limits by so-called statistical nontangential and statistical radial limits (which are weaker than the usual one) and define generalizations

of some Engliš’s algebras. It turns out that the same results are true for these generalized Engliš’s algebras. Before giving our results, let us recall the definition of the statistical convergence of real or complex numbers sequence.

If K is a subset of the positive integers \mathbb{N} , the K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the number of elements in K_n . The natural density of K (see ((Niven & Zuckerman, 1980), Chapter 11) is given by $\delta(K) = \lim_{n \rightarrow \infty} \frac{|K_n|}{n}$. A sequence $(x_k : k = 1, 2, \dots)$ of (real or complex) numbers is said to be statistically convergent to some number L if for each $\varepsilon > 0$ the set $K_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero; in this case we write $st - \lim_k x_k = L$. In what follows statistical convergence studied in many further papers (see, for instance, (Braha et al., 2014), (Mursaleen et al., 2014)).

The following notion is due to Fridy (Fridy, 1985). A sequence (x_k) is said to be statistically Cauchy if for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - x_N| \geq \varepsilon\}| = 0.$$

We recall that (see (Fridy, 1985)) for two sequences $x = (x_k)$ and $y = (y_k)$ the notion ” $x_k = y_k$ for almost all k ” means that $\delta(\{k : x_k \neq y_k\}) = 0$. Fridy proved the following main result of this theory (Fridy, 1985).

Theorem F (Fridy). *The following statements are equivalent:*

- (1) (x_k) is a statistically convergent sequence;
- (2) (x_k) is a statistically Cauchy sequence;
- (3) (x_k) is a sequence for which there is a convergent sequence (y_k) such that $x_k = y_k$ for almost all k .

The following result is immediate from Theorem F.

Corollary 1.1. *If (x_k) is a sequence such that $st - \lim_k x_k = L$, then (x_k) has a subsequence (y_k) such that $\lim_k y_k = L$ (in the usual sense).*

2. On some properties of Berezin Symbols

In this section, we prove some results concerning to Berezin symbols.

2.1. Approach Regions

For $0 < \alpha < 1$, following by Rudin (Rudin, 1974), pp. 240-241, let us define Ω_α to be the union of the disc $D(0; \alpha) := \{z \in \mathbb{C} : |z| < \alpha\}$ and the line segments from $z = 1$ to points of $D(0; \alpha)$. In other words, Ω_α is the smallest convex open set that contains $D(0; \alpha)$ and has the point 1 in its boundary. Near $z = 1$, Ω_α is an angle, bisected by the radius of \mathbb{D} that terminates at 1, of opening 2θ , where $\alpha = \sin \theta$. Curves that approach 1 within Ω_α cannot be tangent to \mathbb{T} . Therefore Ω_α is called a nontangential approach region, with vertex 1. The regions Ω_α expand when α increases. The union is \mathbb{D} , their intersection is the radius $[0, 1)$. Rotated copies of Ω_α , with vertex at e^{it} , will be denoted by $e^{it}\Omega_\alpha$.

2.2. **Statistical Nontangential and Radial Limits.**

A function F , defined in \mathbb{D} , is said to have statistical nontangential limit λ at $e^{i\theta} \in \mathbb{T}$ if, for each $\alpha < 1$,

$$st - \lim_{j \rightarrow \infty} F(z_j) = \lambda$$

for every sequence $\{z_j\}$ that statistically converges to $e^{i\theta}$ and that lies in $e^{i\theta}\Omega_\alpha$. If F is a function on \mathbb{D} , and f a function on \mathbb{T} , we say that F tends to f statistically radially, written

$$st - \lim F(z_n) = f(e^{i\theta})$$

whenever $(z_n) \rightarrow e^{i\theta}$ statistically radially (i.e., if $st - \lim_{n \rightarrow \infty} |z_n| e^{i\theta} = e^{i\theta}$) for almost all $\theta \in [0, 2\pi)$.

2.3. **Some Properties of Berezin Symbols of Operators.**

For any two functions $\varphi, \psi \in L^\infty(\mathbb{T})$, let us denote $[T_\varphi, T_\psi) := T_{\varphi\psi} - T_\varphi T_\psi$, which is called their semicommutator.

Theorem 2.1. For $\varphi \in L^\infty(\mathbb{T})$, $\|H_\varphi \widehat{k}_\lambda\| \rightarrow 0$ statistically radially. Consequently, for any $\varphi, \psi \in L^\infty(\mathbb{T})$, $[T_\varphi, T_\psi) \sim(\lambda) \rightarrow 0$ statistically radially.

Proof. Following the method of the paper (Engliš, 1995), let $y := P_- \varphi$. By considering that $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{H^2}} \in H^\infty$, we may write

$$H_\varphi \widehat{k}_\lambda = P_-(\varphi \widehat{k}_\lambda) = P_-(P_+ \varphi \widehat{k}_\lambda) + P_-(y \widehat{k}_\lambda) = P_-(y \widehat{k}_\lambda) = H_y \widehat{k}_\lambda,$$

and hence

$$\|H_\varphi \widehat{k}_\lambda\|^2 = \|y \widehat{k}_\lambda\|^2 - \|T_y \widehat{k}_\lambda\|^2 = \langle |y|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle - \|T_y \widehat{k}_\lambda\|^2 = \widetilde{|y|^2}(\lambda) - \|T_y \widehat{k}_\lambda\|^2.$$

Then, for any $\beta \in \mathbb{D}$, we have

$$\langle T_y k_\lambda, k_\beta \rangle = \langle k_\lambda, \bar{y} k_\beta \rangle = \bar{y}(\lambda) \langle k_\lambda, k_\beta \rangle,$$

since \bar{y} is the boundary value of an analytic function. It follows that $T_\varphi \widehat{k}_\lambda = \bar{y}(\lambda) \widehat{k}_\lambda$ and

$$\|H_\varphi \widehat{k}_\lambda\|^2 = \widetilde{|y|^2}(\lambda) - |\bar{y}(\lambda)|^2.$$

By Fatou’s theorem, both $\widetilde{|y|^2}$ and $|\bar{y}|^2$ tend radially to $|y|^2$, which implies that they tend statistically radially to $|y|^2$, and so their difference statistically tends to zero and we are done.

Observe that, $[T_\varphi, T_\psi) = H_\varphi^* H_\psi$, and therefore we obtain that

$$\begin{aligned} |[T_\varphi, T_\psi) \sim(\lambda)| &= \left| \langle H_\varphi^* H_\psi \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| = \left| \langle H_\psi \widehat{k}_\lambda, H_\varphi \widehat{k}_\lambda \rangle \right| \leq \\ &\leq \|H_\psi \widehat{k}_\lambda\| \|H_\varphi \widehat{k}_\lambda\| \rightarrow 0 \end{aligned}$$

statistically radially, and the second assertion follows. The theorem is proved. □

Let us call the closed ideal in \mathcal{T} , generated by all semicommutators $[T_\varphi, T_\psi]$, $\varphi, \psi \in L^\infty(\mathbb{T})$, the semicommutator ideal. The following strengthens Theorem 2.1.

Theorem 2.2. *If $T \in \mathcal{B}(H^2)$ belongs to the semicommutator ideal of \mathcal{T} , then $\widetilde{T} \rightarrow 0$ statistically radially.*

Proof. Since the linear combinations of Toeplitz operators of the form

$$T_{\varphi_1} T_{\varphi_2} \dots T_{\varphi_n} [T_a, T_b] T_{\psi_1} T_{\psi_2} \dots T_{\psi_m} \tag{1}$$

form a dense subset of the semicommutator ideal (and therefore it is a statistically dense subset of the ideal in question), it suffices to prove the assertion when T is of the form (1). So, we obtain (see also (Engliš, 1995))

$$\begin{aligned} T_c [T_a, T_b] &= T_c T_{ab} - T_c T_a T_b \\ &= (T_c T_{ab} - T_{cab}) + (T_{cab} - T_{ca} T_b) + (T_{ca} T_b - T_c T_a T_b) \\ &= -[T_c, T_{ab}] + [T_{ca}, T_b] + [T_c, T_a] T_b. \end{aligned}$$

It follows that we may even assume T to be of the form

$$T = [T_\varphi, T_\psi] A = H_\varphi^* H_\psi A, \quad A \in \mathcal{T}, \quad \varphi, \psi \in L^\infty(\mathbb{T}).$$

Now by using that $[T_\varphi, T_\psi] = H_\varphi^* H_\psi$, we have

$$\begin{aligned} |\widetilde{T}(\lambda)| &= \left| \langle T \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| = \left| \langle H_\varphi^* H_\psi A \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \\ &= \left| \langle H_\psi A \widehat{k}_\lambda, H_\varphi \widehat{k}_\lambda \rangle \right| \\ &\leq \|H_\psi A\| \|H_\varphi \widehat{k}_\lambda\| \quad (\text{Cauchy-Schwarz inequality}) \end{aligned}$$

and by considering that $\|H_\varphi \widehat{k}_\lambda\| \rightarrow 0$ statistically radially (see Theorem 2.1), we obtain that $\widetilde{T} \rightarrow 0$ statistically radially, as desired. □

Now we prove more general theorem which improves Engliš’s result (see [4, Theorem 2]) and implies the theorem of Douglas (Theorem D) as an easy corollary.

Theorem 2.3. *For any T in \mathcal{T} , $\widetilde{T} \rightarrow \varphi$ statistically radially for some function $\varphi \in L^\infty(\mathbb{T})$. The mapping*

$$\sigma : \mathcal{T} \rightarrow L^\infty(\mathbb{T}), \quad T \rightarrow \varphi$$

is a C^ -algebra morphism, its kernel is precisely the commutator ideal of \mathcal{T} , and $\sigma(T_\psi) = \psi$ for any Toeplitz operator T_ψ . Thus, σ coincides with the symbol map from Theorem D.*

Proof. Let \mathcal{J} be the semicommutator ideal in \mathcal{T} . As in (Berger & Coburn, 1987), by repeated applications of the identity $AT_a T_b B - AT_{ab} B = -A [T_a, T_b] B$, we have that $T_{\varphi_1} T_{\varphi_2} \dots T_{\varphi_n} - T_{\varphi_1} T_{\varphi_2} \dots \varphi_n \in \mathcal{J}$ for any $\varphi_1, \varphi_2, \dots, \varphi_n \in L^\infty(\mathbb{T})$. By considering that linear combinations of operators of the form

$$T = T_\varphi + S, \quad \varphi \in L^\infty(\mathbb{T}), \quad S \in \mathcal{J}, \tag{2}$$

form a statistically dense subset of \mathcal{T} . According to the fact that $\widetilde{T}_\varphi = \widetilde{\varphi}$ and Theorem 2.2, $\widetilde{T} \rightarrow \varphi$ statistically radially, and thus we have $\|\varphi\|_\infty \leq \|\widetilde{T}\|_\infty \leq \|T\|$. It follows that the mapping

$$\sigma : T_\varphi + S \mapsto \varphi$$

is well defined and extends continuously to the whole of \mathcal{T} , so that, in particular, every operator in \mathcal{T} is of the form (2), because a statistical limit of Toeplitz operators is again a Toeplitz operator. Indeed, if

$$st - \lim_n \|T_{\varphi_n} - X\| = 0$$

for some $X \in \mathcal{B}(H^2)$, then by Corollary 1, the sequence $(\|T_{\varphi_n} - X\|)_{n \geq 1}$ has a subsequence $(\|T_{\varphi_{n_k}} - X\|)_k$ such that $\lim_k \|T_{\varphi_{n_k}} - X\| = 0$ in the usual sense, which easily implies that X is a Toeplitz operator, and so \mathcal{J} is statistically closed.

This mapping is clearly linear, preserves conjugation and, owing to Theorem 2.2, is also multiplicative. It is clear that its kernel is precisely the semicommutator ideal \mathcal{J} of \mathcal{T} . Now it remains only to show that \mathcal{J} coincides with the commutator ideal \mathcal{G} . Since $[T_\varphi, T_\psi] = [T_\psi, T_\varphi] - [T_\varphi, T_\psi]$, the inclusion $\mathcal{G} \subset \mathcal{J}$ is trivial. The reverse inclusion $\mathcal{J} \subset \mathcal{G}$ is proved in (Engliš, 1995), and therefore we omit it. The theorem is proved. \square

Corollary 2.1. *If $A \in \mathcal{T}$, then $[A^*, A]^\sim \rightarrow 0$ statistically radially.*

3. Generalized Engliš algebras and extending the Toeplitz calculus

In the present section, we introduce the concept of generalized Engliš algebra of operators, study some properties and exhibit a family of generalized Engliš C^* -algebras containing \mathcal{T} for which analogs of Theorem 2.3 still hold.

For this reason, let us define the following generalized Engliš algebra:

$$\mathcal{E} := \left\{ H \in \mathcal{B}(H^2) : \|\widehat{Hk}_\lambda\| \text{ and } \|\widehat{H^*k}_\lambda\| \rightarrow 0 \text{ statistically radially} \right\}.$$

In other words, we demand that

$$st - \lim_{r \nearrow 1} \|\widehat{Hk}_{re^{i\theta}}\| = 0$$

for all $\theta \in [0, 2\pi) \setminus E$, where E is a set (depending on H) of zero Lebesgue measure; similarly for H^* . Note that in case of usual radial limits, this algebra is the usual Engliš algebra (Engliš, 1995).

In the following theorem we give some important properties of algebra \mathcal{E} .

Theorem 3.1. *We have:*

- (a) \mathcal{E} is a C^* -algebra;
- (b) If $T_\varphi \in \mathcal{E}$, then $\varphi = 0$;
- (c) For $\varphi, \psi \in L^\infty(\mathbb{T})$, $[T_\varphi, T_\psi] \in \mathcal{E}$;
- (d) \mathcal{E} is an "ideal with respect to Toeplitz operators", that is, $H \in \mathcal{E}$ and $\varphi \in L^\infty(\mathbb{T})$ implies that $HT_\varphi, T_\varphi H \in \mathcal{E}$.

Proof. (a) Everything is trivial, except may be for the implication $A, B \in \mathcal{E} \Rightarrow AB \in \mathcal{E}$. But, $0 \leq \|AB\widehat{k}_\lambda\| \leq \|A\| \|B\widehat{k}_\lambda\| \rightarrow 0$ statistically radially, and similarly for B^*A^* .

(b) Indeed, $\|T_\varphi\widehat{k}_\lambda\| \rightarrow 0$ statistically radially implies that

$$|\widetilde{T}_\varphi(\lambda)| = \left| \langle T_\varphi\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \leq \|T_\varphi\widehat{k}_\lambda\| \rightarrow 0$$

statistically radially, and hence $\widetilde{T}_\varphi(\lambda) \rightarrow 0$ statistically radially. But, we know that $\widetilde{T}_\varphi \rightarrow \varphi$ statistically radially, so $\varphi = 0$.

(c) The proof is immediate from Theorem 2.1 and the equality $[T_\varphi, T_\psi] = H_\varphi^* H_\psi$.

(d) Indeed, we have that

$$0 \leq \|T_\varphi H\widehat{k}_\lambda\| \leq \|T_\varphi\| \|H\widehat{k}_\lambda\| \rightarrow 0$$

statistically radially, and similarly for $T_\varphi H^*\widehat{k}_\lambda$. So, the corresponding assertions for $HT_\varphi\widehat{k}_\lambda$ are immediate from the following fact. □

Proposition 1. *Let $\varphi \in L^\infty(\mathbb{T})$, and denote, as before, by $\widetilde{\varphi}$ its harmonic extension (by the Poisson formula) into \mathbb{D} . Then $T_\varphi\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda \rightarrow 0$ statistically radially, i.e.,*

$$st - \lim_{r \nearrow 1} \left\| T_\varphi\widehat{k}_{re^{i\theta}} - \widetilde{\varphi}(re^{i\theta})\widehat{k}_{re^{i\theta}} \right\| = 0$$

for almost all $t \in [0, 2\pi)$.

The proof of this proposition is immediate from Theorem 6 in [4] and Corollary 1.1 in Section 1.

Denote

$$\mathcal{A}_1 := \{T_\varphi + H : \varphi \in L^\infty(\mathbb{T}), H \in \mathcal{E}\}.$$

The following theorem, which generalizes the Douglas theorem, can be proved by using Corollary 1.1 and the method of the proof of Theorem 7 in (Engliš, 1995) and therefore its proof is omitted.

Theorem 3.2. *We have;*

(i) \mathcal{A}_1 is a C^* -algebra.

(ii) For any $T \in \mathcal{A}_1$, there exists a statistical radial limit, denoted $\sigma_{st}(T)$, of $\widetilde{T}(\lambda)$:

$$\widetilde{T} \rightarrow \sigma_{st}(T) \in L^\infty(\mathbb{T}) \text{ statistically radially.}$$

(iii) $\sigma_{st} : \mathcal{A}_1 \rightarrow L^\infty(\mathbb{T})$ induces a C^* -isomorphism of $\mathcal{A}_1/\mathcal{E}$ onto $L^\infty(\mathbb{T})$ which maps T_φ into φ , for any $\varphi \in L^\infty(\mathbb{T})$.

The following result shows that Theorem 3.2 can also be used for the characterization of the class \mathcal{T} .

Proposition 2. Let $A \in \mathcal{B}(H^2)$ be an operator. If $A \in \mathcal{T}$, then

- (i) \widetilde{A} has statistical radial limit : $\widetilde{A} \xrightarrow{\text{st}} \varphi$ statistically radially for some $\varphi \in L^\infty(\mathbb{T})$; and
- (ii) $\|A\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\|$ and $\|A^*\widehat{k}_\lambda - \overline{\widetilde{\varphi}(\lambda)}\widehat{k}_\lambda\| \rightarrow 0$ statistically radially.

Proof. Let $A \in \mathcal{T} \subset \mathcal{A}_1$. Then clearly $A = T_\varphi + H$ for some $\varphi \in L^\infty(\mathbb{T})$ and $H \in \mathcal{E}$. Hence $\widetilde{A} \xrightarrow{\text{st}} \varphi$ statistically radially, which proves (i), and $\|A\widehat{k}_\lambda - T_\varphi\widehat{k}_\lambda\| = \|H\widehat{k}_\lambda\| \rightarrow 0$ statistically radially. By virtue of Proposition 1, this is equivalent to $\|A\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\| \rightarrow 0$ statistically radially, because

$$\begin{aligned} \|A\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\| &\leq \|A\widehat{k}_\lambda - T_\varphi\widehat{k}_\lambda\| + \|T_\varphi\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\| \\ &\leq \|H\widehat{k}_\lambda\| + \|T_\varphi\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\| \rightarrow 0 \end{aligned}$$

statistically radially. Similarly, it can be proved that $\|A^*\widehat{k}_\lambda - \overline{\widetilde{\varphi}(\lambda)}\widehat{k}_\lambda\| \rightarrow 0$ statistically radially, which completes the proof of (ii). So, the proposition is proved. \square

Below we give some results further extending the result of Engliš from (Engliš, 1995) by means of statistical Banach limits. The proofs of them are slight modification of analogous results of the paper (Engliš, 1995), and therefore omitted.

Following by (Engliš, 1995), recall that if $BC[0, 1)$ is the C^* -algebra of all bounded continuous functions on the half-open interval $[0, 1)$. It is known by Gelfand theory that $BC[0, 1) \simeq C(M)$, where M is the maximal ideal space of $BC[0, 1)$. M consist of a homeomorphic copy of $[0, 1)$ plus a certain fiber, denoted M_1 , over the point 1. Each multiplicative linear functional $Lim \in M_1$ will be called a Banach limit. For $f \in L^\infty(\mathbb{D})$ and $\varphi \in L^\infty(\mathbb{T})$, we say that f tends to φ statistically radially with respect to Lim , written $f \xrightarrow{Lim} \varphi$ statistically radially, when

$$Lim(r \xrightarrow{st} f(re^{i\theta})) = \varphi(e^{i\theta})$$

for all $\theta \in [0, 2\pi)$ except for a set of measure zero. Define

$\mathcal{E}_{st-Lim} := \{H \in \mathcal{B}(H^2) : \|H\widehat{k}_\lambda\| \text{ and } \|H^*\widehat{k}_\lambda\| \xrightarrow{Lim} 0 \text{ statistically radially}\}$. It can be easily shown that all of assertions in Theorem 3.1 remain in force when \mathcal{E} is replaced by \mathcal{E}_{st-Lim} , and thus we obtain the following analogs of Theorem 3.2 and Proposition 2.

Theorem 3.3. Let $\mathcal{A}_{st-Lim} := \{T_\varphi + H : \varphi \in L^\infty(\mathbb{T}), H \in \mathcal{E}_{st-Lim}\}$. Then

- (i) \mathcal{A}_{st-Lim} is a C^* -algebra;
- (ii) $\forall T \in \mathcal{A}_{st-Lim} \exists \varphi \in L^\infty(\mathbb{T})$ such that $\widetilde{T} \xrightarrow{Lim} \varphi$ statistically radially;
- (iii) The mapping $T \mapsto \varphi$ is a C^* morphism of $\mathcal{A}_{st-Lim}/\mathcal{E}_{st-Lim}$ onto $L^\infty(\mathbb{T})$ which maps T_φ onto φ .

Proposition 3. Let A be an operator on H^2 . A necessary and sufficient condition for $A \in \mathcal{A}_{st-Lim}$ is that

- (i) there is $\varphi \in L^\infty(\mathbb{T})$ such that $\widetilde{A} \xrightarrow{Lim} \varphi$ statistically radially;
- (ii) $\|A\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\| \xrightarrow{Lim} 0$ statistically radially and $\|A^*\widehat{k}_\lambda - \overline{\widetilde{\varphi}(\lambda)}\widehat{k}_\lambda\| \xrightarrow{Lim} 0$ statistically radially.

In conclusion we define one more generalized Engliš algebra (which in case of usual radial limits is defined by Engliš (Engliš, 1995)) and study its some properties.

Define

$$\mathcal{A}_{st} := \left\{ T \in \mathcal{B}(H^2) : \left\| T\widehat{k}_\lambda \right\|^2 - \left| \langle T\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \text{ and } \left\| T^*\widehat{k}_\lambda \right\|^2 - \left| \langle T\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \rightarrow 0 \text{ statistically radially} \right\} \quad (3)$$

or, in other words, $\left\| T\widehat{k}_\lambda \right\|^2 - |\widetilde{T}(\lambda)|^2 \rightarrow 0$ statistically radially and similarly for T^* . Following by the arguments of the paper (Engliš, 1995), if we decompose $T\widehat{k}_\lambda$ as

$$T\widehat{k}_\lambda = c_\lambda \widehat{k}_\lambda + d_\lambda, \quad c_\lambda \in \mathbb{C}, \quad d_\lambda \perp \widehat{k}_\lambda \quad (4)$$

then $\widetilde{T}(\lambda) = \langle T\widehat{k}_\lambda, \widehat{k}_\lambda \rangle = c_\lambda$ and $\left\| T\widehat{k}_\lambda \right\|^2 = |c_\lambda|^2 + \|d_\lambda\|^2$, so the first condition in (3) reads just

$$d_\lambda \rightarrow 0 \text{ statistically radially} \quad (5)$$

Proposition 4. \mathcal{A}_{st} is a C^* -algebra.

Proof. In fact, it follows from (4) and (5) that \mathcal{A}_{st} is linear, and from (3) that (see (Engliš, 1995)) it is self-adjoint and statistically norm closed. If $T, T' \in \mathcal{A}_{st}$ and $T\widehat{k}_\lambda = c_\lambda \widehat{k}_\lambda + d_\lambda$, $T'\widehat{k}_\lambda = c'_\lambda \widehat{k}_\lambda + d'_\lambda$ are the decomposition (4), then

$$TT'\widehat{k}_\lambda = Td'_\lambda + c'_\lambda c_\lambda \widehat{k}_\lambda + c'_\lambda d_\lambda.$$

Let $TT'\widehat{k}_\lambda = c''_\lambda \widehat{k}_\lambda + d''_\lambda$ be the decomposition (4) for TT' . Then

$$(c'_\lambda c_\lambda - c''_\lambda) \widehat{k}_\lambda = (d''_\lambda - c'_\lambda d_\lambda) - Td'_\lambda. \quad (6)$$

Since $d_\lambda \perp \widehat{k}_\lambda$ and $d''_\lambda \perp \widehat{k}_\lambda$, taking the inner product with \widehat{k}_λ on both sides gives

$$\begin{aligned} |c'_\lambda c_\lambda - c''_\lambda| &= \left| \langle Td'_\lambda, \widehat{k}_\lambda \rangle \right| \\ &\leq \|T\| \|d'_\lambda\| \rightarrow \text{statistically radially (by (5)).} \end{aligned} \quad (7)$$

Now putting this back into (6) shows that

$$d''_\lambda - (c'_\lambda d_\lambda + Td'_\lambda) \rightarrow 0 \text{ statistically radially.}$$

Now $|c'_\lambda| \leq \|T'\|$, so by (5) and the boundedness of T and T' it follows that

$$c'_\lambda d_\lambda + Td'_\lambda \rightarrow 0 \text{ statistically radially.}$$

Consequently, $d''_\lambda \rightarrow 0$ statistically radially. By a similar arguments for T'^*T^* it can be easily proved that $TT' \in \mathcal{A}_{st}$, that is \mathcal{A}_{st} is also closed under multiplication. So, \mathcal{A}_{st} is a C^* -algebra, as desired. \square

Proposition 5. For $\varphi \in L^\infty(\mathbb{T})$, $T_\varphi \in \mathcal{A}_{st}$.

Proof. Immediate from Proposition 1 and the fact that $\widetilde{T}_\varphi = \widetilde{\varphi}$. □

Proposition 6. If $H \in \mathcal{E}$, then $H \in \mathcal{A}_{st}$.

Proof. The assertion that $\|H\widehat{k}_\lambda\| \rightarrow 0$ statistically radially implies that $\langle H\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \rightarrow 0$ statistically radially (because $|\langle H\widehat{k}_\lambda, \widehat{k}_\lambda \rangle| \leq \|H\widehat{k}_\lambda\|$), and hence $\|H\widehat{k}_\lambda\|^2 - |\langle H\widehat{k}_\lambda, \widehat{k}_\lambda \rangle| \rightarrow 0$ statistically radially as well; similarly for H^* , and thus $\mathcal{E} \subset \mathcal{A}_{st}$, as desired. □

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On a Subclass of Harmonic Univalent Functions Based on Subordination

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Abstract

In this paper, we introduce a new class of harmonic univalent functions defined by subordination with a linear operator. Certain properties of this class are discussed.

Keywords: Harmonic functions, univalent functions, Hadamard product, modified generalized Sălăgean operator, subordination and modified multiplier transformation.

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1. Introduction

Let H denote the family of continuous complex valued harmonic functions which are harmonic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in U . A function harmonic in U may be written as $f = h + \bar{g}$, where h and g are members of A . We call h the analytic part and g co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in U is that $|h'(z)| > |g'(z)|$ (see Clunie and Sheil-Small (Clunie & Sheil-Small, 1984)). To this end, without loss of generality, we may write

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=2}^{\infty} b_k z^k. \quad (1.1)$$

Let SH denote the family of functions $f = h + \bar{g}$ which are harmonic, univalent, and sense-preserving in U for which $f(0) = f_z(0) - 1 = 0$. The subclass SH^0 of SH consists of all functions in SH which have the additional property $f_{\bar{z}}(0) = 0$.

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In 1984 Clunie and Sheil-Small (Clunie & Sheil-Small, 1984) investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on SH and its subclasses. Also note that SH reduces to the class S of normalized analytic univalent functions in U , if the co-analytic part of f is identically zero.

For $f \in S$, the differential operator D^n ($n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$) of f was introduced by Sălăgean (Sălăgean, 1983). For $f = h + \bar{g}$ given by (1.1), Jahangiri et al. (Jahangiri et al., 2002) defined the modified Sălăgean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)},$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad D^n g(z) = \sum_{k=2}^{\infty} k^n b_k z^k.$$

Next, for functions $f \in A$, Al-Oboudi (Al-Oboudi, 2004) defined multiplier transformations. For $n \in \mathbb{N}_0$, $\lambda \geq 1$ and $f \in SH^0$ of the form (1.1), Yaşar and Yalçın (Yaşar & Yalcin, 2012) defined the modified Al-Oboudi operator $D_\lambda^n : SH^0 \rightarrow SH^0$ by

$$D_\lambda^0 f(z) = D^0 f(z) = h(z) + \overline{g(z)},$$

$$D_\lambda^1 f(z) = (1 - \lambda) D^0 f(z) + \lambda D^1 f(z), \quad \lambda \geq 1, \tag{1.2}$$

$$D_\lambda^n f(z) = D_\lambda^1 (D_\lambda^{n-1} f(z)). \tag{1.3}$$

If f is given by (1.1), then from (1.2) and (1.3) we see that (see (Yaşar & Yalcin, 2012))

$$D_{1,\lambda}^n h(z) = z + \sum_{k=2}^{\infty} [\lambda(k - 1) + 1]^n a_k z^k,$$

$$D_{2,\lambda}^n g(z) = \sum_{k=2}^{\infty} [\lambda(k + 1) - 1]^n b_k z^k,$$

$$D_\lambda^n f(z) = D_{1,\lambda}^n h(z) + (-1)^n \overline{D_{2,\lambda}^n g(z)}$$

or

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [\lambda(k - 1) + 1]^n a_k z^k + (-1)^n \sum_{k=2}^{\infty} [\lambda(k + 1) - 1]^n \overline{b_k z^k}. \tag{1.4}$$

When $\lambda = 1$, we get modified Sălăgean differential operator (Jahangiri et al., 2002). If we take the co-analytic part of $f = h + g$ of the form (1.1) is identically zero, $D_\lambda^n f$ reduces to the Al-Oboudi operator (Al-Oboudi, 2004).

The Hadamard product (or convolution) of functions f_1 and f_2 of the form

$$f_t(z) = z + \sum_{k=2}^{\infty} a_{t,k} z^k + \sum_{k=2}^{\infty} \overline{b_{t,k} z^k} \quad (z \in U, t = \{1, 2\})$$

is defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{1,k} a_{2,k} z^k + \sum_{k=2}^{\infty} \overline{b_{1,k} b_{2,k}} z^k \quad (z \in U)$$

Also if f is given by (1.1), then we have

$$\begin{aligned} D_{\lambda}^n f(z) & : = f(z) * \underbrace{(\phi_1(z) + \overline{\phi_2(z)}) * \dots * (\phi_1(z) + \overline{\phi_2(z)})}_{n \text{ times}}, \\ & = h(z) * \underbrace{\phi_1(z) * \dots * \phi_1(z)}_{n \text{ times}} + \overline{g(z)} * \underbrace{\overline{\phi_2(z)} * \dots * \overline{\phi_2(z)}}_{n \text{ times}}. \end{aligned}$$

where

$$\phi_1(z) = \frac{(\lambda - 1)z^2 + z}{(1 - z)^2}, \quad \phi_2(z) = \frac{(\lambda - 1)z^2 + (1 - 2\lambda)z}{(1 - z)^2}.$$

We say that a function $f : U \rightarrow \mathbb{C}$ is subordinate to a function $g : U \rightarrow \mathbb{C}$, and write $f(z) \prec g(z)$, if there exists a complex valued function w which maps U into itself with $w(0) = 0$, such that

$$f(z) = g(w(z)) \quad (z \in U).$$

Furthermore, if the function g is univalent in U , then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Denote by $SH^0(\lambda, n, A, B)$ the subclass of SH^0 consisting of functions f of the form (1.1) that satisfy the condition

$$\frac{D_{\lambda}^{n+1} f(z)}{D_{\lambda}^n f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad -B \leq A < B \leq 1 \tag{1.5}$$

where $D_{\lambda}^n f(z)$ is defined by (1.4).

By suitably specializing the parameters, the classes $SH^0(\lambda, n, A, B)$ reduces to the various subclasses of harmonic univalent functions. Such as,

- (i) $SH^0(1, \lambda, A, B) = H_{\lambda}(A, B)$, $\lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ (Dziok et al. (2016)),
- (ii) $SH^0(1, 1, A, B) = S_H^*(A, B) \cap SH^0$ ((Dziok, 2015a)),
- (iii) $SH^0(\lambda, n, 2\alpha - 1, 1) = SH(\lambda, n, \alpha) \cap SH^0$ (Yaşar & Yalcin, 2012),
- (iv) $SH^0(1, n, 2\alpha - 1, 1) = H^0(n, \alpha)$ (Jahangiri et al., 2002),
- (v) $SH^0(1, 0, 2\alpha - 1, 1) = S_{H^0}^*(\alpha)$ ((Jahangiri, 1999), (Silverman, 1998), (Silverman & Silvia, 1999)),
- (vi) $SH^0(1, 1, 2\alpha - 1, 1) = S_{H^0}^c(\alpha)$ ((Jahangiri, 1999)),
- (vii) $SH^0(\lambda, n, 2\alpha - 1, 1) = \overline{SH}(\lambda, 1 - \lambda, n, \alpha)$ ((Bayram & Yalcin, 2017)).

Making use of the techniques and methodology used by Dziok (see (Dziok, 2015a), (Dziok, 2015b)), Dziok et al. (Dziok et al., 2016), in this paper we find necessary and sufficient conditions, distortion bounds, radii of starlikeness and convexity, compactness and extreme points for the above defined class $SH^0(\lambda, n, A, B)$.

2. Main Results

First, we provides a necessary and sufficient convolution condition for the harmonic functions in $SH^0(\lambda, n, A, B)$.

Theorem 2.1. For $z \in U \setminus \{0\}$, let $f \in SH^0$. Then $f \in SH^0(\lambda, n, A, B)$ if and only if

$$D_\lambda^n f(z) * \varphi(z; \zeta) \neq 0 \quad (\zeta \in \mathbb{C}, |\zeta| = 1),$$

where

$$\begin{aligned} \varphi(z; \zeta) = & \frac{[(A - B)\zeta + \lambda(1 + B\zeta)]z^2 + (B - A)\zeta z}{(1 - z)^2} \\ & - (-1)^n \frac{[-\lambda(1 + B\zeta) + (B - A)\zeta]\bar{z}^2 + [2\lambda(1 + B\zeta) + (A - B)\zeta]\bar{z}}{(1 - \bar{z})^2}. \end{aligned}$$

Proof. Let $f \in SH^0$. Then $f \in SH^0(\lambda, n, A, B)$ if and only if the condition (1.5) holds or equivalently

$$\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} \neq \frac{1 + A\zeta}{1 + B\zeta} \quad (\zeta \in \mathbb{C}, |\zeta| = 1). \quad (2.1)$$

Now for

$$D_\lambda^n f(z) = D_\lambda^n f(z) * \left(\frac{z}{1 - z} + \frac{\bar{z}}{1 - \bar{z}} \right),$$

and

$$D_\lambda^{n+1} f(z) = D_\lambda^n f(z) * (\phi_1(z) + \overline{\phi_2(z)}),$$

the inequality (2.1) yields

$$\begin{aligned} & (1 + B\zeta) D_\lambda^{n+1} f(z) - (1 + A\zeta) D_\lambda^n f(z) \\ = & D_\lambda^n f(z) * \left\{ \frac{(1 + B\zeta)[(\lambda - 1)z^2 + z]}{(1 - z)^2} + \frac{(1 + B\zeta)[(\lambda - 1)\bar{z}^2 + (1 - 2\lambda)\bar{z}]}{(1 - \bar{z})^2} \right\} \\ & - D_\lambda^n f(z) * \left\{ \frac{(1 + A\zeta)z}{1 - z} + \frac{(1 + A\zeta)\bar{z}}{1 - \bar{z}} \right\} \\ = & D_\lambda^n f(z) * \left\{ \frac{[(A - B)\zeta + \lambda(1 + B\zeta)]z^2 + (B - A)\zeta z}{(1 - z)^2} \right. \\ & \left. - \frac{[2 - \lambda(1 + B\zeta) + (B + A)\zeta]\bar{z}^2 + [-2 + 2\lambda(1 + B\zeta) + (B - A)\zeta]\bar{z}}{(1 - \bar{z})^2} \right\} \\ = & D_\lambda^n f(z) * \varphi(z; \zeta) \neq 0. \end{aligned}$$

□

A sufficient coefficient for the functions in $f \in SH^0(\lambda, n, A, B)$ is provided in the following.

Theorem 2.2. Let $f = h + \bar{g}$ be so that h and g are given by (1.1). Then $f \in SH^0(\lambda, n, A, B)$, if

$$\sum_{k=2}^{\infty} (M_k |a_k| + N_k |b_k|) \leq B - A \tag{2.2}$$

where

$$M_k = [(k - 1) \lambda + 1]^n [\lambda(k - 1)(B + 1) + B - A] \tag{2.3}$$

and

$$N_k = [(k + 1) \lambda - 1]^n [\lambda(k + 1)(B + 1) + A - B]. \tag{2.4}$$

Proof. It is easy to see that the theorem is true for $f(z) = z$. So, we assume that $a_k \neq 0$ or $b_k \neq 0$ for $k \geq 2$. Since $M_k \geq k(B - A)$ and $N_k \geq k(B - A)$ by (2.2), we obtain

$$\begin{aligned} |h'(z)| - |g'(z)| &\geq 1 - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} - \sum_{k=2}^{\infty} k |b_k| |z|^{k-1} \\ &\geq 1 - |z| \sum_{k=2}^{\infty} (k |a_k| + k |b_k|) \\ &\geq 1 - \frac{|z|}{B - A} \sum_{k=2}^{\infty} (M_k |a_k| + N_k |b_k|) \\ &\geq 1 - |z| > 0. \end{aligned}$$

Therefore f is sense preserving and locally univalent in U . For the univalence condition, consider $z_1, z_2 \in U$ so that $z_1 \neq z_2$. Then

$$\left| \frac{z_1^k - z_2^k}{z_1 - z_2} \right| = \left| \sum_{m=1}^k z_1^{m-1} z_2^{k-m} \right| \leq \sum_{m=1}^k |z_1^{m-1}| |z_2^{k-m}| < k, \quad k \geq 2.$$

Hence

$$\begin{aligned} \left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \left| \frac{\sum_{k=2}^{\infty} b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)} \right| \\ &> 1 - \frac{\sum_{k=2}^{\infty} k |b_k|}{1 - \sum_{k=2}^{\infty} k |a_k|} \geq 1 - \frac{\sum_{k=2}^{\infty} \frac{N_k}{B-A} |b_k|}{1 - \sum_{k=2}^{\infty} \frac{M_k}{B-A} |a_k|} \geq 0 \end{aligned}$$

which proves univalence.

On the other hand, $f \in SH^0(\lambda, n, A, B)$ if and only if there exists a complex valued function w ; $w(0) = 0, |w(z)| < 1 (z \in U)$ such that

$$\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}$$

or equivalently

$$\left| \frac{D_\lambda^{n+1} f(z) - D_\lambda^n f(z)}{BD_\lambda^{n+1} f(z) - AD_\lambda^n f(z)} \right| < 1, (z \in U). \tag{2.5}$$

The above inequality (2.5) holds, since for $|z| = r (0 < r < 1)$ we obtain

$$\begin{aligned} & \left| D_\lambda^{n+1} f(z) - D_\lambda^n f(z) \right| - \left| BD_\lambda^{n+1} f(z) - AD_\lambda^n f(z) \right| \\ &= \left| \sum_{k=2}^{\infty} (k-1)\lambda [(k-1)\lambda + 1]^n a_k z^k - (-1)^n \sum_{k=2}^{\infty} (k+1)\lambda [(k+1)\lambda - 1]^n \overline{b_k z^k} \right| \\ & \quad - \left| (B-A)z + \sum_{k=2}^{\infty} [(k-1)\lambda B + B - A] [(k-1)\lambda + 1]^n a_k z^k \right. \\ & \quad \left. - (-1)^n \sum_{k=2}^{\infty} [(k+1)\lambda B + A - B] [(k+1)\lambda - 1]^n \overline{b_k z^k} \right| \\ &\leq \sum_{k=2}^{\infty} (k-1)\lambda [(k-1)\lambda + 1]^n |a_k| r^k + \sum_{k=2}^{\infty} (k+1)\lambda [(k+1)\lambda - 1]^n |b_k| r^k - (B-A)r \\ & \quad + \sum_{k=2}^{\infty} [(k-1)\lambda B + B - A] [(k-1)\lambda + 1]^n |a_k| r^k + \sum_{k=2}^{\infty} [(k+1)\lambda B + A - B] [(k+1)\lambda - 1]^n |b_k| r^k \\ &\leq r \left\{ \sum_{k=2}^{\infty} (M_k |a_k| + N_k |b_k|) r^{k-1} - (B-A) \right\} < 0, \end{aligned}$$

therefore $f \in SH^0(\lambda, n, A, B)$, and so the proof is complete. □

Next we show that the condition (2.2) is also necessary for the functions $f \in H$ to be in the class $SH_T^0(\lambda, n, A, B) = T^n \cap SH^0(\lambda, n, A, B)$ where T^n is the class of functions $f = h + \bar{g} \in SH^0$ so that

$$f = h + \bar{g} = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=2}^{\infty} |b_k| \bar{z}^k (z \in U). \tag{2.6}$$

Theorem 2.3. *Let $f = h + \bar{g}$ be defined by (2.6). Then $f \in SH_T^0(\lambda, n, A, B)$ if and only if the condition (2.2) holds.*

Proof. The ”if” part follows from Theorem 2.2. For the ”only-if” part, assume that $f \in SH_T^0(\lambda, n, A, B)$, then by (2.5) we have

$$\left| \frac{\sum_{k=2}^{\infty} (k-1)\lambda[(k-1)\lambda+1]^n |a_k| z^k + (k+1)\lambda[(k+1)\lambda-1]^n |b_k| \bar{z}^k}{(B-A)z - \sum_{k=2}^{\infty} [(k-1)\lambda B+B-A][(k-1)\lambda+1]^n |a_k| z^k + [(k+1)\lambda B+A-B][(k+1)\lambda-1]^n |b_k| \bar{z}^k} \right| < 1.$$

For $z = r < 1$ we obtain

$$\frac{\sum_{k=2}^{\infty} \{(k-1)\lambda[(k-1)\lambda+1]^n |a_k| + (k+1)\lambda[(k+1)\lambda-1]^n |b_k|\} r^{k-1}}{B-A - \sum_{k=2}^{\infty} \{[(k-1)\lambda B+B-A][(k-1)\lambda+1]^n |a_k| + [(k+1)\lambda B+A-B][(k+1)\lambda-1]^n |b_k|\} r^{k-1}} < 1.$$

Thus, for M_k and N_k as defined by (2.3) and (2.4), we have

$$\sum_{k=2}^{\infty} [M_k |a_k| + N_k |b_k|] r^{k-1} < B - A \quad (0 \leq r < 1). \tag{2.7}$$

Let $\{\sigma_k\}$ be the sequence of partial sums of the series

$$\sum_{k=2}^{\infty} [M_k |a_k| + N_k |b_k|].$$

Then $\{\sigma_k\}$ is a nondecreasing sequence and by (2.7) it is bounded above by $B - A$. Thus, it is convergent and

$$\sum_{k=2}^{\infty} [M_k |a_k| + N_k |b_k|] = \lim_{k \rightarrow \infty} \sigma_k \leq B - A.$$

This gives the condition (2.2). □

In the following we show that the class of functions of the form (2.6) is convex and compact.

Theorem 2.4. *The class $SH_T^0(\lambda, n, A, B)$ is a convex and compact subset of SH .*

Proof. Let $f_t \in SH_T^0(\lambda, n, A, B)$, where

$$f_t(z) = z - \sum_{k=2}^{\infty} |a_{t,k}| z^k + (-1)^n \sum_{k=2}^{\infty} |b_{t,k}| \bar{z}^k \quad (z \in U, t \in \mathbb{N}). \tag{2.8}$$

Then $0 \leq \eta \leq 1$, let $f_1, f_2 \in SH_T^0(\lambda, n, A, B)$ be defined by (2.8). Then

$$\begin{aligned} \kappa(z) &= \eta f_1(z) + (1 - \eta) f_2(z) \\ &= z - \sum_{k=2}^{\infty} (\eta |a_{1,k}| + (1 - \eta) |a_{2,k}|) z^k \\ &\quad + (-1)^n \sum_{k=2}^{\infty} (\eta |b_{1,k}| + (1 - \eta) |b_{2,k}|) \bar{z}^k \end{aligned}$$

and

$$\begin{aligned} & \sum_{k=2}^{\infty} \left\{ M_k \left[\eta |a_{1,k}| + (1-\eta) |a_{2,k}| \right] + N_k \left[\eta |b_{1,k}| + (1-\eta) |b_{2,k}| \right] \right\} \\ &= \eta \sum_{k=2}^{\infty} \left\{ M_k |a_{1,k}| + N_k |b_{1,k}| \right\} + (1-\eta) \sum_{k=2}^{\infty} \left\{ M_k |a_{2,k}| + N_k |b_{2,k}| \right\} \\ &\leq \eta(B-A) + (1-\eta)(B-A) = B-A. \end{aligned}$$

Thus, the function $\kappa = \eta f_1 + (1-\eta)f_2$ belongs to the class $SH_T^0(\lambda, n, A, B)$. This means that the class $SH_T^0(\lambda, n, A, B)$ is convex.

On the other hand, for $f_t \in SH_T^0(\lambda, n, A, B)$, $t \in \mathbb{N}$ and $|z| \leq r$ ($0 < r < 1$), we get

$$\begin{aligned} |f_t(z)| &\leq r + \sum_{k=2}^{\infty} \left\{ |a_{t,k}| + |b_{t,k}| \right\} r^k \\ &\leq r + \sum_{k=2}^{\infty} \left\{ M_k |a_{t,k}| + N_k |b_{t,k}| \right\} r^k \\ &\leq r + (B-A)r^2. \end{aligned}$$

Therefore, $SH_T^0(\lambda, n, A, B)$ is locally uniformly bounded. Let

$$f_t(z) = z - \sum_{k=2}^{\infty} |a_{t,k}| z^k + (-1)^n \sum_{k=2}^{\infty} |b_{t,k}| \bar{z}^k \quad (z \in U, t \in \mathbb{N})$$

and let $f = h + \bar{g}$ be so that h and g are given by (1.1). Using Theorem 2.3 we obtain

$$\sum_{k=2}^{\infty} \left\{ M_k |a_{t,k}| + N_k |b_{t,k}| \right\} \leq (B-A). \quad (2.9)$$

If we assume that $f_t \rightarrow f$, then we conclude that $|a_{t,k}| \rightarrow |a_k|$ and $|b_{t,k}| \rightarrow |b_k|$ as $k \rightarrow \infty$ ($t \in \mathbb{N}$). Let $\{\sigma_k\}$ be the sequence of partial sums of the series $\sum_{k=2}^{\infty} \{M_k |a_k| + N_k |b_k|\}$. Then $\{\sigma_k\}$ is a nondecreasing sequence and by (2.9) it is bounded above by $B-A$. Thus, it is convergent and

$$\sum_{k=2}^{\infty} \{M_k |a_k| + N_k |b_k|\} = \lim_{k \rightarrow \infty} \sigma_k \leq B-A.$$

Therefore $f \in SH_T^0(\lambda, n, A, B)$ and therefore the class $SH_T^0(\lambda, n, A, B)$ is closed. In consequence, the class $SH_T^0(\lambda, n, A, B)$ is compact subset of SH , which completes the proof. \square

We continue with the following lemma due to Jahangiri (Jahangiri, 1999).

Lemma 2.1. Let $f = h + \bar{g}$ be so that h and g are given by (1.1). Furthermore, let

$$\sum_{k=2}^{\infty} \left\{ \frac{k-\alpha}{1-\alpha} |a_k| + \frac{k+\alpha}{1-\alpha} |b_k| \right\} \leq 1 \quad (z \in U)$$

where $0 \leq \alpha < 1$. Then f is harmonic, orientation preserving, univalent in U and f is starlike of order α .

In the following theorems we obtain the radii of starlikeness and convexity for functions in the class $SH_T^0(\lambda, n, A, B)$.

Theorem 2.5. Let $0 \leq \alpha < 1$, M_k and N_k be defined by (2.3) and (2.4). Then

$$r_{\alpha}^*(SH_T^0(\lambda, n, A, B)) = \inf_{k \geq 2} \left[\frac{1-\alpha}{B-A} \min \left\{ \frac{M_k}{k-\alpha}, \frac{N_k}{k+\alpha} \right\} \right]^{\frac{1}{k-1}}. \tag{2.10}$$

Proof. Let $f \in SH_T^0(\lambda, n, A, B)$ be of the form (2.6). Then, for $|z| = r < 1$, we get

$$\begin{aligned} & \left| \frac{D_1 f(z) - (1+\alpha)f(z)}{D_1 f(z) + (1-\alpha)f(z)} \right| \\ &= \left| \frac{-\alpha z - \sum_{k=2}^{\infty} (k-1-\alpha) |a_k| z^k - (-1)^n \sum_{k=2}^{\infty} (k+1+\alpha) |b_k| \bar{z}^k}{(2-\alpha)z - \sum_{k=2}^{\infty} (k+1-\alpha) |a_k| z^k - (-1)^n \sum_{k=2}^{\infty} (k-1+\alpha) |b_k| \bar{z}^k} \right| \\ &\leq \frac{\alpha + \sum_{k=2}^{\infty} \{ (k-1-\alpha) |a_k| + (k+1+\alpha) |b_k| \} r^{k-1}}{2-\alpha - \sum_{k=2}^{\infty} \{ (k+1-\alpha) |a_k| + (k-1+\alpha) |b_k| \} r^{k-1}}. \end{aligned}$$

Note (see Lemma 2.1) that f is starlike of order α in U_r if and only if

$$\left| \frac{D_1 f(z) - (1+\alpha)f(z)}{D_1 f(z) + (1-\alpha)f(z)} \right| < 1, \quad z \in U_r$$

or

$$\sum_{k=2}^{\infty} \left\{ \frac{k-\alpha}{1-\alpha} |a_k| + \frac{k+\alpha}{1-\alpha} |b_k| \right\} r^{k-1} \leq 1. \tag{2.11}$$

Moreover, by Theorem 2.2, we have

$$\sum_{k=2}^{\infty} \left\{ \frac{M_k}{B-A} |a_k| + \frac{N_k}{B-A} |b_k| \right\} r^{k-1} \leq 1.$$

The condition (2.11) is true if

$$\frac{k-\alpha}{1-\alpha} r^{k-1} \leq \frac{M_k}{B-A} r^{k-1},$$

$$\frac{k + \alpha}{1 - \alpha} r^{k-1} \leq \frac{N_k}{B - A} r^{k-1} \quad (k = 2, 3, \dots)$$

or if

$$r \leq \frac{1 - \alpha}{B - A} \min \left\{ \frac{M_k}{k - \alpha}, \frac{N_k}{k + \alpha} \right\}^{\frac{1}{k-1}} \quad (k = 2, 3, \dots).$$

It follows that the function f is starlike of order α in the disk $U_{r_\alpha^*}$ where

$$r_\alpha^* := \inf_{k \geq 2} \left[\frac{1 - \alpha}{B - A} \min \left\{ \frac{M_k}{k - \alpha}, \frac{N_k}{k + \alpha} \right\} \right]^{\frac{1}{k-1}}.$$

The function

$$f_k(z) = h_k(z) + \overline{g_k(z)} = z - \frac{B - A}{M_k} z^k + (-1)^n \frac{B - A}{N_k} \overline{z^k}$$

proves that the radius r_α^* cannot be any larger. Thus we have (2.10). □

Using a similar argument as above we obtain the following.

Theorem 2.6. Let $0 \leq \alpha < 1$ and M_k and N_k be defined by (2.3) and (2.4). Then

$$r_\alpha^c(SH_T^0(\lambda, n, A, B)) = \inf_{k \geq 2} \left[\frac{1 - \alpha}{B - A} \min \left\{ \frac{M_k}{k(k - \alpha)}, \frac{N_k}{k(k + \alpha)} \right\} \right]^{\frac{1}{k-1}}.$$

Our next theorem is on the extreme points of $SH_T^0(\lambda, n, A, B)$.

Theorem 2.7. Extreme points of the class $SH_T^0(\lambda, n, A, B)$ are the functions f of the form (1.1) where $h = h_k$ and $g = g_k$ are of the form

$$\begin{aligned} h_1(z) &= z, & h_k(z) &= z - \frac{B - A}{M_k} z^k, \\ g_k(z) &= (-1)^n \frac{B - A}{N_k} \overline{z^k} \quad (z \in U, k \geq 2). \end{aligned} \tag{2.12}$$

Proof. Let $g_k = \eta f_1 + (1 - \eta) f_2$ where $0 < \eta < 1$ and $f_1, f_2 \in SH_T^0(\lambda, n, A, B)$ are functions of the form

$$f_t(z) = z - \sum_{k=2}^{\infty} |a_{t,k}| z^k + (-1)^n \sum_{k=2}^{\infty} |b_{t,k}| \overline{z^k} \quad (z \in U, t \in \{1, 2\}).$$

Then, by (2.12), we have

$$|b_{1,k}| = |b_{2,k}| = \frac{B - A}{N_k},$$

and therefore $a_{1,t} = a_{2,t} = 0$ for $t \in \{2, 3, \dots\}$ and $b_{1,t} = b_{2,t} = 0$ for $t \in \{2, 3, \dots\} \setminus \{k\}$. It follows that $g_k(z) = f_1(z) = f_2(z)$ and g_k are in the class of extreme points of the function class $SH_T^0(\lambda, n, A, B)$. Similarly, we can verify that the functions $h_k(z)$ are the extreme points of the class $SH_T^0(\lambda, n, A, B)$. Now, suppose that a function f of the form (1.1) is in the family of extreme points of the class $SH_T^0(\lambda, n, A, B)$ and f is not of the form (2.12). Then there exists $m \in \{2, 3, \dots\}$ such that

$$0 < |a_m| < \frac{B - A}{[(m - 1)\lambda + 1]^n [\lambda(m - 1)(B + 1) + B - A]}$$

or

$$0 < |b_m| < \frac{B - A}{[(m + 1)\lambda - 1]^n [\lambda(m + 1)(B + 1) + A - B]}.$$

If

$$0 < |a_m| < \frac{B - A}{[(m - 1)\lambda + 1]^n [\lambda(m - 1)(B + 1) + B - A]},$$

then putting

$$\eta = \frac{|a_m| [(m - 1)\lambda + 1]^n [\lambda(m - 1)(B + 1) + B - A]}{B - A}$$

and

$$\phi = \frac{f - \eta h_m}{1 - \eta},$$

we have $0 < \eta < 1$, $h_m \neq \phi$.

Therefore, f is not in the family of extreme points of the class $SH_T^0(\lambda, n, A, B)$. Similarly, if

$$0 < |b_m| < \frac{B - A}{[(m + 1)\lambda - 1]^n [\lambda(m + 1)(B + 1) + A - B]},$$

then putting

$$\eta = \frac{|b_m| [(m + 1)\lambda - 1]^n [\lambda(m + 1)(B + 1) + A - B]}{B - A}$$

and

$$\phi = \frac{f - \eta g_m}{1 - \eta},$$

we have $0 < \eta < 1$, $g_m \neq \phi$.

It follows that f is not in the family of extreme points of the class $SH_T^0(\lambda, n, A, B)$ and so the proof is completed. \square

Therefore, by Theorem 2.7, we have the following corollary.

Corollary 2.1. Let $f \in SH_T^0(\lambda, n, A, B)$ and $|z| = r < 1$. Then

$$r - \frac{B - A}{(\lambda + 1)^n [\lambda(B + 1) + A - B]} r^2 \leq |f(z)| \leq r + \frac{B - A}{(\lambda + 1)^n [\lambda(B + 1) + A - B]} r^2.$$

The following covering result follows from Corollary 2.1.

Corollary 2.2. If $f \in SH_T^0(\lambda, n, A, B)$ then $U_r \subset f(U)$ where

$$r = 1 - \frac{B - A}{(\lambda + 1)^n [\lambda(B + 1) + A - B]}.$$

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Non-Computable, Indiscernible and Uncountable Mathematical Constructions. Sub-Cardinals and Related Paradoxes

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Abstract

One of the most important achievements of the last century is the knowledge of the existence of non-countable sets. The proof by Cantor's diagonal method requires the assumption of actual infinity. By two paradoxes we show that this method sometimes proves nothing because of it can involve self-referential definitions. To avoid this inconvenient, we introduce another proving method based upon the information in the involved object definitions. We also introduce the concepts of indiscernible mathematical construction and sub-cardinal. In addition, we show that the existence of indiscernible mathematical constructions is an unavoidable consequence of uncountability.

Keywords: Remote-cardinals, sub-cardinals, uncountability, indiscernibility, paradoxes.
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1. Introduction and preliminaries

In this article that two sets X and Y have different cardinalities only means that it is impossible to define one bijection between them. The existence of a bijection between two sets leads to size equality, but the converse implication need not be true, at least, with respect to infinite sets. The term “size” is ambiguous when applied to infinite sets. Recall that there is always a bijection between every infinite set and some proper subset of it. Bijections not only compare set-sizes but information and complexity in the definitions of their members. In particular, we consider those mathematical constructions that cannot be defined or determined by any finite expression. We say these objects to be *indiscernible*.

If $f : X \rightarrow Y$ is a bijection, the expression $y = f(x)$ specifies the object y uniquely. Thus, we can consider $y = f(x)$ as a definition for y , whenever both f and x are definable. As a consequence, if the information lying in any definition for y is always greater than the one in both definitions of f and x , we cannot assume the local equality $y = f(x)$ at x . We can find this situation when y is

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an indiscernible object. This method leads to non-existence of a bijection $f : X \rightarrow Y$ when there is, at least, one indiscernible object in Y . Even singletons can satisfy this condition. However, it is trivial to define a bijection between two singletons, what suggests that this proving method is wrong. Fortunately, in Theorem 2.2 we show that, if a set E contains one indiscernible object, then E must be infinite, and satisfies the following relation.

$$“E \text{ contains an indiscernible object}” \Rightarrow (“E \text{ is infinite}” \wedge “E \text{ is uncountable}.”)$$

Analogously, by virtue of Theorem 2.1,

$$“E \text{ is uncountable}” \Rightarrow$$

$$“E \text{ contains an infinite subset each member of which is indiscernible}.”$$

We can find similar facts using complexity instead of information. For instance, let Ω denote the Chaitin’s constant (Chaitin, 2012). If $f : \mathbb{N} \rightarrow E \subseteq \mathbb{R}$ is a bijection, being computable by a Turing machine T , then E cannot contain Ω ; otherwise, T could compute Ω , simply, as the value of f at some positive integer $n \in \mathbb{N}$. Accordingly, if $\Omega \in E$, then there is no bijection between \mathbb{N} and E , being computable by any Turing machine. Nevertheless, depending on their nature, we could define a non-computable bijection F between \mathbb{N} and E . This fact allows us to define the concept of sub-cardinal. Thus, comparing sets by computable bijections, we define sub-cardinals as sub-equivalences of those that we can state by definable one-to-one maps.

1.1. Preliminaries

To avoid any ambiguity, we use the concepts of “extensible language,” “identifier” and “discernible mathematical construction” with particular meanings that we explain below.

Let A be a finite alphabet. We term “vocabulary generated by A ” every nonempty subset of the class $\text{Voc}(A)$ of all finite and infinite sequences of symbols in A . Likewise, we term “language” every partial (syntactic) free-monoid $\mathcal{L}(\text{Voc}(A))$, generated by $\text{Voc}(A)$, provided that we assign a meaning to each of its members. Thus, $\mathcal{L}(\text{Voc}(A))$ consists of sentences. Two sentences s_1 and s_2 are equivalent when both denote the same concept.

Definition 1.1. We say that a language $\mathcal{L}(\text{Voc}(A))$ is “extensible” when it satisfies the following conditions.

1. If there is a set E of sentences in some language \mathcal{L} that have no equivalent in $\mathcal{L}(\text{Voc}(A))$, we can add sentences and symbols to $\mathcal{L}(\text{Voc}(A))$ denoting all members of E .
2. When we extend $\mathcal{L}(\text{Voc}(A))$ with a set W of new symbols and sentences, the way that we assign meanings and interpret each member of W can be described by “finite” sentences in $\mathcal{L}(\text{Voc}(A))$.

The second condition is important for our purposes. For instance, consider the sentence: $S = “We can compute the area of a polygon of three angles.”$ Using the word “triangle” we can transform S into the shorter sentence: “We can compute the area of a triangle.” Since we can describe the meaning of the term “triangle” by a finite expression, this substitution satisfies the former definition.

Definition 1.2. We say that a sentence S in an extensible language is an “identifier” for a mathematical construction X , provided that S specifies X uniquely. Thus, S can denote a definition, a predicate, or any procedure that can be associated with X in an ambiguity-free way.

Identifiers can be infinite, for instance, the binary expansion $0.d_1d_2d_3\dots$ of an irrational number consists of an infinite symbol sequence.

Definition 1.3. We say that a mathematical construction X is discernible when there is, at least, one finite identifier for it; otherwise, we say that X is indiscernible.

To illustrate the concept in the former definition, consider the real number π . There are endless expressions that identify π . For instance, the infinite digit sequence $3.14159\dots$ of its decimal expansion. In spite of being an endless symbol sequence, the number π is discernible because it can be specified by finite sentences. For instance,

$$S = \text{“the ratio of a circle’s circumference to its diameter.”}$$

Although the symbol π denotes the infinite expression $3.141592\dots$, the language extension obtained adding π satisfies the second condition in Definition 1.1 because its meaning can be described by the finite sentence S . Likewise, every algebraic number is discernible by any equation that it satisfies. Every computable mathematical construction X is discernible because any algorithm (finite) that calculates X also identifies it.

A noticeable property related with indiscernible mathematical constructions is the impossibility of handling them individually. We show this topic in Theorem 2.2. There are sets of indiscernible objects that can be denoted by finite expressions. For instance, “the subset K of all indiscernible members of $[0, 1]$ ” is a finite definition; hence K is discernible, but each of its members is not. Likewise, we show in Theorem 3.1 that, if there is one non-computable number in $[0, 1]$, there is also an infinite subset no member of which is computable.

From now on, we frequently use the term “infinite,” that can be regarded either as potential or actual. By potential infinity, we understand the concept stated by Poincaré (1854-1912) in the following quotation.

Actual infinity does not exist. What we call infinite is only the endless possibility of creating new objects no matter how many exist already.

When we do not state the concept of infinity as “potential,” we implicitly mean that it is actual.

2. Discernible mathematical constructions

In this section, we show that every uncountable set contains an infinite subset no member of which is discernible. In addition, indiscernible mathematical constructions cannot be identified and compared to each other by finite procedures. This property gives rise to some undecidable statements together with the impossibility of proving that a map satisfies the one-to-one property. By undecidable, we mean that either its truthfulness or falsehood cannot be proved by some finite procedure. Thus, if we assume any undecidable statement, there is no finite procedure to reject or confirm our assumption.

Lemma 2.1. *Every computable mathematical construction is discernible.*

Proof. Let O be a mathematical construction that is computable by a Turing machine T . The transition map of T is determined by a finite matrix T_M . We can denote T_M by a finite symbol sequence in any extensible language $\mathcal{L}(\text{Voc}(A))$, simply, adding some end-row symbol to A , whenever it does not contain one.

Now, suppose that there is no Turing machine computing O , but we can compute it by some more complex algorithm Alg , for instance, a super-recursive one. In this case, we need a finite definition Def for Alg . Accordingly, in some extensible language there is a finite symbol sequence S that denotes Def . Thus, S determines Def ; this denotes Alg ; the later determines O . As a consequence, O is discernible through S . \square

The concept of discernibility stated in Definition 1.3 can be either intrinsic or circumstantial. For instance, the expression $E = \text{“the positive solution of the equation } x^2 - 2 = 0\text{”}$ determines $\sqrt{2}$. Since E is finite, $\sqrt{2}$ is discernible. Likewise, every algebraic number is discernible. The discernibility of $\sqrt{2}$ is intrinsic because arises from an intrinsic property of $\sqrt{2}$. By contrast, the expression

$$E = \text{“the second object in the sequence } S = \square, \sqrt{2}, 6, \gamma\text{”}$$

determines $\sqrt{2}$ by a circumstance (position) that can be satisfied by any object O , simply, substituting $\sqrt{2}$ by O in S . This is a positional determination in the scenario denoted by S . If S can be defined by a finite symbol sequence, then this description together with the expression S is also an identifier for $\sqrt{2}$.

Theorem 2.1. *Every uncountable set X contains an uncountable subset each member of which is indiscernible. In addition, the subset of all discernible members of X is countable.*

Proof. Let X be a nonempty set each member of which is discernible, hence, for every $x \in X$, there is a finite expression $S_x = \alpha_1 \alpha_2 \dots \alpha_{k_x}$ in some extensible language $\mathcal{L}(\text{Voc}(A))$ that determines x . Let $\beta : A \rightarrow \mathbb{N}$ be an injective map and $\{p_1, p_2, p_3 \dots\}$ the set of all positive prime integers. We can define a numbering function γ that sends each symbol sequence $\alpha_1 \alpha_2 \dots \alpha_{k_x}$ in $\mathcal{L}(\text{Voc}(A))$ into the positive integer

$$\gamma(\alpha_1 \alpha_2 \dots \alpha_{k_x}) = \prod_{i=1}^{k_x} p_i^{\beta(\alpha_i)} \in \mathbb{N} \quad (2.1)$$

By construction, γ is injective. Since, by definition, there is a finite expression determining each discernible object x , the set X is countable because its image under γ is a subset of \mathbb{N} . As a consequence, to be uncountable, X must contain a nonempty set U of indiscernible objects. Since each member of $X \setminus U$ is discernible, as we have just seen, it is a countable subset of X . If U were also countable, $X = (X \setminus U) \cup U$ would be the union of two countable sets. Consequently, U must be uncountable. \square

A straightforward consequence of the former theorem is the existence of indiscernible real numbers in the unit interval $[0, 1]$ because it is uncountable. This is an example of discernible set

that contains indiscernible members. The set $[0, 1]$ is discernible because it can be defined by a finite sequence of finite expressions in any extensible language. For instance, “the set of limits of all convergent sequences of rational numbers that are greater than or equal to 0 and less than or equal to 1.”

Axiom 2.1. Let O_1 and O_2 be two mathematical constructions. If $O_1 \neq O_2$, there is, at least, one finitely representable predicate $p(x)$, in some extensible language, that O_1 satisfies and O_2 does not.

The former axiom is widely satisfied. As the lemma below shows, the real number set satisfies it.

Lemma 2.2. The real number set \mathbb{R} satisfies Axiom 2.1.

Proof. Let r_1 and r_2 be two real numbers. If

$$r_1 = c_1c_2 \cdots c_k \cdot c_{k+1}c_{k+2}c_{k+3} \cdots$$

and

$$r_2 = d_1d_2 \cdots d_j \cdot d_{j+1}d_{j+2}d_{j+3} \cdots$$

are their decimal expansions and $r_1 \neq r_2$, for some $n \in \mathbb{N}$, the inequality $c_n \neq d_n$ holds. Thus, r_1 is the only of them that satisfies the following predicate:

$$p(x) = \text{“The } n\text{-th digit of the decimal expansion of } x \text{ is } c_n\text{.”}$$

□

Lemma 2.3. Let $E = \{O_n \mid n \in \{0, 1, \dots, k\}\}$ be a finite subset of \mathbb{N} . If the members of E satisfy Axiom 2.1 pairwise, for every positive integer $n \leq k$, there is a predicate $q_n(x)$ that O_n satisfies and any other member of E does not.

Proof. By Axiom 2.1, for every positive integer $m \leq k$, if $m \neq n$, there is a predicate $p_{n,m}(x)$ that O_n satisfies and O_m does not. Thus, O_n is the only member of E that satisfies the conjunction

$$q_n(x) = \bigwedge_{\substack{m \neq n \\ m \in \{0, 1, \dots, k\}}} p_{n,m}(x) \tag{2.2}$$

□

Now, we show that every finitely representable predicate that is satisfied by one indiscernible member of a finitely definable set K , it is also satisfied by every member of an infinite subset U of K ; therefore K must be infinite.

Theorem 2.2. Let X be a set the members of which satisfy Axiom 2.1 pairwise. If X is finitely definable in an extensible language $\mathcal{L}(\text{Voc}(A))$, and the subset U of all indiscernible members of X is nonempty, the following statements are true.

1. Let $p(x)$ be a finitely representable predicate in $\mathcal{L}(\text{Voc}(A))$. If a member O_0 of U satisfies $p(x)$, then there is an infinite subset U_0 of U each member of which satisfies $p(x)$ too; therefore U must be infinite.
2. With the same assumptions as in the preceding statement, there is no finitely definable bijective map from any nonempty subset \mathbf{K} of \mathbb{N} onto U_0 .

Proof.

1. Suppose that O_0 is the only member of U that satisfies $p(x)$. In this case, $p(x)$ determines O_0 uniquely. Since, by assumption, $p(x)$ can be represented by a finite expression E_1 in $\mathcal{L}(\text{Voc}(A))$, then O_0 is discernible and belongs to $X \setminus U$; which contradicts our assumption. Thus, O_0 cannot be the only object that satisfies $p(x)$. Since we assume that X is finitely definable in $\mathcal{L}(\text{Voc}(A))$ by a finite expression E_0 , the subset U of X is also finitely representable in $\mathcal{L}(\text{Voc}(A))$ by the expression

$E_2 = \text{“The subset of all indiscernible members of the set denoted by } E_0\text{.”}$

If U does not contain any other member of the set $\{O \in X \mid p(O)\}$, both finite expressions “ x belongs to the set denoted by E_2 ” and “ x satisfies the predicate denoted by E_1 ” form a finite expression that determines O_0 , and both expressions E_1 and E_2 consist of finite symbol sequences in A . As in the previous case, O_0 would be discernible. Accordingly, there is at least one $O_1 \in U$ that satisfies $p(x)$ too. By Axiom 2.1 and taking into account Lemma 2.3, there is a finitely representable predicate $p_1(x)$ that O_0 satisfies and O_1 does not. The conjunction of both predicates $p(x) \wedge p_1(x)$ determines O_0 , unless there is $O_2 \in U$ that also satisfies this conjunction. Iterating the procedure, we obtain an infinite subset U_0 of U each member of which satisfies $p(x)$.

2. If there is a bijection $f : \mathbb{N} \rightarrow U_0$ that can be defined by a finite expression E_f in $\mathcal{L}(\text{Voc}(A))$, for every $n \in \mathbf{K}$, the expression

$\text{“}f(n)\text{ is the image of the integer }n\text{ under the map }f\text{ defined by }E_f\text{”}$

determines $f(n) \in U_0$ uniquely. Since, by hypothesis, E_f can be denoted by a finite symbol sequence, the former expression is finite, and $f(n)$ is discernible, for every $n \in \mathbb{N}$. Thus, f cannot be surjective. Accordingly, the existence of any finitely definable bijection f is not compatible with the indiscernibility of the members of U_0 .

□

Corollary 2.1. *With the same assumptions as in the preceding theorem, if a discernible set E contains one indiscernible member O , then there is an infinite subset U of E each member of which is also indiscernible; hence E must be infinite. In addition, if E is uncountable, so is U .*

Proof. If E is discernible, there is at least one identifier Q for E that can be described by a finite symbol sequence in some extensible language. Now, let $P(x)$ denote the predicate:

$P(x) = \text{“}x\text{ is a member of the set that }Q\text{ specifies.”}$

It is straightforward that an object O satisfies $P(x)$ if and only if it is a member of E . Since $P(x)$ can be denoted by a finite expression, by virtue of the preceding theorem, there is an infinite set U each of its members satisfies also $P(x)$; hence $U \subseteq E$. Thus, E must be also infinite.

Finally, if E is uncountable, by virtue of Theorem 2.1, the subset $V \subseteq E$ of all discernible members of E is countable. As a consequence, U must be uncountable; otherwise $E = V \cup U$ is the union of two countable sets, which contradicts our assumption of the uncountability of E . \square

Remark. As a consequence of Statement 1 in the former theorem, we can only handle infinite sets of indiscernible objects. Procedures involving indiscernible singletons require endless expressions. For instance, the short sentence

$$S = \text{“the subset of all indiscernible real numbers,”}$$

denotes a subset \mathbf{K} of \mathbb{R} . In spite of being discernible, because we can denote \mathbf{K} by the short sentence S , each of its members requires an endless expression to be handled or identified. As a consequence, when \mathbf{K} belongs to the image of a map f , by no finite method or procedure is possible to discern whether f is a one-to-one map because of the indiscernibility of the members of \mathbf{K} .

Each non-finitely definable map $f : X \rightarrow Y$ between infinite sets can be stated through a two column table, each of its rows consists of a member of X in the first column followed by its image under f in the second one. Its indiscernibility can be a consequence of its infinite size whenever by no predictable pattern we can determine its values. If for every finite subset K of X the restriction of f to K is finitely representable, we say f to be “first-kind indiscernible”. In this case, each restriction of f to any finite subset of its domain is discernible. By contrast, if for some $x_0 \in X$ the image $f(x_0)$ can only be denoted by an infinite symbol sequence in any language, then we say that f is “second-kind indiscernible”.

Theorem 2.3. *Let X be a set that contains a nonempty subset of indiscernible objects. If the members of X satisfy Axiom 2.1 pairwise, for every one-to-one map $f : \mathbb{N} \rightarrow X$, the following statements are true.*

1. *If f is a finitely definable map in some language $\mathcal{L}(\text{Voc}(A))$, $\forall K \in \mathbb{N}$, there is no positive integer $n \leq K$ such that its image $y = f(n)$ is indiscernible.*
2. *If f is a first-kind indiscernible map, the image of f does not contain any indiscernible member of X consequently f cannot be surjective.*

Proof.

1. By hypothesis, the map f is finitely definable. Thus, the predicate $p(n, y)$ denoted by the expression

$$p(n, y) = \text{“}y \text{ is the image of } n \text{ under } f\text{,”}$$

satisfies Definition 1.1; therefore it is also finitely definable, for every $n \in \mathbb{N}$. As a consequence of Statement (1) in Theorem 2.2, there is an infinite subset of indiscernible members of X each member of which satisfies $p(n, y)$ too, which contradicts our hypothesis because f would be a relation, but not a map.

2. Let T be a table that describes f , and M any positive integer. The restriction of f to the finite set $\{1, 2, 3 \dots M\}$ consists of a finite sub-table T_M of T . Since, by hypothesis, f is first-kind indiscernible, the finite sub-table T_M can be described by a finite sequence S_M of symbols in an extensible language $\mathcal{L}(\text{Voc}(A))$. The result of substituting S_M by the corresponding finite symbol sequence in the expression

$$E = \text{“}x \text{ is the member of } X \text{ denoted by the expression lying in the second column and the } n\text{-th row of the table denoted by } S_M \text{”}$$

is also finite and determines the image of f at n , for each $n \leq M$. Thus, the image of every positive integer n under f is discernible. As a consequence, f cannot be surjective, whenever its codomain contains some nonempty subset of indiscernible members. By hypothesis, X satisfies this condition. □

It is worth pointing out that Theorem 2.3 is very similar to the well-known Cantor’s Theorem. However, the non-existence of a first-kind indiscernible bijection is a consequence of indiscernible members of X , which is a local property. By contrast, the diagonal proof is built under the assumption of actual infinity and depends on the complete bijection domain. In the proof of Cantor’s Theorem, there is no mention of the nature of the table denoting the map $f : \mathbb{N} \rightarrow [0, 1]$. We suppose that, at most, f must be a first-kind indiscernible map; otherwise, assuming that f cannot be injective, to reject this claim we need an endless procedure.

To proceed more accurately, we say that a set X is ω -countable to denote the cardinality equivalence between X and \mathbb{N} is based on the existence of a *first-kind indiscernible* bijection $f : \mathbb{N} \rightarrow X$.

Theorem 2.3 is built by a pointwise method, which does not depend on the set sizes. We are aware that this method can seem dark because, in finite sets, cardinality strongly depends on size. This is why, in the section below, we show similar results from another viewpoint. In any case, it is worth mentioning that, by virtue of Corollary 2.1, every finitely definable set that contains some indiscernible member must be infinite. The following result shows the existence of indiscernible numbers in the unit interval $[0, 1]$.

Theorem 2.4. *If there is one non-computable number ϖ in the unit interval $[0, 1]$, there is also an infinite subset $U \subseteq [0, 1]$, each member of which is indiscernible.*

Proof. For each $n \in \mathbb{N}$, let $p_n(x, d_n)$ denote the predicate

$$p_n(x, d_n) = \text{“The } n\text{-th digit of the decimal mantissa of } x \text{ is } d_n \text{.”}$$

It is straightforward that, for each map $\lambda : \mathbb{N} \rightarrow \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, the conjunction

$$\bigwedge_{n \in \mathbb{N}} p_n(x, \lambda(n))$$

together with the predicate $Q(x) = \text{“}x \text{ belongs to } [0, 1]\text{”}$ define a unique member of $[0, 1]$.

Now, for each $r \in [0, 1]$, let $Y(n, r, \bigwedge_{m \in \mathbb{N}} p_m(x, c_m))$ be the conjunction obtained by substituting $p_n(x, c_n)$ by $p_n(x, d_n)$ in $\bigwedge_{m \in \mathbb{N}} p_m(x, c_m)$, where the members of $\{d_n \mid n \in \mathbb{N}\}$ are the digits of the decimal expansion of r .

Let $r_1 = 0.c_1c_2c_3 \dots \in [0, 1]$ be a number the figures of which are chosen at random. Taking into account Theorem 2.2, if r is indiscernible the theorem is true. If it is not, consider the number r_2 defined by the conjunction $P_1(x) = \Upsilon(2, \varpi, \bigwedge_{m \in \mathbb{N}} p_m(x, c_m))$. If there is no finitely representable predicate being equivalent to $P_1(x)$, then the number r_2 that it defines is indiscernible; otherwise, let $P_2(x)$ be the conjunction $\Upsilon(4, \varpi, P_1(x))$. Once again, if there is no finitely representable predicate being equivalent to $P_2(x)$, the number r_3 that it defines is indiscernible, and the theorem is true. Iterating the process, we obtain a sequence $\mathbf{S} = P_1(x), P_2(x), P_3(x) \dots$ defined recursively as follows.

$$\begin{cases} P_1(x) = \Upsilon(2, \varpi, \bigwedge_{m \in \mathbb{N}} p_m(x, c_m)) \\ P_{n+1} = \Upsilon(2^{(n+1)}, \varpi, P_n(x)) \end{cases} \quad (2.3)$$

The process stops when, for some $n \in \mathbb{N}$, the predicate $P_n(x)$ defines an indiscernible number, and the theorem is true. Otherwise, the process becomes infinite. In this case, if the infinite sequence \mathbf{S} defines a member r of $[0, 1]$, since this is an endless definition, r is indiscernible. Notice, that by no finite method is possible to build the sequence \mathbf{S} because it involves the digits of ϖ and, by hypothesis, this is not computable. Now, we show that \mathbf{S} defines a real number in $[0, 1]$.

Let $S = r_1, r_2, r_3 \dots$ be the sequence that the members of \mathbf{S} define. As a consequence of (2.3), for every positive integer $m > 0$, and each pair $(i, j) \in \mathbb{N} \times \mathbb{N}$, if $i, j > 2^m$, the 2^m -th first digits of the mantissas of both r_i and r_j are the same. Accordingly, for each couple r_i and r_j in $[0, 1]$, the relation $i, j > 2^m$ leads to

$$|r_i - r_j| \leq \frac{1}{10^{2^m}}. \quad (2.4)$$

Thus, S is a Cauchy sequence and converges to some $r \in \mathbb{R}$. Since $[0, 1]$ is closed, $r \in [0, 1]$. Taking into account Theorem 2.2, the subset U of indiscernible members of $[0, 1]$ is infinite. \square

It is worth pointing out, that the information in the sequence \mathbf{S} is infinite. No finite expression can contain the same information as \mathbf{S} . Since the definition of \mathbf{S} involves figures obtained at random, in each instance determines different numbers. As a consequence, \mathbf{S} defines a class of indiscernible objects. Recall that, by finite methods, we only can handle infinite sets of indiscernible objects (Theorem 2.2). Finally, by virtue of Theorem 2.3, the former result leads to the ω -uncountability of $[0, 1]$.

2.1. Remote-cardinals and identifier supports

Let \mathfrak{R} be a class of discernible maps and relations, and X the union of all domains and parameter sets of all members of \mathfrak{R} . We assume that X can contain hidden parameters too (Palomar Tarancón, 2016). From now on, we denote by $\mathfrak{E}[X, \mathfrak{R}]$ the class of all mathematical constructions such that, for each of which, we can build an identifier consisting of “finite compositions” of maps and relations in \mathfrak{R} . Likewise, for each mathematical construction K , we say each class $\mathfrak{E}[X, \mathfrak{R}]$ that contains K , to be an identifier support for it, provided that \mathfrak{R} is finite and X is nonempty.

NOTATION. For every mathematical construction K , we denote by $\text{Supp}(K)$ the class of all identifier supports for K .

Definition 2.1. We say that $\mathfrak{E}[X, \mathfrak{R}] \in \text{Supp}(K)$ is a *basic identifier support for K* provided that X satisfies the relation

$$\forall \mathfrak{E}(Y, \mathfrak{S}) \in \text{Supp}(K) : \quad \#(X) \leq \#(Y) \tag{2.5}$$

In this case, we also say that $\#(X)$ is the remote-cardinal of K , that we denote by the symbol \flat . In this case, $\flat(K) = \#(X)$.

Lemma 2.4. *The remote-cardinal of the set \mathbb{Z} of all integers and the one of every of its members is 1.*

Proof. We can build every positive integer n in \mathbb{Z} iterating n -times the map $\text{suc} : m \rightarrow (m + 1)$ with the only argument 0. Likewise, we can build each negative integer iterating the inverse map suc^{-1} . To obtain 0 we can apply the identity map. Thus, if $X = \{0\}$ and $\mathfrak{R} = \{\text{suc}, \text{suc}^{-1}, \text{id}_{\mathbb{Z}}\}$, then $\mathfrak{E}[X, \mathfrak{R}]$ is an identifier support for \mathbb{Z} , and so is also for every of its members. Since $\#\{0\} = 1$ is the smallest possible cardinal of any nonempty set, $\flat(\mathbb{Z}) = \#\{0\} = 1$. Analogously, $\forall n \in \mathbb{Z} : \flat(n) = \#\{0\} = 1$. □

Theorem 2.5. *If $f : X \rightarrow Y$ is a discernible bijection, and for every $x \in X$ there are both remote-cardinals $\flat(x)$ and $\flat(f(x))$, then*

$$\forall x \in X : \quad \flat(x) = \flat(f(x)). \tag{2.6}$$

Proof. First, we show that $\flat(f(x)) \leq \flat(x)$. Let $\mathfrak{E}(A_0, \mathfrak{R}_0)$ be a basic identifier support for x , and $\mathfrak{E}(A_1, \mathfrak{R}_1)$ a basic one for $f(x)$; hence $\flat(x) = \#(A_0)$, and $\flat(f(x)) = \#(A_1)$. Since we can obtain $f(x)$, simply, applying the map f to x , and by assumption f is discernible, the expression $\mathfrak{E}(A_0, \mathfrak{R}_0 \cup \{f\})$ is also an identifier support for $f(x)$. Now, taking into account (2.5), the following relation holds.

$$\flat(f(x)) = \#(A_1) \leq \#(A_0) = \flat(x). \tag{2.7}$$

Likewise, because of f is a bijection there is the inverse f^{-1} ; therefore we can also show that

$$\#(A_0) = \flat(x) = \flat(f^{-1}(f(x))) \leq \flat(f(x)) = \#(A_1) \tag{2.8}$$

Both equations (2.7) and (2.8) lead to (2.6). □

Definition 2.2. We say that a prime factorization $p_{k_1}^{n_1} p_{k_2}^{n_2} \cdots p_{k_j}^{n_j}$ of an integer N is exhaustive if it contains every prime smaller than p_{k_j} , perhaps with exponent 0.

For instance, $20 = 2^2 \cdot 5^1$, but the exhaustive factorization is $20 = 2^2 \cdot 3^0 \cdot 5^1$, which contains every prime smaller than 5.

Corollary 2.2. *The remote-cardinal of every discernible mathematical construction is 1.*

Proof. If X is a discernible mathematical construction, the map γ defined in (2.1) specifies X by an integer $M = \prod_{i=1}^m p_i^{\beta(\alpha_i)}$ because we can obtain the parameters $\alpha_1 \alpha_2 \dots \alpha_m$, simply, through the prime factorization of M as follows.

Let Φ denote the map that sends each integer M into the m -tuple of its factors

$$(p_1^{k_1}, p_2^{k_2}, \dots, p_m^{k_m}) = (p_1^{\beta(\alpha_1)}, p_2^{\beta(\alpha_2)}, \dots, p_m^{\beta(\alpha_m)})$$

in its exhaustive prime factorization, and ω the map

$$(p_1^{\beta(\alpha_1)}, p_2^{\beta(\alpha_2)}, \dots, p_m^{\beta(\alpha_m)}) \mapsto \alpha_1 \alpha_2 \dots \alpha_m.$$

Accordingly, $\mathfrak{E}[\{M\}, \{\Phi, \omega\}]$ is an identifier support for X . Since $\{M\}$ is a singleton, $b(X) = \#\{M\} = 1$.

We can also show this result as a corollary of Theorem 2.5. By Theorem 2.1 we know that, for every nonempty set E , if every of its members is discernible, the discernible injective map γ sends E into a subset K of \mathbb{N} . Since the restriction of γ to its image is bijective, by virtue of Theorem 2.5, for each X in E , $b(X) = 1$. \square

Lemma 2.5. *The remote-cardinal of every indiscernible real number is \aleph_0 .*

Proof. On the one hand, a real number r is defined as the limit of a sequence $(a_n)_{n \in \mathbb{N}}$ of rationals. On the other hand, since \mathbb{R} with the standard topology T is a Hausdorff space, the binary relation Lim_T between each converging sequence and its limit is a map. If $f : \mathbb{N} \rightarrow \mathbb{Q}$ is the map $n \mapsto a_n$, then $\mathfrak{E}[\text{img}(f), \{\text{Lim}_T, f\}]$ is an identifier support for r . Now, we show that, for every indiscernible real number, this support is basic.

If r is indiscernible, every of its identifiers must be infinite. By definition, for every $\mathfrak{E}[A, \mathfrak{R}] \in \text{Supp}(r)$, each identifier E for r , associated with $\mathfrak{E}[A, \mathfrak{R}]$, must be built by finite compositions of members of \mathfrak{R} , and each of which is discernible. Under this condition, E can only be infinite if so is A . Accordingly, $\#\mathbb{N} = \aleph_0 \leq \#(A)$. Since $\text{img}(f)$ is countable, then $\#\text{img}(f) = \aleph_0$, and $\mathfrak{E}[\text{img}(f), \{\text{Lim}_T, f\}]$ is basic. Thus, $b(r) = \#\text{img}(f) = \#\mathbb{N} = \aleph_0$. \square

Corollary 2.3. *If the set $[0, 1]$ contains any indiscernible number, there is no discernible bijection between \mathbb{N} and $[0, 1]$.*

Proof. Let $f : \mathbb{N} \rightarrow E \subseteq [0, 1]$ be a bijection. As a consequence of Theorem 2.5 and Lemma 2.5, for every n in \mathbb{N} , $b(n) = 1$ and $b(f(n)) = 1$; hence $\text{img}(f)$ does not contain any indiscernible real number. Consequently, $E = \text{img}(f)$ is a proper subset of $[0, 1]$ and f cannot be surjective. \square

3. Non-computable real numbers

In this section, we say that a real number r is non-computable, whenever there is no Turing machine computing r . We show that the existence of any non-computable number in $[0, 1]$ leads to an infinite set of them. To this end, we assign an automaton to each positive integer, which accepts every binary representation of integers. We simplify their structures using one-way automaton classes. Consider the equivalence between two-way and one-way finite automata (Hulden, 2015).

Definition 3.1. Let $\mathbf{Aut}(\mathbb{N})$ be the class of all automata each of its members is a 7-tuple $\mathbf{A}_n = (Q_n, \Gamma, \emptyset, \Sigma, \delta, q_0, F)$; where

- $Q_n = \{p_1, p_2, \dots, p_n\} \cup \{HALT\}$ is the set of states, for some positive integer n ; where each member is associated with a prime integer p_k , and $q_0 = p_1$ is the initial state.
- $\Gamma = \{0, 1, \emptyset\}$ is the tape alphabet, and \emptyset is the blank symbol.
- $\Sigma \subset \Gamma$ is the set $\{0, 1\}$ of input symbols.
- $\delta : (Q_n \setminus F) \times \Gamma \rightarrow Q_n \times \Gamma \times \{R\}$ is the transition map, the codomain of which only contains the move R (right).
- $F = \{HALT\} \subseteq Q_n$ is the subset of final states.

From now on, we assume that each member of $\mathbf{Aut}(\mathbb{N})$ satisfies the following conditions.

1. For every $k \in \mathbb{N}$, p_k denotes the k -th prime integer. Thus, the initial state is $q_0 = p_1 = 2$. Likewise, $p_2 = 3$, $p_3 = 5$ and so on.
2. For every state $s \in (Q_n \setminus F)$: $\delta(s, \emptyset) = (HALT, 1, R)$.
3. The image $(p_j, y, R) = \delta(p_k, x)$ under δ of every pair $(p_k, x) \in (Q_n \setminus F) \times \Gamma$ satisfies the following conditions.

$$p_j = \begin{cases} p_{k+1} & \text{if } k < n \text{ and } x \neq \emptyset, \\ p_n & \text{if } k = n \text{ and } x \neq \emptyset, \\ HALT & \text{if } x = \emptyset. \end{cases} \tag{3.1}$$

Now, we assign a Turing machine to each non-negative integer through the map $\Psi : \mathbb{N} \rightarrow \mathbf{Aut}(\mathbb{N})$ defined as follows.

- If $n \leq 2$, then $\Phi(n)$ is the Turing-machine in $\mathbf{Aut}(\mathbb{N})$ with the only states p_1 and $HALT$, and with the transition map

$$\delta(p_1, x) = \begin{cases} (p_1, x, R) & \text{if } x \in \{0, 1\} \\ (HALT, 1, R) & \text{otherwise.} \end{cases} \tag{3.2}$$

- If $n > 2$ and $p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_m^{k_m} = n$ is its exhaustive factorization into prime integers, then $\Phi(n)$ is the member of $\mathbf{Aut}(\mathbb{N})$ with the transition map δ defined in the following table.

x	state p_1	state p_2	...	state p_m	(3.3)
0	$(p_2, f_{k_1}(0), R)$	$(p_3, f_{k_2}(0), R)$...	$(p_m, f_{k_m}(0))$	
1	$(p_2, f_{k_1}(1), R)$	$(p_3, f_{k_2}(1), R)$...	$(p_m, f_{k_m}(1))$	
\emptyset	$(HALT, 1, R)$	$(HALT, 1, R)$...	$(HALT, 1, R)$	

where for each $k \in \mathbb{N}$, $f_k : \{0, 1\} \rightarrow \{0, 1\}$ is the function defined as follows.

$$f_m(x) = \begin{cases} 0 & \text{if } m \equiv 0 \pmod{4} \\ x & \text{if } m \equiv 1 \pmod{4} \\ 1 & \text{if } x = 0 \text{ and } m \equiv 2 \pmod{4} \\ 0 & \text{if } x = 1 \text{ and } m \equiv 2 \pmod{4} \\ 1 & \text{if } m \equiv 3 \pmod{4}. \end{cases} \tag{3.4}$$

Notation. For every positive integer n and each finite tape-symbol sequence $c_1c_2 \dots c_j$, we denote by $\Phi(n)[c_1c_2 \dots c_j]$ the output that $\Phi(n)$ carries out from the initial tape configuration $c_1c_2 \dots c_j$.

Lemma 3.1. *For every $n \in \mathbb{N}$, the Turing machine $\Phi(n)$ accepts every finite tape-symbol sequence.*

Proof. Let $s_1, s_2, s_3 \dots s_m$ be any finite symbol sequence in the tape of $\Phi(n)$. Since the only move is R , after a finite step sequence the focused cell is the one containing the symbol s_m . In the following step, the head reads the $(m + 1)$ -th cell that must contain the blank symbol \emptyset . According to (3.3), \emptyset is substituted by 1, and the process gets the final state HALT. \square

Theorem 3.1. *The set $[0, 1] \subset \mathbb{R}$ contains an infinite subset of non-computable discernible numbers.*

Proof. We show that if there is one non-computable number in $[0, 1]$, it also contains an infinite subset $\mathbf{K} \subset [0, 1]$ of them. Let $r = 0.d_1d_2d_3 \dots d_j \dots$ a non-computable real number in $[0, 1]$ written in the binary numeration system. For each integer $k > 0$, let n_k be the positive one the binary figures of which are

$$d_{10k+1}d_{10k+2} \dots d_{(10k+10)} \tag{3.5}$$

Likewise, let $c_{k,1}c_{k,2} \dots c_{k,j_k}$ be the result of the computation

$$\Phi(n_k)[d_{10k+1}d_{10k+2} \dots d_{(10k+10)}] = c_{k,1}c_{k,2} \dots c_{k,j_k} \tag{3.6}$$

The real number

$$r^* = 0.c_{1,1}c_{1,2} \dots c_{1,j_1}c_{2,1}c_{2,2} \dots c_{2,j_2} \dots c_{k,1}c_{k,2} \dots c_{k,j_k} \dots \tag{3.7}$$

is non-computable because its binary figures depend on the ones of r , and by hypothesis so is r . In addition, the set of Turing machines

$$\{\Phi(n_1), \Phi(n_2) \dots \Phi(n_k) \dots\}$$

by which we compute r^* , is infinite and requires to know previously the binary expansion of r .

Iterating the process over r^* , we obtain an infinite sequence of non-computable real numbers in $[0, 1]$. Although in (3.5) we take sequences of ten binary figures, with figure sequences of other length we can also build infinite sets of non-computable real numbers.

Finally, the non-computable numbers involved in this proof can be determined by the finite expressions (3.5), (3.6) and (3.7); hence each of them is discernible. \square

4. Sub-cardinals

From now on, we denote by **Dsc** the subset of all discernible members of $[0, 1]$. Likewise, by **Cmp** we denote the subset of all computable members of **Dsc**. As we have just seen in Theorem 3.1, there exist non-computable numbers that are discernible; hence, **Cmp** is a proper subset of **Dsc**.

Lemma 4.1. *There is no computable bijective map between any nonempty subset \mathbf{K} of \mathbb{N} and $\mathbb{C}_{[0,1]}\mathbf{Cmp}$.*

Proof. Suppose that there is a computable bijection $f : \mathbf{K} \rightarrow \mathbb{C}_{[0,1]}\mathbf{Cmp}$. For every positive integer n_0 in \mathbf{K} , $f(n_0)$ would be computable, which contradicts $f(n_0) \in \mathbb{C}_{[0,1]}\mathbf{Cmp}$. \square

Both results, Statement (2) in Theorem 2.2 and the former lemma, allow us to compare infinite sets through several criteria based upon the nature of the comparing bijections. According to Theorem 2.1 and Lemma 2.1, both sets **Dsc** $\subset [0, 1]$ and **Cmp** $\subset [0, 1]$ are countable. However, there is no “computable” bijection between these subsets of $[0, 1]$. This peculiarity allows us to state the following definitions.

Definition 4.1. Two sets X and Y are of the same λ -cardinality if there is, at least, one “computable” bijection between them (Radó, 1962). Likewise, we say X and Y to have the same δ -cardinal whenever there is a “discernible” bijection $f : X \rightarrow Y$.

Remark. The set **Cmp** of all computable real numbers in $[0, 1]$ is a proper subset of the set **Dsc** of all finitely definable members of $[0, 1]$. As we have seen in Lemma 2.1 and Theorem 2.1, both are countable subsets of $[0, 1]$, hence there is, at least, one finitely definable bijection $f : \mathbf{Cmp} \rightarrow \mathbf{Dsc}$. However, as a consequence of Lemma 4.1, there is no computable bijection $k : \mathbf{Cmp} \rightarrow \mathbf{Dsc}$. In other words, although both sets have the same cardinal \aleph_0 , they are not of the same λ -cardinality. That is a consequence of the information that each member of **Cmp** and **Dsc** contains. Both, the information and complexity of non-computable numbers are greater than that lying in any computable one (Li & Vitányi, 2008).

Taking into account these properties, we can consider the λ -cardinal of **Cmp** as a sub-cardinal of $\aleph_0 = \#(\mathbb{N})$, which we denote by \aleph_{-2} . Since $\aleph_{-2} \neq \aleph_0$, then \aleph_{-2} is a “proper” sub-cardinal of \aleph_0 . By sub-cardinal we do not mean “smaller than.” According to both Theorem 2.2 and Lemma 4.1, cardinalities are, simply, equivalence relations based on the comparing bijection nature. This viewpoint needs some explanation. The existence of an isomorphism between structured sets leads to size equality (Hume’s principle). The converse statement need not be true. The size equality between structured sets need not lead to the existence of isomorphisms. Nevertheless, we can define bijections between the underlying sets, which are structure-free constructions. However, we cannot always forget the structure of any mathematical construction. For instance, real numbers are defined as limits of rational sequences. If we forget the topology of \mathbb{R} , the concept of limit vanishes, and we lose the real number definition. Indiscernibility is also an unforgettable structure property because it is an intrinsic object-definition attribute.

We denote by the expression \aleph_{-1} the cardinal of **Dsc**. Since **Cmp** is a proper subset of **Dsc**, \aleph_{-1} is a sub-cardinal of \aleph_0 ; and \aleph_{-2} is a sub-cardinal of \aleph_{-1} . Finally, we denote by \aleph_ω the cardinal of each set E such that there is a first-kind indiscernible bijection between \mathbb{N} and E .

Conjecture 4.1. *Each of the following incompatible statements is undecidable.*

- 1) \aleph_ω is a proper sub-cardinal of \aleph_0 .
- 2) $\aleph_\omega = \aleph_0$.

In the former conjecture, by undecidable we mean the impossibility of rejecting or proving any of these statements by finite procedures. Take into account that, in general, by no finite procedure we can discern the equality or inequality of two indiscernible objects. Discerning the equality is necessary to know whether any map is a one-to-one correspondence. This is a consequence of Theorem 2.2.

5. Some paradoxes arising from Cantor’s method

In this section, we introduce three paradoxes to show that Cantor’s methods can involve inadequate self-referential definitions. The first paradox is built through an instance of the diagonal method, which leads to a contradiction. This situation occurs if the method involves any self-referential definition that fits into the following pattern.

$$X = \text{“Expression that contains } X\text{”} \tag{5.1}$$

The former expression is an abstract equation. Since there are unsolvable equations, it is possible that some self-referential definitions define nothing. Self-referential definitions can be implicitly stated as in the following pattern.

$$X = \text{“Expression that contains a class } E \text{ containing } X \text{ implicitly.”} \tag{5.2}$$

This is the case of Cantor’s diagonal method under the assumption of actual infinity. For instance, let the following table denote a bijection $f : \mathbb{N} \rightarrow X \subseteq [0, 1]$.

$$\left\{ \begin{array}{l} f(1) = 0.d_{11}d_{12}d_{13} \dots \\ f(2) = 0.d_{21}d_{22}d_{23} \dots \\ \dots\dots\dots \\ f(n) = 0.d_{n1}d_{n2}d_{n3} \dots \\ \dots\dots\dots \end{array} \right. \tag{5.3}$$

By the diagonal method we define a member $r = 0.c_1c_2c_3 \dots$ of $[0, 1]$ that satisfies the condition

$$\forall n \in \mathbb{N} : c_n \neq d_{nn} \tag{5.4}$$

If $r \notin X$ the statement above is an adequate definition, but if $r \in X$, is implicitly self-referential.

Paradox 5.1. According to Theorem 2.1, the subset **Dsc** of all discernible members or $[0, 1]$ is countable. Let $\gamma : \mathbf{Dsc} \rightarrow \mathbb{N}$ be the injective map finitely defined in (2.1), and $\gamma_0 : \mathbf{Dsc} \rightarrow \text{img}(\gamma)$ the restriction to its image, which is bijective; hence there exists its inverse

$$\gamma_0^{-1} : \text{img}(\gamma) \subseteq \mathbb{N} \rightarrow \mathbf{Dsc}.$$

Since γ is finitely defined in (2.1), so are both γ_0 and γ_0^{-1} ; consequently, the three maps are discernible.

Now, for every positive integer n , let $0.c_{n1}c_{n2}c_{n3} \dots$ be the decimal mantissa of $\gamma_0^{-1}(n)$. By Cantor’s diagonal method we can build a real number $r = 0.d_1d_2d_3 \dots$ in $[0, 1]$ as follows.

$$\forall n \in \mathbb{N} : d_n = \begin{cases} c_{nn} + 1 & \text{if } c_{nn} < 9 \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

The former equation is a finite expression that determines r uniquely; hence it is discernible, and the relation

$$r \in \mathbf{Dsc}. \tag{5.6}$$

is true. Nevertheless, equation (5.5) leads to $\forall n \in \mathbb{N} : r \neq \gamma_0^{-1}(n)$, because $\forall n \in \mathbb{N} : c_{nn} \neq d_n$; consequently

$$r \notin \mathbf{Dsc}, \tag{5.7}$$

which contradicts (5.6).

We can solve the former paradox easily. Recall that, in the scope of actual infinity, every infinite set is a construction that can be completed. In this case, when the image of an indiscernible bijection $f : \mathbb{N} \rightarrow E$ contains every number r_0 that can be obtained by Cantor’s diagonal method, this method involves a self-referential definition for r_0 . This is the case in the former paradox. To better understand this topic, we state the paradox below by the inverse method. We choose the number $r_0 \in [0, 1]$ and build a bijection that does not contain it.

Paradox 5.2. Let $f : \mathbb{N} \rightarrow \mathbf{M} \subseteq [0, 1]$ be a bijection that satisfies the following conditions.

1. Let \mathbb{E} denote the set $\{2 \cdot n \mid n \in \mathbb{N}\}$ of all positive even integers. The restriction $f|_{\mathbb{E}} : \mathbb{E} \rightarrow \mathbf{M}$ of f to \mathbb{E} is the subset $\mathbb{Q} \cap [0, 1]$ of all rationals in $[0, 1]$. Recall that both \mathbb{E} and \mathbb{Q} are countable; so then we can build the map f under the condition $\text{img}(f|_{\mathbb{E}}) = \mathbb{Q} \cap [0, 1]$.
2. The image $f(n)$ of each odd integer n is an irrational member of \mathbf{M} . Thus, $\text{img}(f)$ contains every rational in $[0, 1]$ and an infinite and countable subset of irrationals.

Let $r \in \mathbb{Q} \cap [0, 1] \subset \text{img}(f)$ be a rational number and $0.d_1d_2d_3 \dots$ the figures of its decimal expansion. Suppose that, for every $n \in \mathbb{N}$, the expansion of $f(n)$ is $0.c_{n1}c_{n2}c_{n3} \dots$. Now, we define a new bijection f_1 as follows. If $d_1 \neq c_{11}$, then $f_1 = f$; otherwise, if n_1 is the smallest integer such that $c_{n1} \neq d_1$, then

$$f_1(n) = \begin{cases} f(n_1) & \text{if } n = 1 \\ f(1) & \text{if } n = n_1 \\ f(n) & \text{otherwise;} \end{cases} \tag{5.8}$$

We obtain the bijection f_1 , simply, by the transposition of $f(1)$ and $f(n_1)$ in the image of f , hence $\text{img}(f) = \text{img}(f_1)$, besides, the first figure in the mantissa of $f_1(1)$ differs from d_1 .

Now, we define another bijection f_2 by a similar procedure. If the second figure in the mantissa of $f_1(2)$ is not equal to d_2 , then $f_2 = f_1$; otherwise, if n_2 is the smallest integer such that $n_2 > 2$ and $c_{n_2} \neq d_2$, then

$$f_2(n) = \begin{cases} f(n_2) & \text{if } n = 2 \\ f(2) & \text{if } n = n_2 \\ f(n) & \text{otherwise;} \end{cases} \quad (5.9)$$

As in the preceding case, the bijection f_2 is obtained, simply, by the transposition of $f(2)$ and $f(n_2)$; hence $\text{img}(f) = \text{img}(f_1) = \text{img}(f_2)$, besides, the second figure in the mantissa of $f_1(2)$ differs from d_2 .

Iterating the procedure m -times, we obtain a bijection $f_m : \mathbb{N} \rightarrow \mathbf{M}$ such that $\text{img}(f_m) = \text{img}(f)$, and $\forall n \leq m: r \neq f_m(n)$. Under the scope of potential infinity, the iteration never ends. By contrast, assuming infinity as an actual entity, the iteration can be completed obtaining a bijection $f_\infty : \mathbb{N} \rightarrow \mathbf{M}$ with $\text{img}(f_\infty) = \text{img}(f) = \mathbf{M}$, and $\forall n \in \mathbb{N} : r \neq f_\infty(n)$, hence $r \notin \mathbf{M}$; which contradicts that $r \in \mathbb{Q} \cap [0, 1] \subseteq \mathbf{M}$, that we assume implicitly in statement (1), and f satisfies this condition.

Paradox 5.3. Let $\mathcal{F}(\mathbb{N}) \subseteq \mathcal{P}(\mathbb{N})$ be the subset of all finitely definable subsets of \mathbb{N} . Since we can denote every member of $\mathcal{F}(\mathbb{N})$ by a finite symbol sequence in some extensible language, we can also define a map $\zeta : \mathcal{F}(\mathbb{N}) \rightarrow \mathbb{N}$ similar to the one γ defined in (2.1). Let $\zeta_0 : \mathcal{F}(\mathbb{N}) \rightarrow \text{img}(\zeta)$ be the restriction of ζ to its image, which is a bijection.

Now, we define the subset \mathbf{K} of \mathbb{N} by the finite expression

$$\mathbf{K} = \{n \in \text{img}(\zeta) \subseteq \mathbb{N} \mid n \notin \zeta_0^{-1}(n)\} \quad (5.10)$$

If $\zeta(\mathbf{K}) \in \mathbf{K}$, by virtue of (5.10), this relation leads to $\zeta(\mathbf{K}) \notin \mathbf{K}$, and vice versa. We can solve this contradiction supposing that ζ is not defined at \mathbf{K} . In other words: $\mathbf{K} \notin \mathcal{F}(\mathbb{N})$. However, \mathbf{K} is a subset of \mathbb{N} finitely defined in (5.10); hence $\mathbf{K} \in \mathcal{F}(\mathbb{N})$.

It is worth pointing out that, as in both paradoxes 5.1 and 5.2, the former one requires the assumption of actual infinity. Thus, if we assume that every infinity is potential, these paradoxes vanish.

6. Conclusion

On the one hand, we should not ignore the existence of indiscernible mathematical constructions, unless we reject the existence of uncountable sets. Unfortunately, as a consequence of Theorem 2.2, we can only handle infinite sets of indiscernible objects. Thus, when working with real world problems, we never can be involved with any “finite” set of indiscernible numbers.

On the other hand, in the scope of actual infinity, Cantor’s diagonal method sometimes proves nothing. Fortunately, taking into account Corollary 2.1 and Theorem 2.3, we can deduce the existence of uncountable sets from indiscernible mathematical constructions. In addition, if Conjecture 4.1 is true, a set theory including it becomes incomplete, and we can apply Gödel’s incompleteness theorems to it.

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Proximal Čech Complexes in Approximating Digital Image Object Shapes. Theory and Application

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Abstract

This article introduces proximal Čech complexes in approximating object shapes in digital images. The theoretical framework is based on Čech complexes and proximity spaces. Several topological structures are defined for the Čech nerve based covers of a finite region of Euclidean plane. We define k -petals and k -corollas which are the generalizations of spokes and maximal nuclear clusters. We extend the classical notion of a proximity as a binary relation, to arbitrary number of sets. A new shape signature based on the distribution of orders of Čech nerves is defined. A practical application of this framework in approximating object shapes in digital images is given.

Keywords: Čech complex, Digital image, Nerve, Object shape

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1. Introduction

Understanding of shapes of objects in images is an important aspect of artificial intelligence, with numerous applications. Some of the applications detection of specific patterns in images (Hettiarachchi *et al.*, 2014), quantifying information content of images using Voronoï tessellations (A-iyeh & Peters, 2016) and detecting outliers from images of different classes (Shamir, 2013). In this paper we aim to understand the shape of the objects in a digital image, by covering it with sets of known geometrical properties. Moreover, we aim at enriching the conventional notion of shape

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as its contour (as used in the prevalent literature) to include the interior as well. For this purpose we will combine the notions of topology and geometry with proximity spaces.

E. Čech introduced Čech complexes during a seminar at Brno(1936-39) (Čech, 1966, §A.5). Recent works define the Čech complexes as collections of intersecting closed geometric balls of a radius of r (Edelsbrunner & Harer, 2010); (Peters, 2017a). In this study we focus on approximating shapes of objects in planar digital images. This leads to the restriction of the Čech complexes to a finite bounded region of the Euclidean Plane. In a recent study some open problems regarding planar shapes(shapes of objects in Euclidean plane) have been posed (Peters, 2017a) . The viability of Čech complexes as a method to approximate image object shapes has been shown in (Peters, 2017b).

A previous study formulates the notion of an object in the digital image as a topological space (Ahmad & Peters, 2017a). The notion of a nerve introduced by Alexandroff (Alexandroff, 1965) has been generalized to the notion of spoke complexes. The basic building blocks of the topology are assumed to be curvilinear triangulations. Another study builds on this notion and studies the covering properties of the curvilinear triangulation using the notion of area and geodesic diameter (Ahmad & Peters, 2017b). Moreover, a notion of the frequency of nerves of different order as a possible signature of the image objects was also introduced. The notion of proximity was also defined on triangulated spaces (Ahmad & Peters, 2017a) and Čech nerves (Peters, 2017b).

The subject of the current study is to extend the topological framework for object spaces formulated in (Ahmad & Peters, 2017a) to Čech complexes. In addition to this the framework of assessing the covering properties of nerves and object spaces similar to (Ahmad & Peters, 2017b) is also developed. A notion of proximity will be introduced on the resulting object spaces. This yields a relator space of objects in the digital images.

2. Basic Definitions

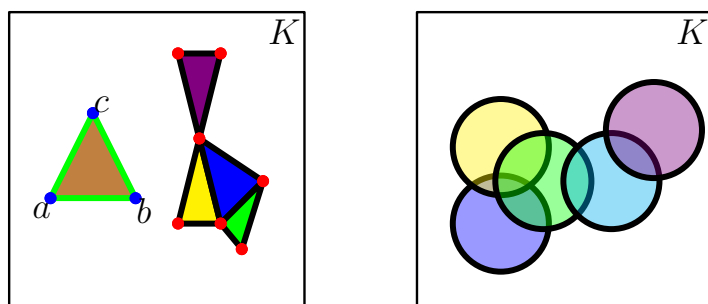
In this section, we will present some basic definitions. Let $X = \mathbb{R}^2$ be the Euclidean plane then $K \in 2^X$ is a finite bounded region in X . The basic building block of Čech complex is the closed geometric ball, $B_r(x)$, with center $x \in X$ and radius $r > 0$ defined as $B_r(x) = \{y \in X : \|x - y\| \leq r\}$.

In this paper, we will study the topology of the objects in a planar digital image. Let us first define **abstract simplicial complex**.

Definition 1. (Ghrist, 2014, §2.1) *Let S be a discrete set. Then the collection of finite subsets of S represented by X is called an abstract simplicial complex, provided for each $\sigma \in X$, all subsets of σ are also in X . This means that X is closed under this restriction.*

Example 1. *Consider a geometric realization of an abstract simplicial complex in form of collections of triangles. Let us first look at the set $X = \{a, b, c\}$, representing the vertices of a triangle as shown in Fig. 1.1. Then the powerset of all the subsets of the set X is,*

$$2^X = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$



1.1: Collections of triangles

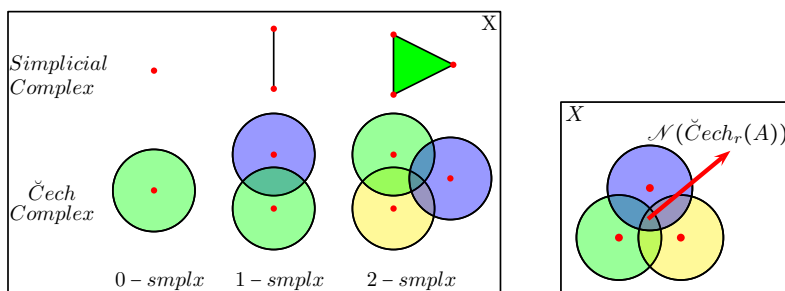
1.2: Collections of Geometric Balls

Figure 1. This figure shows the geometric realizations of abstract simplicial complexes as collections of connected triangles and intersecting closed geometric balls

It can be seen that the set 2^X is closed under restriction as per Def.1. The singleton elements of the set 2^X , i.e. $\{a\}$, $\{b\}$ and $\{c\}$ represent each of the vertices or the 0-simplices. These are shown in blue on the isolated triangle. The elements $\{a,b\}$, $\{b,c\}$ and $\{a,c\}$ represent the edges or the 1-simplices. These are shown in green on the isolated triangle. The element $\{a,b,c\}$ represents the filled triangle or the 2-simplex. This is shown with brown color on the isolated triangle. One can easily verify that the connected triangles in Fig. 1.1, also form a simplicial complex. This follows from the fact that each of the triangles in itself is a simplicial complex determined by the power set of the set of its vertices. The whole collection of the triangles is represented by the set Y , which is the union of their respective power sets. Every possible subset of the elements in set Y is contained in itself. Thus set Y is also a simplicial complex as per Def. 1. ■

Keeping in view the previous example it can be seen that the collection of geometric balls ($B_r(x)$ for $x \in K$) with a finite number of intersections can be seen as a realization of the abstract simplicial complex. For the purpose of visualization consider the Fig. 1.2 The individual balls are the 0-simplices. The pairs of balls with non-empty intersection are the 1-simplices. The examples are the blue and the violet ball, green and the blue ball etc. If three balls have a common intersection, they lead to a 2-simplex. The yellow, green and blue balls in Fig. 1.2 are an example of a 2-simplex. It can be seen that every pair of balls included in the 2-simplex form a 1-simplex and each constituent ball of this resulting 1-simplex is a 0-simplex. Hence, the collection of closed geometric balls ($B_r(x)$, $x \in K$) is also a geometric realization of an abstract simplicial complex. The 1-simplex which is a line segment in the simplicial complex, is analogous to two Čech balls having common intersection. The 2-simplex is a triangle in the simplicial complex and three Čech balls with a common intersection, in the Čech complex.

Moreover, it can be seen from Fig. 1.2 that one can draw analogs between the simplicial complex (generalization of a triangle to arbitrary dimensions) and the simplices in a Čech complex. It can be seen that each ball in a Čech complex is a 0-simplex and hence similar to the vertex in a simplicial complex. Let us define the notion of a Čech nerve.



2.1: Comparison of corresponding simplices in a simplicial complex and Čech complex 2.2: Nucleus of a Čech nerve

Figure 2. This figure explains the notion of a Čech complex (Def. 3) and compares it with the simplicial complex. Moreover, the notion of the nerve (Def. 11) of a Čech complex is illustrated.

Definition 2. A Čech nerve, is a collection of sets having a non-empty intersection denoted by $(\check{C}ech_r(K))$ i.e.,

$$\check{C}ech_r(K) = Nrv\{B_r(x) : x \in K\} = \{B_r(x) : \bigcap B_r(x) \neq \emptyset\}.$$

Then, we can define the notion of the **order** of a nerve. It is denoted by $|\check{C}ech_r(K)|$, and is defined as the number of balls in the nerve. We define a set with no non-empty intersections as being a nerve of order 1. Using this notion, we can formulate a definition of the Čech complex in a space X .

Definition 3. Let X be a finite, bounded planar region $X \in \mathbb{R}^2$ and let $K \in X$ be a nonempty collection of points. A Čech complex is a collection of Čech nerves. The Čech nerves, $\check{C}ech_r(x \in K)$ of order k , form the $k - 1$ -simplices. The Čech complex is denoted by $cxK = 2^{\check{C}ech_r(K)}, K \in 2^{\mathbb{R}^2}$.

The basic building block of a Čech complex is a closed geometric ball $B_r(x)$. The view of a digital image as a topological space has been detailed in (Peters, 2014). In current work we extend the conventional notion of a Čech nerve as follows.

Definition 4. A **strong Čech nerve**, denoted by $\check{C}ech_r^s(K)$, is a collection of sets whose interiors have a non-empty intersection i.e.,

$$\check{C}ech_r^s(K) = Nrv^s\{B_r(x) : x \in K\} = \{B_r(x) : \bigcap int(B_r(x)) \neq \emptyset\}.$$

In doing so we have also extended the classical notion of a **nerve** to **strong nerve**. A strong nerve is a collection of sets, whose interiors have a non-empty intersection. The notion of **order** remains unchanged. Similar to the notion of the Čech complex, we define the concept of a **strong Čech complex** on a space X .

Definition 5. Let X be a finite, bounded planar region $X \in \mathbb{R}^2$ and let $K \in X$ be a nonempty collection of points. A strong Čech complex is a collection of strong Čech nerves. The strong Čech nerves, $\check{C}ech_r^s(x \in K)$ of order k form the $k - 1$ -simplices. The strong Čech complex is denoted by $cX^s K = 2^{\check{C}ech_r^s(K)}$, $K \in 2^{\mathbb{R}^2}$.

In this paper we want to approximate the objects in the digital images as topological spaces. This will allow us to talk about shape (Segal & Dydak, 1978) and define invariant signatures for classification (Carlsson *et al.*, 2005); (Chazal *et al.*, 2009). The classical notion of a topology is defined using the notion of open sets.

Definition 6. (Edelsbrunner & Harer, 2010) Topology is an ordered pair (X, τ) , where X is a set and τ is the collection of subsets of X satisfying the following axioms:

- 1^o The empty set, \emptyset , and X belong to τ
- 2^o Union of sets in τ is also in τ
- 3^o Intersection of finite number of members of τ is also in τ

We call this classical notion of topology as **open topology**. Using the De Morgan's laws, we can convert the notion of an open set topology to a closed set topology. Let us define the notion of a complement with respect to a set X . For a set $A \subset X$, the complement of a A denoted by \overline{A} is defined as $\overline{A} = X \setminus A$. The laws state that for two sets A, B :

$$\begin{aligned} \overline{A \cup B} &= \overline{A} \cap \overline{B} \\ \overline{A \cap B} &= \overline{A} \cup \overline{B} \end{aligned}$$

A set A is closed in a topology τ , if its complement \overline{A} is in τ . Using these notions we can extend the definition of a topology with open set as a primitive to using the closed set as the basis. Let us define the notion of topology using closed sets as primitive. To differentiate this we call it a **closed topology**.

Definition 7. Topology is an ordered pair (X, η) , where X is a set and η is the collection of subsets of X satisfying the following axioms:

- 1^o The empty set, \emptyset , and X are closed and belong to η
- 2^o The intersection of any collection of sets in η is closed
- 3^o The union of a collection of finite number of sets in η is closed

Example 2. Let us examine the equivalence of the two definitions of topology based on open sets(Def. 6) and closed sets(Def. 7). It must be noted that as we have defined the set to be closed under the topology if its complement is in the topology. To establish the equivalence of the notions of open and closed topology we define the collections of subsets η to contain the complement of each subset in the collection τ . This complement is taken with respect to the set X . Let us begin with the first axiom in Def. 6. It states that both the \emptyset and X are in τ . Thus, both are open sets. It can be seen that $\overline{\emptyset} = X \in \tau$ and $\overline{X} = \emptyset \in \tau$. Which means that both \emptyset, X are closed and are in η . Thus, this statement is equivalent to the first axiom stated in Def. 7.

The second axiom in the Def.6 can be stated as $\{\bigcup_{i \in \mathcal{I}} A_i : A_i \in \tau\} \in \tau$, where \mathcal{I} is an index set which can be infinite and $A_i \in \tau$. This means that $\{\bigcup_{i \in \mathcal{I}} A_i\}$ is an open set. Which is equivalent to $\overline{\bigcup_{i \in \mathcal{I}} A_i} \in \eta$ being a closed set. Using De Morgan's law we can write this as $\bigcap_{i \in \mathcal{I}} \overline{A_i}$, where $\overline{A_i} \in \eta$ and \mathcal{I} is an index set which can be infinite. Since the complement of an open set must be closed, $\bigcap_{i \in \mathcal{I}} \overline{A_i}$ must be closed. We have the second axiom of Def. 7. Which states that intersection of any collection of sets in η is closed.

The third axiom in Def. 6 is $\{\bigcap_{i \in \mathcal{I}} A_i : A_i \in \tau\} \in \tau$, where \mathcal{I} is a finite index set. This means that $\bigcap_{i \in \mathcal{I}} A_i$ is an open set and its complement is closed. Thus $\overline{\bigcap_{i \in \mathcal{I}} A_i} = \{\bigcup_{i \in \mathcal{I}} \overline{A_i} : \overline{A_i} \in \eta\}$ (using De Morgan's law), is a closed set. This statement is equivalent to the third axiom of Def. 7, which states that the union of a collection of finite number of sets in η is closed. Hence, the Def. 6 and Def. 7 are equivalent. The former uses the open sets as the primitive and the later uses the notion of a closed set. ■

Let us define a topology on the Euclidean plane(\mathbb{R}^2) using the closed geometric balls, $B_r(x)$.

Definition 8. Let U be a closed subset of \mathbb{R}^2 and τ_{std} be a family of closed subsets of \mathbb{R}^2 . Then, $U \in \tau_{std}$ if and only if for all $p \in U$, there exists a positive real number r such that $B_r(p) \subseteq U$.

Now, let us verify that $(\mathbb{R}^2, \tau_{std})$ is a closed topological space.

Lemma 1. The pair $(\mathbb{R}^2, \tau_{std})$ is a closed topological space.

Proof. Let us verify that the family of subsets τ_{std} satisfies the axioms stated in Def. 7.

1^o : Let us verify that $\emptyset \in \tau_{std}$. From Def. 8, if $\emptyset \in \tau_{std}$, then the following condition must be satisfied.

$$\forall p \in \emptyset \Rightarrow \exists r \in \mathbb{R}^+ \text{ s.t. } B_r(p) \subseteq \emptyset$$

It is evident from the definition of implication(\Rightarrow), that $A \Rightarrow B$ is always true if A is false. Since, $p \in \emptyset$ is always false, thus the statement

$$\exists r \in \mathbb{R}^+ \text{ s.t. } B_r(p) \subseteq \emptyset,$$

is true. From this it follows that $\emptyset \in \tau_{std}$. Let us now verify that there exists a positive real number r such that $B_r(x) \subseteq \mathbb{R}^2$, for all $x \in \mathbb{R}^2$. It can be seen from the definition of $B_r(x) = \{y \in \mathbb{R}^2 : \|x - y\| \leq r\}$ that $B_r(x) \subseteq \mathbb{R}^2$ independent of x and r . Thus $\mathbb{R}^2 \in \tau_{std}$.

2^o : Let $\{C_i\}_i$ be a family of sets such that $\forall i, C_i \in \tau_{std}$. Then, if $p \in \bigcap C_i$ then there exists a positive real number r_i for each C_i , such that $B_{r_i}(p) \subseteq C_i$. It can be stated that,

$$B_{r_1}(p) \subseteq C_1 \wedge \dots \wedge B_{r_i}(p) \subseteq C_i.$$

Let us define $r = \min_i(r_i)$. Then it can be seen that,

$$B_r(p) \subseteq B_{r_1}(p) \subseteq C_1 \wedge \dots \wedge B_r(p) \subseteq B_{r_i}(p) \subseteq C_i.$$

From this we can conclude that $B_r(p) \in \cap C_i$. Therefore, $\cap C_i \in \tau_{std}$.

3^o : Let $\{C_i\}_i$ be a finite collection of sets such that $\forall i, C_i \in \tau_{std}$ and $p \in \cup C_i$. Then $p \in C_i$, for some i . Since, $C_i \in \tau_{std}$:

$$\exists r \in \mathbb{R}^+ \text{ s.t. } B_r p \subseteq U \subseteq \cup C_i.$$

From this we can conclude that $\cup C_i \in \tau_{std}$.

Thus, it can be concluded that $(\mathbb{R}^2, \tau_{std})$. □

Remark 1. *The notion of a closed standard topology is important as it is the closed set analogue of the standard topology, used for analysis and the conventional signal and image processing.*

Moving on, we define the notion of the closed subset topology.

Definition 9. *Let (X, τ) be a closed topological space, and S be a closed subset of X , i.e. $S \subseteq X$. Let us define $\tau_S = \{S \cap U \text{ s.t. } U \in \tau\}$. Then the pair (S, τ_S) is a topological space where τ_S is the subset topology.*

Let us verify that (S, τ_S) is indeed a topological space.

Lemma 2. *Let (X, τ) be a topological space and let S be a closed subset of X . Then the pair (S, τ_S) , where $\tau_S = \{S \cap U \text{ s.t. } U \in \tau\}$, is a closed topological space.*

Proof. To prove that the pair (S, τ_S) is a closed topological space it must satisfy the axioms detailed in Def. 9.

1^o : It can be verified from the definition of τ_S , that $S \cap (\emptyset \in \tau) = \emptyset$ is in τ_S . Moreover, it can also be verified from definition of τ_S , that $S \cap (X \in \tau) = S$ is also in τ_S . Thus, both \emptyset and S are in τ_S .

2^o : Let $\{C_i\}_i$ be a collection of sets, such that $\forall i, C_i \in \tau_S$. Then from definition of τ_S it can be inferred that, for each $C_i \in \tau_S$ there exists a set D_i in τ , such that for all $i, S \cap D_i = C_i$. Since τ is a closed topology over the set X , then by Def. 9 the set $\cap D_i$ is also in τ . This leads to the conclusion that $(\cap D_i) \cap S$ is in τ_S by definition. We can see that $(\cap D_i) \cap S$ can be written as $\cap (D_i \cap S)$. As, we defined in this argument that $D_i \cap S = C_i$ we can conclude that $\cap C_i$ is in τ_S .

3^o : Let $\{C_i\}_i$ be a finite collection of sets, such that for all $i, C_i \in \tau_S$. Then from the definition of τ_S , we can infer that for each $C_i \in \tau_S$ there exists a set $D_i \in \tau$, such that $D_i \cap S = C_i$. Since, τ is a closed topology then by definition it satisfies the condition that, $\cup D_i$ is in τ . From the definition of τ_S we can conclude that $S \cap (\cup D_i)$ is in τ_S . From the distributive laws of set algebra we can rewrite this as $\cup (D_i \cap S)$ is in τ_S . We know that $D_i \cap S = C_i$, hence $\cup C_i$ is in τ_S .

Since the pair (S, τ_S) satisfies all the axioms it is a closed topological space over S , with a subset topology τ_S . \square

Now, let us talk about the Čech complex as a closed topological space.

Theorem 1. *Let (\mathbb{R}^2, τ) , be a closed topological space and let cxK be a Čech complex. Then the pair (cxK, τ_{Cech}) is a closed topological subspace, where τ_{Cech} is the subset topology, defined as $\{cxK \cap U \text{ s.t. } U \in \tau\}$.*

Proof. Let us first ascertain that cxK is a subset of \mathbb{R}^2 . This follows from definition as cxK is the union of Čech nerves ($\check{C}ech_r(K)$) as defined in Def. 3. Moreover, the $\check{C}ech_r(K)$ is a union of closed geometric balls as per Def. 2. Next, we can see that the closed geometric balls are in standard closed topology τ_{std} by definition as there exists a positive real number r such that $B_r(p) \subseteq B_r(x)$ for all p in $B_r(x)$. Since, the closed geometric balls are in τ_{std} , thus by the definition of a closed topology, the union of a finite number of closed geometric balls must also be in τ_{std} i.e. closed. Thus, cxK is closed and by the definition it is a subset of \mathbb{R}^2 . From this and Lemma 2, it can be concluded that given a closed topological space (\mathbb{R}^2, τ) , the pair (cxK, τ_{Cech}) , where $\tau_{Cech} = \{cxK \cap U \text{ s.t. } U \in \tau\}$, is a topological space, where τ_{Cech} is the subset topology. \square

Now that we have a notion of Čech nerves as a topological space, we have to draw a correspondence to the objects in digital images. For this purpose, we use specific points from the image called keypoints. There are several algorithms for selecting such points based on their strength in an appropriate feature spaces (Lowe, 1999); (Bronstein & Kokkinos, 2010); (Aubry et al., 2011); (Bruna & Mallat, 2013); (Alahi et al., 2012). We construct Čech nerves with these keypoints as the centers of the geometric closed balls. We assume that the object under consideration, is the most important part of an image in the feature space. Let us now define the topological structures in the strong Čech nerves, that we are going to use as approximations of the objects in an image. We have used the notion of strong Čech nerves because they allow a more rich set of proximity relations, that we will define later on. The same can be done for the Čech nerves.

The first structure that we are going to define is the notion a petal.

Definition 10. *Let $\check{C}ech_r^s(K)$ be a strong Čech nerve, then each closed geometric ball in the nerve $\check{C}ech_r^s(K)$ is called a petal and is denoted by **ptl**.*

Now, we define the notion of a nucleus.

Definition 11. *Let $\check{C}ech_r^s(K)$ be a strong Čech nerve. Then, the nucleus of the nerve is the common intersection of the constituent sets of the nerve. It is defined as $\mathcal{N}(\check{C}ech_r^s A) = \{\cap B_r(x) : B_r(x) \in \check{C}ech_r^s A\}$, where $\check{C}ech_r^s A$ is a strong Čech nerve.*

Example 3. *Let us explain the concept of a strong Čech nerve. For this consider Fig. 2.2. K is a set of points in the finite, bounded region of \mathbb{R}^2 , shown as red points in the figure. The $\check{C}ech_r^s(K)$ in this illustration is the union of the three geometric balls (blue, green and yellow) centered at the points represented as red dots. Each of the individual balls is the petal (**ptl**) of the $\check{C}ech_r^s(K)$. This is represented as **ptl**($\check{C}ech_r^s(K)$). The common intersection of the nerve is called the nucleus. This is represented as $\mathcal{N}(\check{C}ech_r^s(K))$. \blacksquare*

We now generalize the concept of a petal to the notion of a k -petal, defined using a recursive definition.

Definition 12. Let cx^sA be a strong Čech complex on a finite, bounded region of the Euclidean plane \mathbb{R}^2 , $\check{C}ech_r^sA$ is a strong Čech nerve, and $k > 0, k \in \mathbb{Z}$. Then k -petal is defined as the closed geometric ball $B_r(x)$ that has a nonempty intersection with a $(k - 1)$ -petal. The 0-petal is the nucleus. This can be formally written as:

$$ptl_k = \{B_r(x) \in cx^sA \setminus \{\bigcup ptl_{k-1}\} : B_r(x) \cap \{\bigcup ptl_{k-1}\} \neq \emptyset, ptl_0 = \mathcal{N}(\check{C}ech_r^sA)\}.$$

Here, it is important to mention that the nerve of the highest order in the image is very important for object extraction in digital images (Peters & Inan, 2016). Such a nerve is called the **maximal strong Čech nerve** and is denoted as $\mathbf{max}\check{C}ech_r^s(K)$. The nucleus of the maximal strong Čech nerve is called the **maximal nucleus** and is denoted as $\mathbf{max}\mathcal{N}$. It must be noted that there can be multiple maximal strong Čech nerves in a digital image. In this case the $\mathbf{max}\check{C}ech_r^s(k)$ is a set of each of the maximal strong Čech nerves, and the $\mathbf{max}\mathcal{N}$ is the set of the nuclei associated with the maximal nerves.

Next, we consider the analogue of a corolla, from Botany. A corolla is a cluster of petals of a flower, typically forming a whorl within the sepals and enclosing the reproductive organs. This construct works well in analyzing the interior of Čech complexes that cover an image object shape.

Definition 13. Let cxA be a strong Čech complex on a finite, bounded region of the Euclidean plane \mathbb{R}^2 . Then the k -corolla, denoted as \mathbf{crl}_k is defined as: $\mathbf{crl}_k = \bigcup ptl_k$.

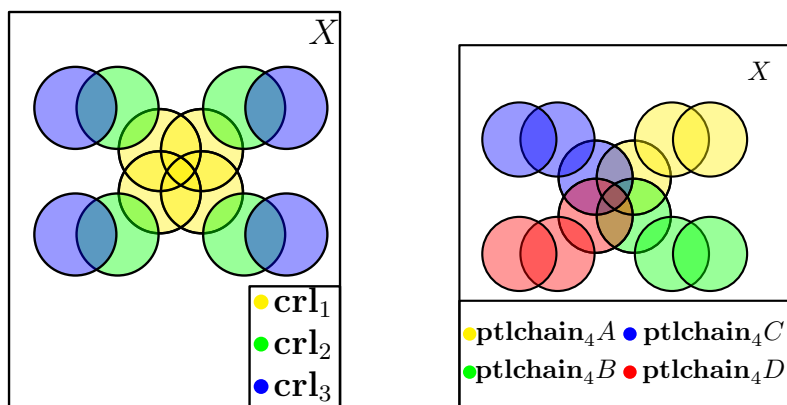
Another notion that we will define using the notion of a k -petal is the k -petal chain. It is a sequence of petals, one for each value of k . Each petal has a nonempty intersection with adjacent petals in the chain. This notion is formalized as follows.

Definition 14. Let cx^sA be a strong Čech complex on a finite, bounded region of Euclidean plane \mathbb{R}^2 . Then the k -petal chain denoted as $\mathbf{ptlchain}_k$ is defined as:

$$k > 1 : \mathbf{ptlchain}_k = \left\{ \bigcup_{\mathcal{I}} A_i : A_i \in \mathbf{crl}_i, A_i \cap A_{i-1} \neq \emptyset, \mathcal{I} = 1, \dots, k \right\}$$

$$k = 0 : \mathbf{ptlchain}_k = \mathcal{N}(\check{C}ech_r^sA)$$

Example 4. Let us explain the topological structures that we have defined above. Let us first explain the notion of a k -petal (\mathbf{ptl}_k) defined in Def. 12, using Fig. 3.1. The \mathbf{ptl}_0 (0-petal) has not been explicitly marked on the figure. It is the intersection of the yellow balls and is also the nucleus of the strong Čech nerve, $\mathcal{N}(\check{C}ech_r^s(K))$. Each of the yellow balls is the \mathbf{ptl}_1 (1-petal). It is evident from the illustration that each of the green balls has a non-empty intersection with the yellow balls (\mathbf{ptl}_1) and no intersection with the nucleus (\mathbf{ptl}_0). Thus each of the green balls is a \mathbf{ptl}_2 (2-petal) as per Def. 12. Each of the blue balls has a non-empty intersection with the green balls (\mathbf{ptl}_2), while has no intersection with the yellow balls (\mathbf{ptl}_1) and as per Def. 12 are the \mathbf{ptl}_3 .



3.1: Collections of triangles

3.2: Collections of Geometric Balls

Figure 3. This figure displays the geometric realizations of the k -corolla, \mathbf{crl}_k (Def. 13), and k -petal chain $\mathbf{ptlchain}_k$ (Def. 14).

Let us now use the notion of a k -petal to define the notion of a k -corolla(\mathbf{crl}_k). We are still using the Fig.3.1. The union of all the \mathbf{ptl}_0 in the image are the \mathbf{crl}_0 . Since, there is one nucleus in the image and it has not been marked, hence the \mathbf{crl}_0 has not been marked. The union of all the yellow balls (\mathbf{ptl}_1) is the \mathbf{crl}_1 (1-corolla). The union of all the green balls(\mathbf{ptl}_2) is the \mathbf{crl}_2 (2-corolla) and the union of all the blue balls(\mathbf{ptl}_3) is the \mathbf{crl}_3 (3-corolla) as per Def. 13.

Now using Figs. 3.1 and 3.2 in conjunction, we will explain the idea of a $\mathbf{ptlchain}_k$ (Def. 14). It must be noted that, similar to each of the structures explained in this example, each of the structures has the nucleus of the nerve($\mathcal{N}(\check{C}ech^s_i)$) as its generating point. The $\mathbf{ptlchain}_0$ is the nucleus of the nerve which is the intersection of the yellow balls in Fig. 3.1. It is not labeled in this image. It is important to note here that each of the yellow balls is the $\mathbf{ptlchain}_1$, from Def. 14. This is because it contains the nucleus and the \mathbf{ptl}_1 (the yellow ball). Thus the $\mathbf{ptlchain}_1$ is the same as the \mathbf{ptl}_1 . There are four possible \mathbf{ptl}_1 and similarly four possible $\mathbf{ptlchain}_1$. Now, by adding to this construction a ball from \mathbf{crl}_2 , that has a non-empty intersection with the $\mathbf{ptl}_1 \in \mathbf{ptlchain}_1$, we can construct a $\mathbf{ptlchain}_2$ as per Def. 14. In a similar fashion we can construct $\mathbf{ptlchain}_3$ by adding to this construction a ball from \mathbf{crl}_3 , having non-empty intersection with the $\mathbf{ptl}_2 \in \mathbf{ptlchain}_2$. It can be seen that we can have four possible $\mathbf{ptlchain}_3$ as illustrated in Fig. 3.2. ■

Using the above definitions we will now define the notion of an object space in the strong Čech complex.

Definition 15. Let cx^sA be a strong Čech complex in a finite, bounded region of an Euclidean plane \mathbb{R}^2 , and let $\hat{k} \in \mathbb{Z}^+$ such that $\mathbf{crl}_{\hat{k}} = \emptyset$. Then, the object space $\mathcal{O}_p^{\check{C}ech}$, for each $p \in \max \mathcal{N}$ is defined as:

$$\mathcal{O}_p^{\check{C}ech} = \left\{ \bigcup_k \mathbf{crl}_k : \mathbf{crl}_0 = p, k = 0, \dots, \hat{k} - 1 \right\}$$

We define the notion of a boundary petal **bdypt**.

Definition 16. Let $\mathcal{O}_p^{\check{c}ech} A$ be an object space over a finite, bounded region of the Euclidean plane, \mathbb{R}^2 . Then the boundary petal of the object space, $\mathbf{bdpt}(\mathcal{O}_p^{\check{c}ech})$ is defined as:

$$\mathbf{bdpt}(\mathcal{O}_p^{\check{c}ech}) = \{x : \forall k x \in \mathbf{crl}_k \text{ and } x \cap \mathbf{crl}_{k+1} = \emptyset\}$$

Now, we introduce a bounding corolla in the object space, denoted as $\mathbf{bdycrl}(\mathcal{O}_p^{\check{c}ech})$.

Definition 17. Let $\mathcal{O}_p^{\check{c}ech} A$ be an object space over a finite, bounded region of the Euclidean plane, \mathbb{R}^2 . Then the bounding corolla of the object space, $\mathbf{bdycrl}(\mathcal{O}_p^{\check{c}ech})$ is defined as:

$$\mathbf{bdycrl}(\mathcal{O}_p^{\check{c}ech}) = \bigcup \mathbf{bdpt}(\mathcal{O}_p^{\check{c}ech})$$

Let us define the notion of a maximal petal chains.

Definition 18. Let $cx^s A$ be a strong Čech complex, and let $\mathbf{ptlchain}_k A$ be a petal chain in $cx A$. Then, $\mathbf{ptlchain}_k A$ is called a maximal chain if $\mathbf{ptlchain}_k A \cap \mathbf{crl}_{k+1} = \emptyset$. The k for which a petal chain becomes maximal is called the **length** of the chain.

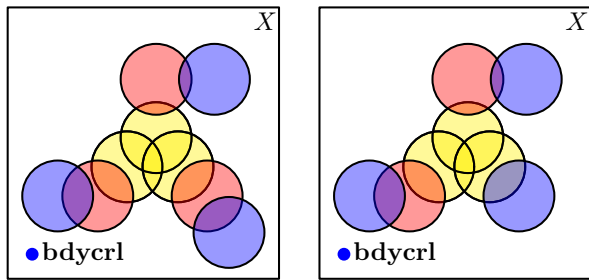
Remark 2. Each maximal petal chain ($\mathbf{maxptlchain}_k$) contains a boundary petal (\mathbf{bdpt}). This follows directly from Def. 16 and Def. 18.

Using this notion of a **length** of petal chain, we define the regularity of an object space $\mathcal{O}_p^{\check{c}ech}$.

Definition 19. Let $\mathcal{O}_p^{\check{c}ech}$ be an object space and A be the set of all the maximal petal chains in the object space. $\mathcal{O}_p^{\check{c}ech}$ is called **regular**, provided the length of all the maximal petal chains in A is the same. The object space is called **irregular** otherwise.

Example 5. Two types of object spaces are shown in Fig. 4.1 and 4.2. According to Def. 16, in Fig. 4.1 each of the blue balls (in \mathbf{crl}_3) is the boundary petal as it has a non-empty intersection with the previous corolla (in \mathbf{crl}_2) but has no intersection with the next corolla (\mathbf{crl}_4). This is due to the fact that the next corolla, \mathbf{crl}_4 , is \emptyset . In Fig. 4.2 each of the blue balls (in \mathbf{crl}_k) is the boundary petal as it has an intersection with the previous corolla (\mathbf{crl}_{k-1}), but has no intersection with the next corolla (\mathbf{crl}_{k+1}). In contrast to the Fig. 4.1 the Fig. 4.2 has boundary petals in different corolla (\mathbf{crl}_k). As per Def. 17 the collection of all the blue balls is the bounding corolla (\mathbf{bdycrl}). From Def. 18 we see that both the figures have three maximal petal chains ($\mathbf{maxptlchain}$). In Fig. 4.1 all the maximal chains are of equal length 3, thus from Def. 19, the object space in this figure is regular. In Fig. 4.2 the maximal chains ($\mathbf{maxptlchain}$) have different length, two chains having lengths of 3 and one chain of length 2. Thus, according to Def. 19 the object space in the figure is an irregular object. ■

Let us consider the order of a nerve as a signature for the $\mathcal{O}_p^{\check{c}ech}$. Based on this notion we define the Čech spectrum.



4.1: A regular object space 4.2: An irregular object space

Figure 4. This figure illustrates the bounding petal (**bdypt**) defined in Def. 16 and the bounding corolla (**bdycrl**) of an object space defined in Def. 17. Moreover, all the maximal petal chains (**maxptchain_k**) can be identified using the Def. 18 and the accompanying remark. The regularity of an object can be determined using the Def. 19

Definition 20. Let cx^sA be a Čech complex over a finite, bounded region of \mathbb{R}^2 , Then the Čech spectrum is defined as:

$$\mathcal{C}(cx^sA) = \{|N^k| : \forall k, N^k = \{\check{C}ech_r^s(cx^sA) : |\check{C}ech_r^s(cx^sA)| = k\}\}.$$

It is important to be noted that we count only the unique nerves, i.e. the nerves that are not completely included in a nerve of higher order.

In this paper we study the **proximity** relations defined over the topological structures defined above. **Proximity** is a measure of nearness between non-empty sets, and a non-empty set endowed with a proximity is known as a **proximity space**. The proximity space is represented as a pair (X, δ) , where X is a non-empty set and the δ is an arbitrary proximity relation on the set X . In this paper we will use the concept of spatial Lodato proximity (δ), strong proximity ($\overset{\wedge}{\delta}$) and descriptive proximity (δ_{ϕ}). Let us list the axioms of each of the proximities. We start with the spatial Lodato proximity (δ).

Definition 21. Let X be a non-empty set and $A, B, C \subset X$ be the subsets of X . Then the spatial Lodato proximity (δ) is a binary relation on the set X , that satisfies the following axioms:

- (P1) $\emptyset \not\delta A, \forall A \subset X$.
- (P2) $A \delta B \Leftrightarrow B \delta A$.
- (P3) $A \cap B \neq \emptyset \Rightarrow A \delta B$.
- (P4) $A \delta (B \cup C) \Leftrightarrow A \delta B \text{ or } A \delta C$.
- (P5) $A \delta B \text{ and } \forall b \in B \{b\} \delta C \Rightarrow A \delta C$.

Now we move on to the definition of strong proximity $(\overset{\wedge}{\delta})$.

Definition 22. Let X be a non-empty set and $A, B, C \subset X$ be the subsets of X . Then the strong proximity $(\overset{\wedge}{\delta})$ is a binary relation on the set X , that satisfies the following axioms:

(snN1) $\emptyset \overset{\wedge}{\delta} A, \forall A \subset X$, and $X \overset{\wedge}{\delta} A, \forall A \subset X$.

(snN2) $A \overset{\wedge}{\delta} B \Leftrightarrow B \overset{\wedge}{\delta} A$.

(snN3) $A \overset{\wedge}{\delta} B \Rightarrow A \cap B \neq \emptyset$.

(snN4) If $\{B_i\}_{i \in I}$ is an arbitrary family of subsets of X and $A \overset{\wedge}{\delta} B_{i^*}$ for some $i^* \in I$ such that $int(B_{i^*}) \neq \emptyset$, then $A \overset{\wedge}{\delta} (\bigcup_{i \in I} B_i)$.

(snN5) $intA \cap intB \neq \emptyset \Rightarrow A \overset{\wedge}{\delta} B$.

(snN6) $x \in int(B) \Rightarrow x \overset{\wedge}{\delta} A$.

(snN7) $\{x\} \overset{\wedge}{\delta} \{y\} \Leftrightarrow x = y$.

Moving on to the notion of a descriptive proximity (δ_ϕ) , we first define the notion of a **probe function**. For a set A , the function ϕ maps a feature vector to each of the elements of A . It is defined as $\phi(A) = \{\phi(x) \in \mathbb{R}^n : x \in A\}$. Based on this let us define the concept of a descriptive intersection $(\overset{\phi}{\cap})$. It is defined as $A \overset{\phi}{\cap} B = \{x \in A \cup B : \phi(x) \in \phi(A) \text{ and } \phi(x) \in \phi(B)\}$. These concepts were introduced and formalized in (Peters, 2007a),(Peters, 2007b). Let us now formalize the notion of a descriptive proximity (δ_ϕ) .

Definition 23. Let X be a non-empty set and $A, B, C \subset X$ be the subsets of the set X . Then the descriptive proximity (δ_ϕ) is a binary relation on the set X , that follows the following axioms:

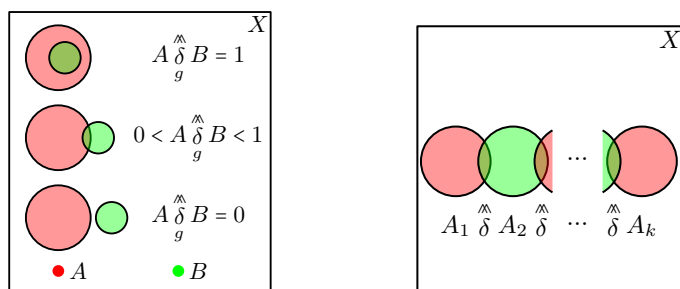
(dP1) $\emptyset \overset{\phi}{\delta} A, \forall A \subset X$.

(dP2) $A \delta_\phi B \Leftrightarrow B \delta_\phi A$.

(dP3) $A \overset{\phi}{\cap} B \neq \emptyset \Rightarrow A \delta_\phi B$.

(dP4) $A \delta_\phi B$ and $\{b\} \delta_\phi C, \forall b \in B \Rightarrow A \delta_\phi C$.

Let us build upon the notion a strong proximity $(\overset{\wedge}{\delta})$. It is a boolean valued relation i.e. it either tells whether two sets A, B are strongly near $(\overset{\wedge}{\delta})$ or strongly far $(\overset{\wedge}{\phi})$. Next, consider an extension of the Smirnov degree-of-closeness measure (Smirnov, 1952). We define a gradation of the strong proximity to make it a continuous valued function. This then gives a notion of the degree of nearness rather than just an indication of being near or not.



5.1: Graduation of a Strong Proximity 5.2: A sequence in a proximal space

Figure 5. This figure illustrates the gradation of proximity relation. It generalizes the usual binary proximity to a continuous valued relation. Moreover, the concept of a sequence in the proximity space is also illustrated.

Definition 24. Let X be a non-empty set and $A, B \in X$ be two subsets of X , then we can define a graded strong proximity as a function $A \overset{\delta}{\underset{g}{\delta}} B : X \times X \rightarrow [0, 1]$.

$$A \overset{\delta}{\underset{g}{\delta}} B = \frac{|A \cap B|}{\min(|A|, |B|)},$$

where $\min(a, b)$ returns the smaller of the two numbers.

Remark 3. It can be seen from the Def. 24, that if $A \overset{\delta}{\underset{g}{\delta}} B = 0$ then this means that $A \cap B = \emptyset$, or $A = \emptyset$ or $B = \emptyset$. Any of these conclusions leads to $A \not\overset{\delta}{\underset{g}{\delta}} B$. Moreover, if $A \overset{\delta}{\underset{g}{\delta}} B = 1$ then either $A \subseteq B$ or $B \subseteq A$. Thus the graded strong proximity ranges from \emptyset (for $\delta = 0$) to \subseteq (for $\delta = 1$).

Let us now talk about a sequence in a proximity space (X, δ) .

Example 6. The notion of proximity as defined in the Def. 22 is a boolean function. This tells us whether two sets are strongly near (δ) or not (\emptyset). In Def. 24, we define a notion of a continuous valued proximity relation. An illustration of the graded strong proximity is presented in the Fig. 5.1. It can be seen that for the case when $A \cap B = \emptyset$, the strong graded proximity is, $A \overset{\delta}{\underset{g}{\delta}} B = \frac{0}{\min(|A|, |B|)} = 0$. The other extreme of the proximity is when $B \subseteq A$, then $A \overset{\delta}{\underset{g}{\delta}} B = \frac{|B|}{|B|} = 1$. All the other cases are in between the two extremes, $0 < A \overset{\delta}{\underset{g}{\delta}} B < 1$, as $0 < |A \cap B| < |B|$. Thus, the graded strong proximity ($\overset{\delta}{\underset{g}{\delta}}$) gives us a more detailed view of the nearness or proximity between two sets. ■

Definition 25. Let X be a non-empty set and δ be an arbitrary proximity relation on the set X . Then (X, δ) is a proximity space. Let $A_i \in X$ for $i \in I$, where I is an index set. The a δ -sequence x_n in the proximity space (X, δ) is defined as:

$$x_n = \{A_i : A_i \delta A_j \text{ for } |j - i| \leq 1, i, j \in I\}$$

where δ represents the proximity relation.

Let us explain the idea of a sequence in a proximal space.

Example 7. Let (X, δ) be a proximity space equipped with an arbitrary proximity relation. We define the idea of a sequence which establishes an arrangement of subsets of X . For an illustration of this concept, let us consider Fig. 5.2. It shows a sequence of sets $A_i \subset X$, where $i \in I$, and I is an index set. An important point to note in this definition of a sequence is that a set A_i is proximal only to adjacent sets (A_{i-1}, A_{i+1}) in the sequence and far (not proximal) from all other sets in the sequence. The idea of a δ -sequence can be extended to other proximities such as the $\overset{\wedge}{\delta}$ and δ_Φ . ■

Till now, we have been discussing the notion of proximity as a binary mapping on a set X , of the form $\delta : X \times X \rightarrow \{0, 1\}$. Now, let us extend this notion to a mapping of n sets, where n is an arbitrary number. This mapping is termed as a proximity of order n , denoted as δ^n . It is non-continuous surjective map of the form $\delta^n : X^n \rightarrow \{0, 1\}$. This generalized notion of a proximity is referred to as **hyper-connectedness**. Let us first define the axioms of a **Lodato hyper-connectedness**. We have used the notation $A \delta B$ to denote that A, B are proximal, and $A \not\delta B$ to denote otherwise. Let us introduce a new notation that will come in handy in the case of hyper-connectedness. If $\delta(A, B, C) = 1$, then A, B are hyper-connected, and if $\delta(A, B, C) = 0$ then the opposite is true.

Definition 26. Let X be a non-empty set and $\{A_i \subset X : i \in I\}$, where I is an index set, and B, C be the non-empty subsets of X . Let $S(D)$ be the set of n -permutations of the elements of the set D , for $2 \leq n \leq |I|$. Then the spatial Lodato hyper-connectedness (δ^k) is a mapping on k subsets of the set X , that satisfies the following axioms for $k \geq 1$:

(hP1) $\forall A_k \subset X, \delta^k(A_1, \dots, A_k) = 0$, if any $A_1, \dots, A_k = \emptyset$.

(hP2) $\delta^k(A_1, \dots, A_k) = 1 \Leftrightarrow \delta^k(Y) = 1, \forall Y \in S(\{A_1, \dots, A_k\})$.

(hP3) $\bigcap_{i=1}^k A_i \neq \emptyset \Rightarrow \delta^k(A_1, \dots, A_k) = 1$.

(hP4) $\delta^k(A_1, \dots, A_{k-1}, B \cup C) = 1 \Leftrightarrow \delta^k(A_1, \dots, A_{k-1}, B) = 1$ or $\delta^k(A_1, \dots, A_{k-1}, C) = 1$.

(hP5) $\delta^k(A_1, \dots, A_{k-1}, B) = 1$ and $\forall b \in B, \delta^2(\{b\}, C) = 1 \Rightarrow \delta^k(A_1, \dots, A_{k-1}, C) = 1$.

(hP6) $\forall A \subset X, \delta^1(A) = 1$, a constant map.

Now, we use an example to better understand the spatial Lodato hyper-connectedness denoted by δ^k .

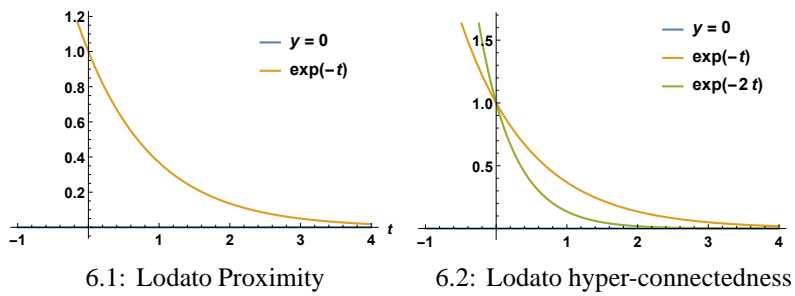
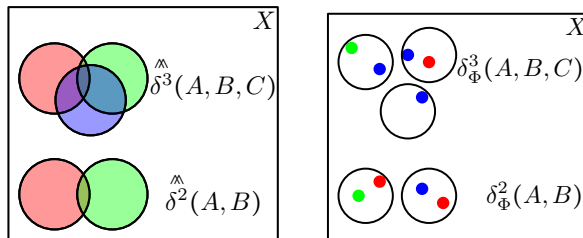


Figure 6. This figure illustrates the spatial Lodato proximity in the form of convergence of exponential to the x-axis. It then generalizes this proximity to a hyper proximity, by considering two exponentials converging to x-axis. Further generalization to n such functions follow naturally.



7.1: Strong hyper-connectedness 7.2: Descriptive hyper-connectedness

Figure 7. The Fig. 7.1 illustrates the extension of the classical binary relation of spatial Lodato proximity to three sets. Such a relation can be generalized to n sets. Fig. 7.2 displays the same for the strong proximity.

Example 8. Let us consider the idea of a spatial Lodato hyper-connectedness(Def. 26) as illustrated in Fig.6.1. This figure uses two sets $A = \{y(t) = 0 : \forall t \in \mathbb{R}\}$ and $B = \{y(t) = e^{-t} : \forall t \in \mathbb{R}\}$. It can be observed that these two sets are proximal in the sense that as t approaches ∞ , the function e^{-t} approaches 0. Thus $A \cap B \neq \emptyset$ in the limit sense. This spatial Lodato hyper-proximity, δ^k , for $k = 2$ is the classical notion of spatial Lodato proximity defined in Def. 21. Let us extend this idea to arbitrary values of k i.e. to the proximity of k non-empty subsets of a set X . For $k = 1$, the notion of a proximity is trivial. $\delta^1(C) = 1 \forall C \subset X$ as every set is proximal to itself. It does not make sense to talk about $k = 0$ as there is nothing to quantify the nearness of. Now moving on to $k = 3$ as shown in Fig. 6.2. The sets used in this illustration are $A = \{y(t) = 0 : t \in \mathbb{R}\}$, $B = \{y(t) = e^{-t} : t \in \mathbb{R}\}$ and $C = \{y(t) = e^{-2t} : t \in \mathbb{R}\}$. It is obvious that all the three sets are proximal in the limit sense as the functions e^{-t} and e^{-2t} approach 0 as t approached ∞ . Thus $\cap(A, B, C) \neq \emptyset$, which implies that $\delta^3(A, B, C) = 1$. Using a similar approach we can generalize this notion for $k > 3$.

We generalize the notion of strong proximity(δ) defined in Def. 21 to the notion of strong hyper-connectedness.

Definition 27. Let X be a non-empty set and $A_i, B, C \subset X$ be subsets of X , where $i \in I, I$ is an index

set. Let $S(D)$ be the set of n -permutations of set D , where $2 \leq n \leq |I|$. Then the strong hyper-connectedness (δ^k) is a mapping on k subsets of the set X , that satisfies the following axioms for $k \geq 1$:

(snhN1) $\forall A_k \subset X, \delta^k(A_1, \dots, A_k) = 0$ if any $A_1, \dots, A_k = \emptyset$ and $\delta^k(X, A_1, \dots, A_{k-1}) = 1, \forall A_i \subset X$.

(snhN2) $\delta^k(A_1, \dots, A_k) = 1 \Leftrightarrow \delta^k(Y) = 1, \forall Y \in S(\{A_1, \dots, A_k\})$.

(snhN3) $\delta^k(A_1, \dots, A_k) = 1 \Rightarrow \bigcap_{i=1}^k A_i \neq \emptyset$.

(snhN4) If $\{B_i\}_{i \in I}$ is an arbitrary family of subsets of X and $\delta^k(A_1, \dots, A_{k-1}, B_{i^*}) = 1$ for some $i^* \in I$ such that $\text{int}(B_{i^*}) \neq \emptyset$, then $\delta^k(A_1, \dots, A_{k-1}, (\bigcup_{i \in I} B_i)) = 1$.

(snhN5) $\bigcap_{i=1}^k \text{int}A_i \neq \emptyset \Rightarrow \delta^k(A_1, \dots, A_k) = 1$.

(snhN6) $x \in \bigcap_{i=1}^{k-1} \text{int}(A_i) \Rightarrow \delta^k(x, A_1, \dots, A_{k-1}) = 1$.

(snhN7) $\delta^k(\{x_1\}, \dots, \{x_k\}) = 1 \Leftrightarrow x_1 = x_2 = \dots = x_n$.

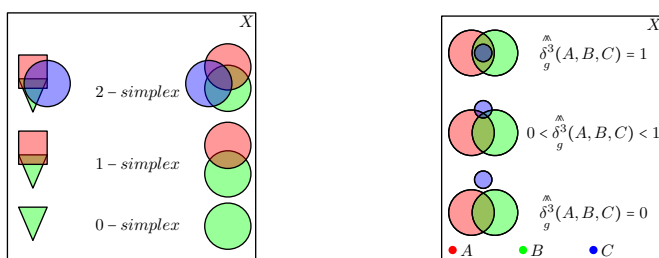
(snhN8) $\forall A \in X, \delta^1(A) = 1$ is a constant map.

To explain the notion of strong hyper-connectedness we present an example.

Example 9. Let us consider the illustration of strong hyper-connectedness, δ^k , in the Fig. 7.1. First we consider the case of $k = 2$. The two geometric balls (red and green) have a non-empty intersection and thus $\delta^2(\text{red ball}, \text{green ball}) = 1$. Let us now generalize this notion to arbitrary values of k . We consider the case of $k = 1$, which is a trivial case. It can be seen that each ball is strongly near it self. The case of $k = 0$ does not make any sense as the notion of proximity requires a set or an object for definition. Let us now consider the case of $k = 3$. We can see that three geometric balls (red, blue and green) have a non-empty intersection, $\bigcap(\text{red ball}, \text{green ball}, \text{blue ball}) \neq \emptyset$, and thus are strongly proximal. Thus we can write that $\delta^3(\text{red ball}, \text{green ball}, \text{blue ball}) = 1$. Using a similar argument we can extend this notion of strong hyper-connectedness (δ^k) to the case of $k > 3$. ■

Now, we extend the notion of a descriptive proximity to descriptive hyper-connectedness, denoted by δ_Φ^k .

Definition 28. Let X be a non-empty set and $A_i, B, C \subset X$ be the subsets of the set X , where $i \in I, I$ is an index set. Let $S(D)$ be the set of all the n -permutations of set D , where $2 \leq n \leq |I|$. Then the descriptive hyper-connectedness (δ_Φ^k) is a mapping on the set X , that satisfies the following axioms for $k \geq 1$:



8.1: Hyper-connected complexes 8.2: Graded strong hyper-connectedness

Figure 8. The Fig. 8.1 displays two different hyper-connected complexes defined in a proximity space X . Fig. 8.2 is a graphical visualization of the graded strong proximity for different values in the interval $[0, 1]$.

(dhP1) $\forall A_i \subset X, \delta_{\Phi}^k(A_1, \dots, A_k) = 0$ if any of the $A_1, \dots, A_k = \emptyset$.

(dhP2) $\delta_{\Phi}^k(A_1, \dots, A_k) = 1 \Leftrightarrow \delta_{\Phi}^k(Y) = 1 \forall Y \in S(\{A_1, \dots, A_k\})$.

(dhP3) $\bigcap_{\Phi} A_i \neq \emptyset \Rightarrow \delta_{\Phi}^k(A_1, \dots, A_k) = 1$.

(dhP4) $\delta_{\Phi}^k(A_1, \dots, A_{k-1}, B) = 1$ and $\forall b \in B, \delta_{\Phi}^2(\{b\}, C) = 1 \Rightarrow \delta_{\Phi}^k(A_1, \dots, A_{k-1}, C) = 1$.

(dhP5) $\forall A \subset X, \delta_{\Phi}^1(A) = 1$ a constant map.

To help explain the notion of descriptive hyper-connectedness δ_{Φ}^k .

Example 10. We use the illustration in Fig. 7.2 to aid in understanding the concept of descriptive hyper proximity (δ_{Φ}^k). To describe the notion of descriptive proximity we require a probe function $\phi : A \rightarrow \mathbb{R}$. In this example we define the probe function such that it returns the hue value (color) of each element of the set. Based on similarity in the hue domain we define the notion of the descriptive proximity (Def. 23). Let us first discuss the case of $k = 2$. Consider the two sets at the bottom of the figure. We can see that each has two elements. Now we see that both sets have an element with red color. Thus the sets have a non-empty descriptive intersection i.e. $A \cap_{\phi} B \neq \emptyset$.

This leads to $\delta_{\Phi}^2(A, B) = 1$. Let us now generalize this concept to arbitrary values of k . Now we consider the case of $k = 1$. It is obvious that every set is descriptively proximal to itself. Now we can generalize to the case of $k = 3$. We can see that the three sets on the top each have an element with blue color. Thus the sets have a non-empty intersection, $\bigcap_{\phi}(A, B, C) \neq \emptyset$, which implies that $\delta_{\Phi}^3(A, B, C) = 1$. Using a similar argument we can extend this notion to arbitrary values of k .

Using the notion of hyper-connectedness, let us define the notion of a hyper-connected complex.

Definition 29. Let S be a discrete set and X be a collection of non-empty subsets of S . X is a hyper-connected complex, $\delta - cx$, if every collection of non-empty subsets $\{A_1, \dots, A_n\} \in X$, of cardinality $|n|$, is hyper-connected with respect to a proximity δ i.e. $\delta^n(A_1, \dots, A_n) = 1$. Every subset of size k is the $k - 1$ simplex in the resulting hyper-connected complex, $\delta - cx$.

To clarify the notion of a hyper-connected complex we present the following example.

Example 11. Let us consider the illustration of a strong hyper-connected complex as depicted in the Fig. 8.1. It can be seen that the bottom row of the figure represents the 0-simplex as per Def. 27, we can see that every set is strongly connected to itself. Thus $\overset{\wedge}{\delta}^1(A) = 1 \forall A$.

Let us move on to the middle row of the illustration. It can be seen that the two sets are strongly connected thus $\overset{\wedge}{\delta}^2(A, B) = 1$ and the remaining constituent sets $\{A\}$ and $\{B\}$ are also strong hyper-connected ($\overset{\wedge}{\delta}^1(A) = \overset{\wedge}{\delta}^1(B) = 1$). Thus, the collection $\{A, B\}$ is a 1-simplex of the strong hyper-connected complex.

Let us now move to the top row of the illustration. It can be seen that due to the common intersection the $\overset{\wedge}{\delta}^3(A, B, C) = 1$ and all of the collections of its subsets are also strongly connected ($\overset{\wedge}{\delta}^2(A, B) = \overset{\wedge}{\delta}^2(A, C) = \overset{\wedge}{\delta}^2(B, C) = \overset{\wedge}{\delta}^1(A) = \overset{\wedge}{\delta}^1(B) = \overset{\wedge}{\delta}^1(C) = 1$). Thus the collection of sets $\{A, B, C\}$ forms a 2-simplex as per Def. 29.

It must be noted that this concept can be generalized to spatial Lodato hyper-connectedness and descriptive hyper-connectedness.

Let us now define the graded strong hyper-connectedness, denoted by $\overset{\wedge}{\delta}_g^k$.

Definition 30. Let X be a non-empty set and $\{A_1, A_2, \dots, A_n\}$ be a family of subsets of X . Then the graded strong hyper-connectedness is a continuous surjection of the form $\overset{\wedge}{\delta}_g^k: X^n \rightarrow [0, 1]$ and is defined as:

$$\overset{\wedge}{\delta}_g^k = \frac{|\cap(A_1, A_2, \dots, A_n)|}{\min(|A_1|, |A_2|, \dots, |A_n|)},$$

where $\min(a_1, a_2, \dots, a_n)$ is the smallest of all the arguments.

Example 12. To explain the notion of graded strong hyper-connectedness, denoted as $\overset{\wedge}{\delta}_g^k$, let us consider the Fig. 8.2. Since we are considering three sets we are going to use $k = 3$ in the following discussion. First consider the case illustrated towards the bottom of the figure. It illustrates the case when $\cap(A, B, C) \neq \emptyset$ or any of the A, B or C is an empty set. In this case the $\overset{\wedge}{\delta}_g^3 = 0$ as per Defn. 30 and the three sets A, B, C are not strong hyper-connected, $\overset{\wedge}{\delta}^3(A, B, C) = 0$.

Let us move on to the case when there is a partial intersection among the three sets as shown in the illustration in the middle of the figure. We can see that the number of elements in the intersection is smaller than the elements in smallest set i.e. C . Thus the value of the graded strong hyper-connectedness is between 0 and 1. The sets A, B and C are strong hyper-connected ($\overset{\wedge}{\delta}^3(A, B, C) = 1$).

Let us now consider the third case in which the smallest of the three sets, namely C is contained in the intersection. This is represented as $C \subset \cap(A, B, C)$. For this case it can be seen from the

Def. 30 that $\delta_g^3(A, B, C) = 1$. We can extend this framework to higher values of k using a similar argument. ■

Last, but not the least we present the Borsuk’s nerve theorem.

Theorem 2. (Borsuk, 1948) *If U is a collection of subsets in a topological space, the nerve complex is homotopy equivalent to the union of the subsets.*

This theorem will be helpful in the study of Čech object spaces ($\mathcal{O}_p^{\check{C}ech}$) undertaken in this paper.

3. Main Results

Lemma 3. *Let $\mathcal{O}_p^{\check{C}ech}$ be an object space, $\mathbf{crl}_i \in \mathcal{O}_p^{\check{C}ech}$ be a corolla, $\mathbf{ptlchain}_k \in \mathcal{O}_p^{\check{C}ech}$ be a petal chain and $\hat{k} \in \mathbb{Z}^+$ such that $\mathbf{crl}_{\hat{k}} = \emptyset$. Then, $\bigcup_i \mathbf{crl}_i = \bigcup \mathbf{maxptlchain}_k$, where $i = 0, 1, 2, \dots, \hat{k} - 1$.*

Proof. From Def. 13 it is obvious that $\mathbf{crl}_i = \bigcup \mathbf{ptl}_i$, thus i -corolla is the union of all the i -petals in $\mathcal{O}_p^{\check{C}ech}$. Thus for $i = 0, 1, 2, \dots, \hat{k} - 1$, the $\bigcup_i \mathbf{crl}_i$ is equivalent to $\bigcup_i \bigcup \mathbf{ptl}_i$. Thus, $\bigcup_i \mathbf{crl}_i, i = 1, 2, \dots, \hat{k} - 1$, is equal to the union of all the possible $\mathbf{ptl}_i \in \mathcal{O}_p^{\check{C}ech}$ (i -petals in the object space).

Let us now look at Def. 14, from which it can be concluded that each k -petal chain in the object space contains one instance of $\mathbf{ptl}_i \in \mathcal{O}_p^{\check{C}ech}$ (i -petal) for each value $i = 0, 1, \dots, k$. Thus if we consider the $\bigcup \mathbf{ptlchain}_k$, for $k = 0, 1, \dots, \hat{k} - 1$, we get the union of all the $\mathbf{ptl}_i \in \mathcal{O}_p^{\check{C}ech}$ (i -petal), where $i = 0, 1, \dots, k$. Which means that union of all $\mathbf{ptlchain}_k \in \mathcal{O}_p^{\check{C}ech}$ consists of all the $\mathbf{ptl}_i \in \mathcal{O}_p^{\check{C}ech}$ for all values of $i = 0, 1, \dots, k$. Thus, from Def. 13 we get $\bigcup \mathbf{ptlchain}_k = \bigcup_i \mathbf{crl}_i$ for $i = 0, 1, \dots, k$. From Def. 18 it can be seen that any $\mathbf{ptlchain}_k$ with the largest value of k is called the **maxptlchain**. From Def. 14 it can be seen that every member of the $\mathbf{ptlchain}_k$ has to be a member of the \mathbf{crl}_i in the object space, for all values of $i = 0, 1, \dots, k$. Thus, the maximum possible value of k for any petal chain can be $\hat{k} - 1$, as $\mathbf{crl}_{\hat{k}} = \emptyset$. It is possible for different maximal petal chains in $\mathcal{O}_p^{\check{C}ech}$ to have different values of k depending on the regularity, as per Def. 19. Keeping this in mind, it is obvious that the $\bigcup \mathbf{maxptlchain}$ equals the union of all the $\mathbf{ptl}_i \in \mathcal{O}_p^{\check{C}ech}$ for $i = 0, 1, \dots, \hat{k} - 1$.

Thus, we can conclude that both $\bigcup_i \mathbf{crl}_i$ for $i = 0, 1, \dots, \hat{k} - 1$, and $\bigcup \mathbf{maxptlchain}$ are equal to the union of all the $\mathbf{ptl}_i \in \mathcal{O}_p^{\check{C}ech}$. Hence we can conclude that, $\bigcup_i \mathbf{crl}_i = \bigcup \mathbf{maxptlchain}_k$. □

Using this we can formulate an other definition of the object space ($\mathcal{O}_p^{breve\check{C}ech}$), equivalent to the Def. 15.

Theorem 3. *Let cx^sA be a strong Čech complex and an arbitrary positive integer $\hat{k} \in \mathbb{Z}^+$ such that $\mathbf{crl}_k \in cx^sA = \emptyset$. The object space, denoted by $\mathcal{O}_p^{\check{C}ech}$ (Def. 15), can be defined as:*

$$\begin{aligned} \mathcal{O}_p^{\check{C}ech} &= \{ \bigcup_k \mathbf{crl}_k : \mathbf{crl}_0 = p, k = 0, \dots, \hat{k} - 1 \} \\ &= \bigcup \mathbf{maxptlchain}. \end{aligned}$$

Proof. From Lemma 3 it can be seen that for $i = 0, 1, \dots, \hat{k} - 1$, $\bigcup_i \mathbf{crl}_i = \bigcup \mathbf{maxptlchain}_k$. Using this conclusion and Def. 15 it is evident that $\mathcal{O}_p^{\check{C}ech} = \bigcup \mathbf{maxptlchain}$. \square

Using this new definition of an object space ($\mathcal{O}_p^{\check{C}ech}$), we can comment on the homotopy type of $\mathcal{O}_p^{\check{C}ech}$.

Theorem 4. *The object space $\mathcal{O}_p^{\check{C}ech}$ has the same homotopy type as the union of all the maximal petal chains ($\mathbf{maxptlchain}$).*

Proof. From Def. 3 the object space is the union of all the maximal petal chains i.e. $\bigcup \mathbf{maxptlchain}$. From the Def. 14 we can conclude that every $\mathbf{ptlchain}_k \in \mathcal{O}_p^{\check{C}ech}$ contains the nucleus p which is $\mathcal{N}(\check{C}ech_r^s(A))$. Thus, we can conclude that $\mathcal{O}_p^{\check{C}ech}$ is a nerve as it is union of sets $\bigcup \mathbf{maxptlchain}$, and the sets have a common intersection $\bigcap \mathbf{maxptlchain} = p$. From Thm. 2, it follows that the homotopy type of the $\mathcal{O}_p^{\check{C}ech}$ is the same as the union of all its $\mathbf{maxptlchain}$ (maximal petal chains). \square

Let us formulate some results for hyper-connectedness relations.

Lemma 4. *Let $C = \{A_1, A_2, \dots, A_n\}$ be a collection of non-empty subsets of a non-empty set X . If $\delta^n(A_1, A_2, \dots, A_n) = 1$, then all the possible non-empty sub-collections are also spatial Lodato hyper-connected. This is represented as $\delta^{|T|}(T) = 1$, where T is a sub-collection of the set C .*

Proof. From axiom (**hp2**) of Def. 26, it can be seen that given $\delta^n(A_1, A_2, \dots, A_n) = 1$ (i.e. is Lodato hyper-connected), then all the k -permutations of the elements in the set $\{A_1, A_2, \dots, A_n\}$ are also Lodato hyper-connected, for $2 \leq k \leq n$. One can see that this axiom is satisfied, provided we introduce an integer $\hat{n} \in \mathbb{Z}^+$, such that $2 \leq k \leq \hat{n} \leq n$. Thus, the axiom is satisfied for all such \hat{n} . We have shown that if $\delta^n(A_1, A_2, \dots, A_n) = 1$, then there exists $\hat{n} \in \mathbb{Z}^+$ and $2 \leq \hat{n} \leq n$, such that $\delta^{\hat{n}}(\check{Y} \in \check{S}(C)) = 1$. $\check{S}(C)$ is the set of all the permutations of elements of set C , taken \hat{n} at a time. Now, we need to prove that this equation is also satisfied for subsets of C of size 1. This follows directly from the axiom (**hp6**) of Def. 26, stating that for all $A \in C$, $\delta^1(A) = 1$. Hence, if $\delta^n(A_1, A_2, \dots, A_n) = 1$, then all non-empty sub-collections (\check{C}) of $C = \{A_1, A_2, \dots, A_n\}$ also satisfy $\delta^{|\check{C}|}(\check{C}) = 1$. \square

Using the lemma we have just formulated, consider next the following result for spatial Lodato hyper-connected complexes..

Theorem 5. *Let $C = \{A_1, A_2, \dots, A_n\}$ be a collection of subsets of a non-empty set X , and $\delta^n(C) = 1$. Then C is a spatial Lodato hyper-connected complex ($\delta - cx$).*

Proof. From Lemma 4 it follows, that if a collection of sets $C \subseteq X$ satisfies $\delta^n(C) = 1$, then $\forall T, \delta^{|T|}(T) = 1$, where T is a non-empty subcollection of the set C . Thus all possible sub-collections of the set C are also Lodato hyper-connected. From this conclusion and the Def. 29, it directly follows that C is a spatial Lodato hyper-connected complex. Every subset $S \subseteq X$ of size k forms the $k - 1$ simplex in the resulting spatial Lodato hyper-connected complex ($\delta - cx$). \square

Lemma 5. Let $C = \{A_1, A_2, \dots, A_n\}$ be a collection of non-empty subsets of a non-empty set X . If $\overset{\wedge}{\delta}^n(A_1, A_2, \dots, A_n) = 1$, then all the possible sub-collections are also strong hyper-connected. This is represented as $\overset{\wedge}{\delta}^{|T|}(T) = 1$, where T is a sub-collection of the set C .

Proof. From axiom (snhN2) of Def. 27, it can be seen that given $\overset{\wedge}{\delta}^n(A_1, A_2, \dots, A_n) = 1$ (i.e. is strongly hyper-connected), then all the k -permutations of the elements in the set $\{A_1, A_2, \dots, A_n\}$ are also strongly hyper-connected, for $2 \leq k \leq n$. One can see that this axiom is satisfied, provided we introduce an integer $\hat{n} \in \mathbb{Z}^+$, such that $2 \leq k \leq \hat{n} \leq n$. Thus, the axiom is satisfied for all such \hat{n} . We have shown that if $\overset{\wedge}{\delta}^n(A_1, A_2, \dots, A_n) = 1$, then there exists $\hat{n} \in \mathbb{Z}^+$ and $2 \leq \hat{n} \leq n$, such that $\overset{\wedge}{\delta}^{\hat{n}}(\hat{Y} \in \hat{S}(C)) = 1$. $\hat{S}(C)$ is the set of all the permutations of elements of set C , taken \hat{n} at a time. Now, we need to prove that this equation is also satisfied for subsets of C of size 1. This follows directly from the axiom (snhN8) of Def. 22, such that for all $A \in C$, $\overset{\wedge}{\delta}^1(A) = 1$. Hence, if $\overset{\wedge}{\delta}^n(A_1, A_2, \dots, A_n) = 1$, then all non-empty sub-collections (\tilde{C}) of $C = \{A_1, A_2, \dots, A_n\}$ also satisfy $\overset{\wedge}{\delta}^{|\tilde{C}|}(\tilde{C}) = 1$. □

Example 13. To explain the Lemma 5, let us consider the Fig.7.1. There are two cases of strong hyper-connectedness shown here, namely the $\overset{\wedge}{\delta}^2(A, B) = 1$ and $\overset{\wedge}{\delta}^3(A, B, C) = 1$.

Let us first look at the case of $\overset{\wedge}{\delta}^2(A, B) = 1$. Here the set $C = \{A, B\}$ and all the possible sub-collections are $\{A\}$ and $\{B\}$. From axiom (snhN8) of Def. 27 it can be seen that $\overset{\wedge}{\delta}^1(A) = 1$ and $\overset{\wedge}{\delta}^1(B) = 1$. Thus, Lemma 5 is satisfied.

Now moving on to the case of $\overset{\wedge}{\delta}^3(A, B, C) = 1$. Here the set $C = \{A, B, C\}$ and all possible sub-collections can be listed as $\{\{A\}, \{B\}, \{C\}, \{A, B\}, \{B, C\}, \{A, C\}, \{A, B, C\}\}$. We can see that from axiom (snhN8) of Def. 27, that $\overset{\wedge}{\delta}^1(A) = \overset{\wedge}{\delta}^1(B) = \overset{\wedge}{\delta}^1(C) = 1$. Moreover, from the figure it can be established that as all the three disks have interior points in common. This leads to the fact that any two of the disks, also have interior points in common. Hence, from axiom (snhN5) of Def. 27 $\overset{\wedge}{\delta}^2(A, B) = \overset{\wedge}{\delta}^2(B, C) = \overset{\wedge}{\delta}^2(A, C) = 1$. Hence, Lemma 5 is satisfied.

The same argument can be extended to higher values of k . ■

From this lemma we obtain the following result.

Theorem 6. Let $C = \{A_1, A_2, \dots, A_n\}$ be a collection of subsets of a non-empty set X , and $\overset{\wedge}{\delta}^n(C) = 1$, then C is a strong hyper-connected complex ($\overset{\wedge}{\delta} - cx$).

Proof. From Lemma 5 it follows, that if the for a collection of sets $C \subseteq X$ satisfying $\overset{\wedge}{\delta}^n(C) = 1$, then $\forall T, \overset{\wedge}{\delta}^{|T|}(T) = 1$, where T is a non-empty subcollection of the set C . Thus all possible sub-collections of the set C are also strong hyper-connected. From this conclusion and the Def. 29, it directly follows that C is a strong hyper-connected complex. Every subset $S \subseteq X$ of size k forms the $k - 1$ simplex in the resulting strong hyper-connected complex ($\overset{\wedge}{\delta} - cx$). □

Lemma 6. Let $C = \{A_1, A_2, \dots, A_n\}$ be a collection of non-empty subsets of a non-empty set X . If $\delta_{\Phi}^n(A_1, A_2, \dots, A_n) = 1$, then all the possible sub-collections are also descriptively hyper-connected. This is represented as $\forall T, \delta_{\Phi}^{|T|}(T) = 1$, where T is a sub-collection of the set C .

Proof. From axiom **(dhP2)** of Def. 28, it can be seen that given $\delta_{\Phi}^n(A_1, A_2, \dots, A_n) = 1$ (i.e. is Lodato hyper-connected), then all the k -permutations of the elements in the set $\{A_1, A_2, \dots, A_n\}$ are also Lodato hyper-connected, for $2 \leq k \leq n$. One can see that this axiom is satisfied if we introduce an integer $\acute{n} \in \mathbb{Z}^+$, such that $2 \leq k \leq \acute{n} \leq n$. Thus, the axiom is satisfied for all such \acute{n} . We have shown that if $\delta_{\Phi}^n(A_1, A_2, \dots, A_n) = 1$, then there exists $\acute{n} \in \mathbb{Z}^+$ and $2 \leq \acute{n} \leq n$, such that $\delta_{\Phi}^{\acute{n}}(\acute{Y} \in \acute{S}(C)) = 1$. $\acute{S}(C)$ is the set of all the permutations of elements of set C , taken \acute{n} at a time. Now, we need to prove that this equation is also satisfied for subsets of C of size 1. This follows directly from the axiom **(dhP6)** of Def. 28, that for all $A \in C, \delta_{\Phi}^1(A) = 1$. Hence, if $\delta_{\Phi}^n(A_1, A_2, \dots, A_n) = 1$, then all non-empty sub-collections(\tilde{C}) of $C = \{A_1, A_2, \dots, A_n\}$ also satisfy $\delta_{\Phi}^{|\tilde{C}|}(\tilde{C}) = 1$. \square

Example 14. To explain the Lemma 6, let us consider the Fig.7.2. There are two cases of descriptive hyper-connectedness shown here, namely the $\delta_{\Phi}^2(A, B) = 1$ and $\delta_{\Phi}^3(A, B, C) = 1$.

Let us first look at the case of $\delta_{\Phi}^2(A, B) = 1$. Here the set $C = \{A, B\}$ and all the possible sub-collections are $\{A\}$ and $\{B\}$. From axiom **(dhP5)** of Def. 28, it can be seen that $\delta_{\Phi}^1(A) = 1$ and $\delta_{\Phi}^1(B) = 1$. Thus, Lemma 5 is satisfied.

Now moving on to the case of $\delta_{\Phi}^3(A, B, C) = 1$. Here the set $C = \{A, B, C\}$ and all possible sub-collections can be listed as $\{\{A\}, \{B\}, \{C\}, \{A, B\}, \{B, C\}, \{A, C\}, \{A, B, C\}\}$. We can see from axiom **(dhP5)** of Def. 28, that $\delta_{\Phi}^1(A) = \delta_{\Phi}^1(B) = \delta_{\Phi}^1(C) = 1$. Moreover, from the figure it can be established that as all the three sets have constituent elements of the same color(blue). This leads to the fact that any two of the sets, will contain constituent elements of same color(blue). Thus, from axiom **(dhP3)** of Def. 28, $\delta_{\Phi}^2(A, B) = \delta_{\Phi}^2(B, C) = \delta_{\Phi}^2(A, C) = 1$. Hence, Lemma 6 is satisfied.

The same argument can be extended to higher values of k . \blacksquare

We now give a result for descriptive hyper-connected complexes..

Theorem 7. Let $C = \{A_1, A_2, \dots, A_n\}$ be a collection of subsets of a non-empty set X , and $\delta_{\Phi}^n(C) = 1$. Then C is a descriptive hyper-connected complex($\delta_{\Phi} - cx$).

Proof. From Lemma 6, it follows, that if the for a collection of sets $C \subseteq X$ satisfying $\delta_{\Phi}^n(C) = 1$, then $\forall T, \delta_{\Phi}^{|T|}(T) = 1$, where T is a non-empty subcollection of the set C . Thus all possible sub-collections of the set C are also descriptively hyper-connected. From this conclusion and the Def. 29, it directly follows that C is a descriptively hyper-connected complex. Every subset $S \subseteq X$ of size k forms the $k-1$ simplex in the resulting descriptvly hyper-connected complex($\delta_{\Phi} - cx$). \square

Theorem 8. Let A_1, A_2, \dots, A_n be a collection of subsets of a non-empty set X then:

$$1^{\circ} : \overset{\wedge}{\delta}^n(A_1, A_2, \dots, A_n) \Rightarrow \delta^n(A_1, A_2, \dots, A_n)$$

$$2^{\circ} : \overset{\wedge}{\delta}^n(A_1, A_2, \dots, A_n) \Rightarrow \delta_{\Phi}^n(A_1, A_2, \dots, A_n)$$

Proof. 1° : It is obvious from axiom **(snhN3)** of Def. 27, that if $\overset{\wedge}{\delta}^n(A_1, A_2, \dots, A_n) = 1$, then $\bigcap_{i=1}^n A_i \neq$

\emptyset . Using this conclusion and axiom **(hP3)** of Def. 26, from $\bigcap_{i=1}^n A_i \neq \emptyset$, we can conclude that $\delta^n(A_1, A_2, \dots, A_n)$.

2° : From axiom **(snhN3)** of Def. 27, that if $\overset{\wedge}{\delta}^n(A_1, A_2, \dots, A_n) = 1$, then $\bigcap_{i=1}^n A_i \neq \emptyset$. Considering $p \in \bigcap_{i=1}^n A_i$, we know that $p \in A_i$ for $i = 1, \dots, n$. If we define a probe function $\phi : X \rightarrow \mathbb{R}$, then it is obvious that $\phi(p) \in \phi(A_i)$ for $i = 1, \dots, n$. Thus $\bigcap_{\Phi} A_i \neq \emptyset$. From axiom **(dhP3)** of Def. 28 it can be concluded that $\delta_{\Phi}^n(A_1, A_2, \dots, A_n) = 1$. □

Example 15. Let us consider the Fig. 7.1, to explain the Thm. 8. It illustrates two cases, namely $\overset{\wedge}{\delta}^2(A, B) = 1$ and $\overset{\wedge}{\delta}^3(A, B, C) = 1$.

It can be seen that if $\overset{\wedge}{\delta}^2(A, B) = 1$, both the sets A, B have interior points in common. Thus, from axiom **(hP3)** of Def. 26, that $\delta^2(A, B) = 1$. The same is true for $\overset{\wedge}{\delta}^3(A, B, C) = 1$. All the three sets have interior points in common which from axiom **(hP3)** of Def. 26, leads to $\delta^3(A, B, C)$.

It can be seen that if $\overset{\wedge}{\delta}^2(A, B) = 1$, both the sets A, B have interior points in common. If we consider a probe function ϕ , which maps the elements of the sets to a description in \mathbb{R} . It can be seen that for a point $p \in A \cap B$, $\phi(p) \in \phi(A) \cap \phi(B)$. Which means that $A \cap_{\Phi} B \neq \emptyset$. Thus, from axiom

(dhP3) of Def. 28, it can be concluded that $\delta_{\Phi}^2(A, B) = 1$. The same is true for $\overset{\wedge}{\delta}^3(A, B, C) = 1$. All the three sets have interior points in common. Thus for a probe function ϕ , if there is a point $p \in \bigcap(A, B, C)$, then $\phi(p) \in \bigcap(A, B, C)$. Thus $\bigcap_{\Phi}(A, B, C) \neq \emptyset$. Thus, from axiom **(dhP3)** of Def. 28, it can be concluded that $\delta_{\Phi}^3(A, B, C) = 1$.

This argument can be generalized for higher values of k .

Theorem 9. Let $C = \{A_1, A_2, \dots, A_n\}$ be a collection of subsets of a non-empty set X , and C is a strong hyper-connected complex i.e. $\overset{\wedge}{\delta}$ -cx. Then,

1° : C is a $\overset{\wedge}{\delta}$ -cx $\Rightarrow C$ is a δ -cx

2° : C is a $\overset{\wedge}{\delta}_{\Phi}$ -cx $\Rightarrow C$ is a δ_{Φ} -cx

Proof. 1° : From Thm. 6 it can be seen that if C is a strong hyper-complex ($sn - cx$), then $\overset{\wedge}{\delta}^n(C) =$

1. From Thm. 8 it can be concluded that as $\overset{\wedge}{\delta}^n(C) = 1$, then $\delta^n(C) = 1$. From this conclusion and the Thm. 5 it can be concluded that as $\delta^n(C) = 1$, C is a Lodato hyper-complex denoted $\delta - cx$.

2° : From Thm. 6 it can be seen that if C is a strong hyper-complex ($sn - cx$), then $\overset{\wedge}{\delta}^n(C) = 1$.

From Thm. 8 it can be concluded that as $\overset{\wedge}{\delta}^n(C) = 1$, then $\delta_{\Phi}^n(C) = 1$. From this conclusion and the Thm. 7 it can be concluded that as $\delta_{\Phi}^n(C) = 1$, C is a descriptive hyper-complex denoted $\delta_{\Phi} - cx$. □

Theorem 10. A Čech complex is:

1° : spatial Lodato hyper-connected complex, $\delta - cx$

2° : descriptive hyper-connected complex, $\delta_{\Phi} - cx$

Proof. 1^o : From Def. 3, we come to know that Čech complex is a collection of Čech nerves represented as $\check{C}ech_r(K), K \in \mathbb{R}^2$. From Def. 2, it can be concluded that $\check{C}ech_r(K)$ is a collection of closed geometric balls, $B_r(x \in K)$, with a non-empty intersection. The $\check{C}ech_r K$ of order k forms the $k - 1$ simplex in the Čech complex, as per Def. 3. From the axiom (dP3) of Def. 26, it can be concluded that every $k - 1$ simplex (or the $\check{C}ech_r(K)$ of order k) is a spatial Lodato hyper-connected collection of sets, i.e. $\delta^{|\check{C}ech_r(K)|}(\check{C}ech_r(K)) = 1$. Using this conclusion and Thm. 5, we can conclude that each $\check{C}ech_r(K)$ is a $\delta - cx$, i.e. a spatial Lodato hyper-connected complex. Similar to the simplicial complex, the union of spatial Lodato hyper-connected complexes is also a spatial Lodato hyper-connected complex.

2^o : From Def. 3, we come to know that Čech complex is a collection of Čech nerves ($\check{C}ech_r(K), K \in \mathbb{R}^2$). From Def. 2, it can be concluded that $\check{C}ech_r(K)$ is a collection of closed geometric balls, $B_r(x \in K)$, with a non-empty intersection. The $\check{C}ech_r K$ of order k forms the $k - 1$ simplex in the Čech complex, as per Def. 3. Consider a probe function $\phi : K \rightarrow \mathbb{R}$. From the definition of intersection, it can be concluded that a point $p \in \bigcap B_r(x \in K)$ exists in each of the individual $B_r(x \in K)$. Now, when we map each ball to a feature space using the probe function ϕ , it is evident that $\phi(p) \in \bigcap \phi(B_r(x \in K))$. From this it can be concluded that if $\bigcap B_r(x \in K) \neq \emptyset$, then $\bigcap_{\phi} \phi(B_r(x \in K)) \neq \emptyset$. Using this result and from the axiom (dhP3) of Def. 28, it can be concluded that every $k - 1$ simplex (or the $\check{C}ech_r(K)$ of order k) is a descriptively hyper-connected collection of sets, i.e. $\delta_{\phi}^{|\check{C}ech_r(K)|}(\check{C}ech_r(K)) = 1$. Using this conclusion and Thm. 5, we can conclude that each $\check{C}ech_r(K)$ is a $\delta_{\phi} - cx$, i.e. a descriptively hyper-connected complex. Similar to the simplicial complex, the union of descriptively hyper-connected complexes is also a descriptively hyper-connected complex. □

Remark 4. *The Čech complex is not guaranteed to be a strong hyper-connected complex, $\overset{\wedge}{\delta} - cx$. This is evident from the definition Def. 3, as it requires the intersection of closed geometric balls, $B_r(x)$, to be non-empty. Since the balls are closed it is possible for the intersection to be points lying on the boundary of the balls. This leads to the spatial Lodato proximity (δ), spatial Lodato hyper-connectedness (δ^n), descriptive proximity (δ_{ϕ}) and descriptive hyper-connectedness (δ_{ϕ}^n). However, this does not allow the notion of a strong proximity ($\overset{\wedge}{\delta}$) and as a consequence the notion of strong hyper-connectedness ($\overset{\wedge}{\delta}^n$). For these two notions the intersection must comprise of the interior points only.*

Based on the above observation we restrict the notion of a Čech complex to the notion of a strong Čech complex as defined in Def. 5. Now, we present the following results for the strong Čech complex.

Lemma 7. *A strong Čech complex is a strong hyper-connected complex, $\overset{\wedge}{\delta} - cx$.*

Proof. It is obvious from Def. 5, that a strong Čech complex is a collection of strong Čech nerves, which are represented as $\check{C}ech_r^s(K)$. From the Def. 4, it is evident that the strong Čech nerve is a collection of closed geometric balls, $B_r(x \in K)$, such that their interiors have a non-empty intersection. This can be written as $\check{C}ech_r^s(K) = \bigcap \text{int}(B_r(x \in K)) \neq \emptyset$. From the axiom (snhN5),

it can be concluded that, $\delta^{|\check{Cech}_r^s(K)|}(\check{Cech}_r^s(K)) = 1$. Using this conclusion and from Thm. 6, it can be concluded that $\check{Cech}_r^s(K)$ is a strong hyper-connected complex, $\delta - cx$. We have proved that, each strong Čech nerve, $\check{Cech}_r^s(K)$ is a $sn - cx$. The union of simplicial complexes is also a simplicial complex. From the Def. 29 that the hyper-connected complex is a case of a simplicial complex (Def. 1), where the simplices are strong hyper-connected collections of sets. Thus, it follows directly that the union of $\delta - cx$ is also a $\delta - cx$. Thus, the collections of strong Čech nerves, or a Čech complex, is a strong hyper-connected complex, $\delta - cx$. \square

Using this lemma, let us formulate the following hyper-connectedness relations for the strong Čech complex.

Theorem 11. A strong Čech complex, $cx^s K$ is:

1^o : spatial Lodato hyper-connected complex, $\delta - cx$

2^o : descriptive hyper-connected complex, $\delta_\Phi - cx$

Proof. 1^o : From Lemma 7, it can be seen that a strong Čech complex, $cx^s K$ is a strong hyper-connected complex, $\delta - cx$. From Thm. 9 it can be concluded that $cx^s K$ is a spatial Lodato hyper-connected complex, $\delta - cx$.

2^o : From Lemma 7, it can be seen that a strong Čech complex, $cx^s K$ is a strong hyper-connected complex, $\delta - cx$. From Thm. 9 it can be concluded that $cx^s K$ is a descriptive hyper-connected complex, $\delta_\Phi - cx$. \square

Now, we formulate some important results regarding equipping an object space with proximity relations.

Lemma 8. Let $(\mathcal{O}_p^{\check{Cech}}, \{\delta, \delta_\Phi\})$ be a proximal relator space, and $\mathbf{crl}_k \in \mathcal{O}_p^{\check{Cech}}$, where $k \in \mathbb{Z}^+$.

Assuming $\mathbf{crl}_a \overset{\delta}{\delta} \mathbf{crl}_b$, then

1^o : $\mathbf{crl}_a \overset{\delta}{\delta} \mathbf{crl}_b \Rightarrow \mathbf{crl}_a \delta \mathbf{crl}_b$

2^o : $\mathbf{crl}_a \overset{\delta}{\delta} \mathbf{crl}_b \Rightarrow \mathbf{crl}_a \delta_\Phi \mathbf{crl}_b$

Proof. 1^o : From axiom (snN3) of Def. 22, it can be concluded that as $\mathbf{crl}_a \overset{\delta}{\delta} \mathbf{crl}_b$, $\mathbf{crl}_a \cap \mathbf{crl}_b \neq \emptyset$. Using the axiom (P3) of Def. 21, we can conclude that as $\mathbf{crl}_a \cap \mathbf{crl}_b \neq \emptyset$, then $\mathbf{crl}_a \delta \mathbf{crl}_b$.

2^o : From axiom (snN3) of Def. 22, it can be concluded that as $\mathbf{crl}_a \overset{\delta}{\delta} \mathbf{crl}_b$, $\mathbf{crl}_a \cap \mathbf{crl}_b \neq \emptyset$. Let us consider a point $p \in \mathbf{crl}_a \cap \mathbf{crl}_b$ and a probe function $\phi : \mathcal{O}_p^{\check{Cech}} \rightarrow \mathbb{R}$. Then it can be seen that $\phi(p) \in \phi(\mathbf{crl}_a) \cap \phi(\mathbf{crl}_b)$, which leads to the fact that $\mathbf{crl}_a \underset{\Phi}{\cap} \mathbf{crl}_b \neq \emptyset$. Using the axiom (dP3) of Def. 23, we can conclude that as $\mathbf{crl}_a \underset{\Phi}{\cap} \mathbf{crl}_b \neq \emptyset$, then $\mathbf{crl}_a \delta_\Phi \mathbf{crl}_b$. \square

Theorem 12. Let $\mathcal{O}_p^{\check{Cech}}$ be an object space and $\mathbf{crl}_k \in \mathcal{O}_p^{\check{Cech}}$, where $k \in \mathbb{Z}^+$. Let \hat{k} be the value of k such that $\mathbf{crl}_{\hat{k}} = \emptyset$. Then, for $0 \leq j < \hat{k} - 1$,

1^o : $\mathbf{crl}_{j+1} \overset{\delta}{\delta} \mathbf{crl}_j$

$$2^\circ : \mathbf{crl}_j \overset{\wedge}{\delta} \mathbf{crl}_{j-1}$$

Proof. 1° : From Def. 13 it can be concluded that $\mathbf{crl}_{j+1} = \bigcup \mathbf{ptl}_{j+1}$ and $\mathbf{crl}_j = \bigcup \mathbf{ptl}_j$. From the Def. 12 it can be seen that a \mathbf{ptl}_{j+1} is a closed geometric ball $B_r(x) \in (cxK \setminus \bigcup \mathbf{ptl}_j)$, such that $B_r(x) \cap \bigcup \mathbf{ptl}_j \neq \emptyset$. Thus each \mathbf{ptl}_{j+1} has a non-empty intersection with $\bigcup \mathbf{ptl}_j = \mathbf{crl}_j$ (Def. 13). Since $\mathbf{crl}_{j+1} = \bigcup \mathbf{ptl}_{j+1}$ (Def. 13) and each $\mathbf{ptl}_{j+1} \cap \mathbf{crl}_j \neq \emptyset$, we can conclude that $\mathbf{crl}_{j+1} \cap \mathbf{crl}_j \neq \emptyset$. Here, our choice to use the strong Čech complex (cx^s) rather than the Čech complex (cx) as the basis of topological approximation of the underlying space, comes in handy. From Def. 15 it can be seen that we consider the object space $(\mathcal{O}_p^{\check{C}ech})$ to be defined over a strong Čech complex (cx^s) . Moreover, we can see from the Def. 12 that the petals are the closed geometric balls in the cx^s , and thus the resulting corolla are a collection of these balls. Moreover, each of the intersections in the cx^s (strong Čech complex) is of the form $\bigcap \text{int}(B_r(x))$, as per Def. 5. Based on this we can conclude that all the intersections in the above argument are also on the interiors of the closed geometric balls. Thus, we can conclude that as the object space is formulated on the cx^s (strong Čech complex), $\text{int}(\mathbf{crl}_{j+1}) \cap \text{int}(\mathbf{crl}_j) \neq \emptyset$. Thus from axiom (snN5) of Def. 22, we can conclude that $\mathbf{crl}_{j+1} \overset{\wedge}{\delta} \mathbf{crl}_j$.

2° : The argument for the previous case of $\mathbf{crl}_{j+1} \overset{\wedge}{\delta} \mathbf{crl}_j$ extends directly to this case of $\mathbf{crl}_j \overset{\wedge}{\delta} \mathbf{crl}_{j-1}$, just by considering a dummy variable $\hat{j} = j + 1$. Thus, the case of the $\mathbf{crl}_{j+1} \overset{\wedge}{\delta} \mathbf{crl}_j$ for $0 \leq j < \hat{k} - 1$ becomes $\mathbf{crl}_{\hat{j}} \overset{\wedge}{\delta} \mathbf{crl}_{\hat{j}-1}$ for $1 \leq \hat{j} < \hat{k}$. This change in the inequality ensures that we only consider the k -corollas for $0 \leq k < \hat{k}$, thus covering a complete range of corollas. □

Based on this theorem we formulate the following important result for the adjacent corollas in an object space.

Theorem 13. Let $(\mathcal{O}_p^{\check{C}ech}, \{\delta, \overset{\wedge}{\delta}, \delta_\Phi\})$ be a proximal relator space, where $\mathbf{crl}_k \in \mathcal{O}_p^{\check{C}ech}$ and $k \in \mathbb{Z}^+$. Let \hat{k} be the value of k such that $\mathbf{crl}_{\hat{k}} = \emptyset$. Then, for $0 < j < \hat{k}$,

$$1^\circ : \mathbf{crl}_{j+1} \overset{\wedge}{\delta} \mathbf{crl}_j \Rightarrow \mathbf{crl}_{j+1} \delta \mathbf{crl}_j$$

$$2^\circ : \mathbf{crl}_j \overset{\wedge}{\delta} \mathbf{crl}_{j-1} \Rightarrow \mathbf{crl}_j \delta \mathbf{crl}_{j-1}$$

$$3^\circ : \mathbf{crl}_{j+1} \overset{\wedge}{\delta} \mathbf{crl}_j \Rightarrow \mathbf{crl}_{j+1} \delta_\Phi \mathbf{crl}_j$$

$$4^\circ : \mathbf{crl}_j \overset{\wedge}{\delta} \mathbf{crl}_{j-1} \Rightarrow \mathbf{crl}_j \delta_\Phi \mathbf{crl}_{j-1}$$

Proof. 1° : We can conclude from Thm. 12, that $\mathbf{crl}_{j+1} \overset{\wedge}{\delta} \mathbf{crl}_j$, and then from Lemma 8 we can conclude that $\mathbf{crl}_{j+1} \delta \mathbf{crl}_j$.

2° : We can conclude from Thm. 12, that $\mathbf{crl}_j \overset{\wedge}{\delta} \mathbf{crl}_{j-1}$, and then from Lemma 8 we can conclude that $\mathbf{crl}_j \delta \mathbf{crl}_{j-1}$.

3° : We can conclude from Thm. 12, that $\mathbf{crl}_{j+1} \overset{\wedge}{\delta} \mathbf{crl}_j$, and then from Lemma 8 we can conclude that $\mathbf{crl}_{j+1} \delta_\Phi \mathbf{crl}_j$.

4° : We can conclude from Thm. 12, that $\mathbf{crl}_j \overset{\wedge}{\delta} \mathbf{crl}_{j-1}$, and then from Lemma 8 we can conclude that $\mathbf{crl}_j \delta_\Phi \mathbf{crl}_{j-1}$. □

Let us move on to discuss some important results regarding sequences in a proximity space (X, δ) .

Lemma 9. Let $(X, \{\delta, \overset{\wedge}{\delta}, \delta_\Phi\})$ be a proximal relator space, and A_1, A_2, \dots, A_n be subsets of X . Then, if the collection of sets $C = \{A_1, A_2, \dots, A_n\}$ is $\overset{\wedge}{\delta}$ -sequence:

$1^\circ : C$ is a $\overset{\wedge}{\delta}$ -sequence $\Rightarrow C$ is a δ -sequence

$2^\circ : C$ is a $\overset{\wedge}{\delta}$ -sequence $\Rightarrow C$ is a δ_Φ -sequence

Proof. 1° : If C is a $\overset{\wedge}{\delta}$ -sequence, then from Def. 25 it can be concluded that for all $i = 1, \dots, n$, $A_i \overset{\wedge}{\delta} A_j$ for $|j - 1| \leq 1$. Using this conclusion and axiom (snN3) of Def. 22, we can establish that $A_i \cap A_j \neq \emptyset$ for $|j - 1| \leq 1$. Using axiom (P3) of Def. 21, it can be concluded that $A_i \delta A_j$ for $|j - 1| \leq 1$. Thus from Def. 25, we can conclude that C is a δ -sequence.

2° : If C is a $\overset{\wedge}{\delta}$ -sequence, then from Def. 25 it can be concluded that for all $i = 1, \dots, n$, $A_i \overset{\wedge}{\delta} A_j$ for $|j - 1| \leq 1$. Using this conclusion and axiom (snN3) of Def. 22, we can establish that $A_i \cap A_j \neq \emptyset$ for $|j - 1| \leq 1$. Now let us consider a point $p \in A_i \cap A_j$ for $|j - i| \leq 1$ and a probe function $\phi : C \rightarrow \mathbb{R}$. It can be seen that $\phi(p) \in \phi(A_i) \cap \phi(A_j)$ for $|j - i| \leq 1$, thus $A_i \underset{\Phi}{\cap} A_j \neq \emptyset$. Using axiom (dP3) of Def. 23, it can be concluded that $A_i \delta_\Phi A_j$ for $|j - 1| \leq 1$. Thus from Def. 25, we can conclude that C is a δ_Φ -sequence. □

Let us now consider the petal chain, $\mathbf{ptlchain}_k$, as a sequence in a proximity space.

Theorem 14. Let $(\mathcal{O}_p^{\check{C}ech}, \overset{\wedge}{\delta})$ be a proximal object space and $\mathbf{ptlchain}_k \in \mathcal{O}_p^{\check{C}ech}$, for $k \in \mathbb{Z}^+$, be a petal chain contained in it. Then $\mathbf{ptlchain}_k$ is a $\overset{\wedge}{\delta}$ -sequence.

Proof. From Def. 14 it can be seen that a $\mathbf{ptlchain}_k$ is a collection of subsets A_i of cx^s (a strong Čech complex). These subsets satisfy the condition that each $A_i \in \mathbf{crl}_i$ and that $A_i \cap A_{i-1} \neq \emptyset$. Since we consider the strong Čech complex (Def. 5), thus all the intersections of the interiors of the subsets of cx^s are non-empty. Thus we can conclude that $\text{int}(A_i) \cap \text{int}(A_{i-1}) \neq \emptyset$. From axiom (snN5) of Def. 22, it can be concluded that $A_i \overset{\wedge}{\delta} A_{i-1}$. Moreover, substituting $\hat{i} = i - 1$, we get $A_{\hat{i}+1} \overset{\wedge}{\delta} A_{\hat{i}}$, thus $A_j \overset{\wedge}{\delta} A_i$ for $|j - i| \leq 1$. Thus, from the Def. 25 it can be seen that as $\mathbf{ptlchain}_k = \{\bigcup_{\mathcal{I}} A_i : A_i \in \mathbf{crl}_i, A_j \overset{\wedge}{\delta} A_i \text{ for } |j - i| \leq 1, i, j \in \mathcal{I}\}$. Hence, $\mathbf{ptlchain}_k$ is a $\overset{\wedge}{\delta}$ -sequence with an additional condition that each $A_i \in \mathbf{crl}_i$. □

Using these results we formulate proximity relations for petal chains ($\mathbf{ptlchain}_k$) in an object space $(\mathcal{O}_p^{\check{C}ech})$.

Theorem 15. Let $(\mathcal{O}_p^{\check{C}ech}, \{\delta, \overset{\wedge}{\delta}, \delta_\Phi\})$ be a proximal relator space, $\mathbf{ptlchain}_k \in \mathcal{O}_p^{\check{C}ech}$ be a petal chain, for $k \in \mathbb{Z}^+$. Then.

$1^\circ : \mathbf{ptlchain}_k$ is a δ -sequence

$2^\circ : \mathbf{ptlchain}_k$ is a δ_Φ -sequence

Proof. 1° : From Thm. 14 we can see that $\mathbf{ptlchain}_k$ is a $\overset{\wedge}{\delta}$ -sequence and using Lemma 9 we can conclude that $\mathbf{ptlchain}_k$ is a δ -sequence.

2° : From Thm. 14 we can see that $\mathbf{ptlchain}_k$ is a $\overset{\wedge}{\delta}$ -sequence and using Lemma 9 we can conclude that $\mathbf{ptlchain}_k$ is a δ_Φ -sequence. □

Let, us now move on to defining proximity relations on the whole object space, $\mathcal{O}_p^{\check{C}ech}$.

Lemma 10. *Let $(\mathcal{O}_p^{\check{C}ech}, \overset{\wedge}{\delta})$ be a proximal object space and $C = \{\mathbf{crl}_k : \mathbf{crl}_k \in \mathcal{O}_p^{\check{C}ech} \text{ and } k = 0, 1, 2, \dots, j\}$ is the set of k -corollas of the object space. Suppose $\hat{k} \in \mathbb{Z}^+$ such that $\mathbf{crl}_{\hat{k}} = \emptyset$. Then for $0 < j < \hat{k}$, the collection of subsets C is a $\overset{\wedge}{\delta}$ -sequence.*

Proof. From Thm. 13 it can be seen that $\mathbf{crl}_{j+1} \overset{\wedge}{\delta} \mathbf{crl}_j$ and $\mathbf{crl}_j \overset{\wedge}{\delta} \mathbf{crl}_{j-1}$ for $0 < j < \hat{k}$. Thus it can be concluded that $\mathbf{crl}_j \overset{\wedge}{\delta} \mathbf{crl}_i$ for $|j - i| \leq 1$ where $0 \leq j, i \leq 1$. From Def. 25 it can be concluded that the set $C = \{\mathbf{crl}_k : \mathbf{crl}_k \in \mathcal{O}_p^{\check{C}ech} \text{ and } k = 0, 1, \dots, \hat{k}\}$ is a $\overset{\wedge}{\delta}$ -sequence. □

Let us now extend this result to the object space, $\mathcal{O}_p^{\check{C}ech}$.

Theorem 16. *Let $(\mathcal{O}_p^{\check{C}ech}, \overset{\wedge}{\delta})$ be a proximal object space, then the object space i.e. $\mathcal{O}_p^{\check{C}ech}$ is a $\overset{\wedge}{\delta}$ -sequence.*

Proof. From Def. 15 it can be seen that $\mathcal{O}_p^{\check{C}ech} = \{\bigcup_k \mathbf{crl}_k : \mathbf{crl}_0 = p, k = 0, 1, \dots, \hat{k}\}$, which from the definition of union is equivalent to $\{\mathbf{crl}_i : \mathbf{crl}_i \in \mathcal{O}_p^{\check{C}ech}, k = 0, 1, \dots, \hat{k}\}$. Thus, from Lemma 10 it is evident that $\mathcal{O}_p^{\check{C}ech}$ is a $\overset{\wedge}{\delta}$ -sequence. □

Using this theorem, we detail the following proximity relations on the object space, $\mathcal{O}_p^{\check{C}ech}$.

Theorem 17. *Let $(\mathcal{O}_p^{\check{C}ech}, \overset{\wedge}{\delta})$ be a proximal object space, and the $\mathcal{O}_p^{\check{C}ech}$ is a $\overset{\wedge}{\delta}$ -sequence. Then,*

1° : $\mathcal{O}_p^{\check{C}ech}$ is a δ -sequence

2° : $\mathcal{O}_p^{\check{C}ech}$ is a δ_Φ -sequence

Proof. 1° : From Thm. 16 it can be concluded that $\mathcal{O}_p^{\check{C}ech}$ is a $\overset{\wedge}{\delta}$ -sequence and from Lemma 9 it is evident that $\mathcal{O}_p^{\check{C}ech}$ is a δ -sequence.

2° : $\mathcal{O}_p^{\check{C}ech}$ is a $\overset{\wedge}{\delta}$ -sequence and from Lemma 9 it is evident that $\mathcal{O}_p^{\check{C}ech}$ is a δ_Φ -sequence. □

4. Computational Experimentation

In this section, we will consider the applications of the topological structures defined in this paper. The aim of this study is to define a topological framework for approximating and extracting the shapes of objects in a digital image. As discussed earlier we require the choice of a selected keypoints from the image. First, we use the scale invariant feature transform(SIFT) based keypoints defined by Lowe (Lowe, 1999).



9.1: Original Image



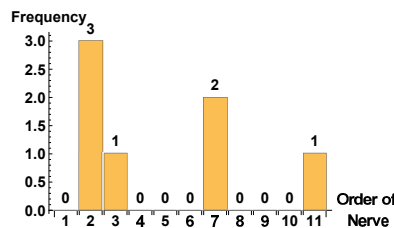
9.2: Original image pre-processed



9.3: SIFT keypoints



9.4: Čech Nerves of different orders



9.5: Čech Spectrum



9.6: 1-Corolla(\mathbf{crl}_1) of the maximal nerve

Figure 9. This figure displays the image of a boat(Fig. 9.1), and the same image after preprocessing to aid in extracting topological structures(Fig. 9.2). Fig. 9.3 shows the locations of SIFT keypoints. In Fig. 9.4, the Čech nerve are superimposed atop the image. Fig. 9.5 displays the Čech spectrum of the object space, denoted by $\mathcal{L}(\mathcal{O}_p^{\check{C}ech})$. Fig. 9.6 displays the 1-corolla(\mathbf{crl}_1) of the object space or the maximal Čech nerve.

4.1. SIFT keypoints based Čech complex

Let us first consider the image shown in Fig. 9.1. The main object of focus in this image is a boat. There are many other objects e.g. mountain, beach, sea and people in the image, that can be considered as noise for current application. To aid in approximating the shape of the main object i.e. the boat, we pre-process the image to remove all other objects. The output of this pre-processing is displayed in Fig. 9.2. It can be observed that almost everything apart from the distinctive features of the boat have been removed. Some of the parts of the boat have also been removed because of the similarity with the objects that were considered to be noise.

After the pre-processing stage we extract the SIFT keypoints from the image, represented as a set S . Based on these points we can now construct a Čech complex(Def. 3) by considering a collection of balls of radius r , $\{B_r(s) : s \in S\}$ super-imposed on to the image. Every set of k balls with a non-empty intersection is the $k - 1$ simplex in the resulting abstract simplicial complex.

Algorithm 1: Čech Complex representation of Image Objects**Input** : digital image img , Keypoints S , Čech radius r **Output:** Čech complex on the image $\check{C}ech_r(S)$, Čech nerve of maximal order $\max\check{C}ech_r(S)$, Čech spectrum \mathcal{C}

```

1 foreach  $s \in S$  do
2    $\check{C}ech_r(S) := \check{C}ech_r S \cup B_r(s)$ ;
3   /*Calculating the Čech spectrum*/;
4    $\mathcal{S} := S$ ;
5    $Nrv(0) := S$ ;
6   Continue  $\leftarrow$  True;  $k \leftarrow 1$ ;
7   while Continue = True do
8      $\mathcal{S} := (k + 1)$ -Combination of  $S$ ;
9     foreach  $comb \in \mathcal{S}$  do
10       $bnddsk \leftarrow$  minimal bounding disk of the  $(k + 1)$  points in  $comb$ ;
11       $rad \leftarrow$  radius of  $bnddsk$ ;
12      if  $rad \leq r$  then
13        |  $nerve \leftarrow comb$ ;
14      else
15        | /*Continue*/;
16      if  $nerve \neq \emptyset$  then
17        |  $\hat{S} \leftarrow$  all the unique points  $s \in S$  present in  $nerve$ ;
18        |  $Nrv(k - 1) := Nrv(k - 1) \setminus \{(k) - \text{Combination of } \hat{S}\}$ ;
19        |  $\acute{S} := S \setminus \hat{S}$ ;
20        |  $k \leftarrow k + 1$ ;
21      else
22        | Continue  $\leftarrow$  False;
23        |  $maximalorder \leftarrow k$ ;
24    $j \leftarrow 1$ ;
25   while  $j \leq maximalorder$  do
26     |  $Nrv(j) \mapsto$  number of elements;
27     |  $(j) \leftarrow$  number of elements
28    $\max\check{C}ech_r(S) := Nrv(maximalorder)$ ;
29    $\check{C}ech_r(S) \mapsto img$ ;

```

Moreover, it can be seen that every set of k geometric balls with a non-empty intersection is called a Čech nerve, denoted by $\check{C}ech_r(S)$. The common intersection is termed as the nucleus and the number of balls k in the nerve is termed its order. An overview of the algorithm to generate the Čech complex is presented in Alg. 1. In Fig. 9.4 all the $\{\check{C}ech_{r,s} : s \in S\}$ in the Čech complex are illustrated on the image. The nerves have been color coded with respect to their order.

Here we can see that for the current choice of radius r , the collection of balls form two complexes as there is no intersection in between them. One of the complexes is on the front side of the boat while the other complex is on the mast and the rear end of the boat. If we increase the radius both these complexes will merge in to one complex. This is an important point to note and is common to this and other frameworks which aim to model the data using topological constructs (Ghrist, 2008). The topological features of the approximation built using Čech and related complexes is dependent on the radius. The appropriate choice of radius will be discussed in a future work.

Here we present a possible topological signature of the shape. It is termed as the Čech spectrum and is defined in Def. 20. It is defined as a sequence of numbers which represents the number of nerves of a particular order in an image. A similar shape signature was considered for the approximation of object space via curved and rectilinear triangulations (Ahmad & Peters, 2017b). It can be seen that this is related to the spatial distribution of the keypoint locations and the radius of the geometric balls. Nerves of higher order are a result of a large number of keypoints proximal to each other. Thus, based on how the keypoints are selected the nerves of different order represent different concentrations(or clusters) of specific features in an image. Based on this we can assume that the region in the image, where the highest number of keypoints are mutually proximal is the most important region.

This brings us to the concept of a maximal Čech nerve. For the image under consideration we can see from the Čech spectrum that the order of the maximal nerve is 11 and there is only one of them in the image. Let us look at the location of this nerve on the image. It is shown in Fig. 9.6. We can see that this nerve lies on the safety tube hanging on the mast towards the rear end of the boat. The reason for this is the the tube is a compact structure that is highly differentiated from its background. Thus there is a high distribution of SIFT keypoints. Moreover, there are keypoints corresponding to the rear hull of the boat. The combination of these points results in the existence of the maximal nerve in this region of the image.

Each of the balls in this image is the maximal nerve is the 1-petal, denoted as \mathbf{ptl}_1 . The union of all the \mathbf{ptl}_1 in the maximal nerve is called the \mathbf{crl}_1 . Thus the maximal Čech nerve is equivalent to the \mathbf{crl}_1 . All the geometric balls in the image that has a non-empty intersection with the 1-corolla(\mathbf{crl}_1) is called the 2-petal, denoted as \mathbf{ptl}_2 . The union of all the \mathbf{ptl}_2 in the image is called the 2-corolla(\mathbf{crl}_2). The concept can be generalized to higher values of k in a similar fashion. Thus, the image shown in Fig. 9.6 also represents the 1-corolla or the \mathbf{crl}_1 .

Let us consider the image of a car as shown in the Fig. 10.1. This image contains a black car, with many other objects such as a human, building, partial parts of a car and bus. Thus, the image contains the focal object and a lot of other objects which for the purpose of this study we consider to be noise. After the pre-processing to remove all other objects apart from the focal object we obtain the image shown in Fig. 10.2. We use this image to select keypoints(set S) which will then be used to construct a Čech complex and superimpose it on the image. The nerves in the Čech complex($\{\check{C}ech_r(s) : s \in S\}$) are color coded with respect to number of sets in them(order). This result is shown in Fig. 10.4. In contrast to the result for the image of the boat as shown in Fig. 9.4, we can notice that the collection of all the geometric balls of radius r form a single Čech complex.

Let us now move on to the newly proposed shape signature, namely the Čech spectrum. For the approximation of the shape of the car with Čech complex, the Čech spectrum is displayed in

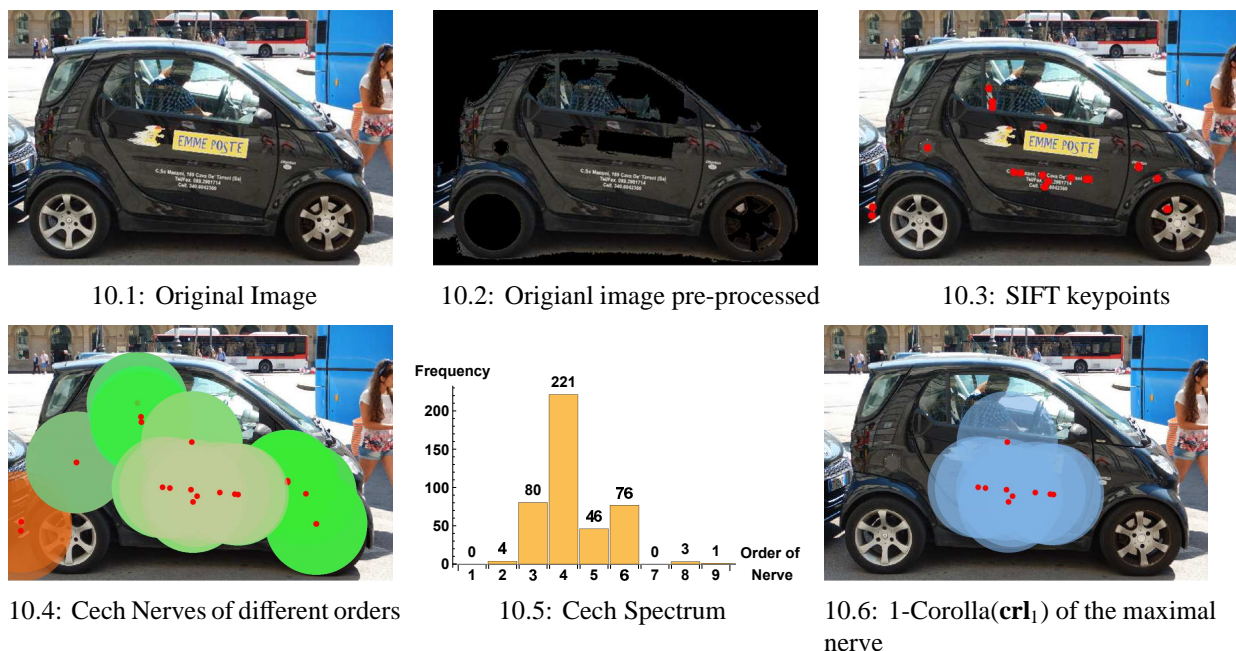


Figure 10. This figure displays the image of a car (Fig. 10.1), and the same image after preprocessing to aid in extracting topological structures(Fig. 10.2). The keypoints extracted by the SIFT algorithm are shown in Fig. 10.3. In Fig. 10.4, the Čech nerve are superimposed atop the image. Fig. 10.5 displays the Čech spectrum of the object space, denoted by $\mathcal{C}(\mathcal{O}_p^{\check{C}})$. Fig. 10.6 displays the 1-corolla(\mathbf{cr}_1) of the object space or the maximal Čech nerve.

Fig. 10.5. It can also be observed that the Čech spectrum for the car is different from that of the image of the boat(Fig. 9.5). The order of the maximal Čech nerve was 11 for the boat and is 9 for the car. The number of nerves of maximal order is again one. There can be multiple maximal Čech nerves in an image. In that case we can either consider them to be multiple objects or the different(in terms of features) regions in the same image. This choice is dependent on whether we consider our image to contain a single focal object or multiple ones.

We plot the maximal Čech nerve on the image and display it in Fig. 10.6. It can be seen that the maximal nerve lies on the interior of the car on its front door. Moreover, as discussed earlier it the maximal Čech nerve is the same as the 1-corolla(\mathbf{cr}_1) and each of the balls in it is the 1-petal(\mathbf{pt}_1). All the geometric balls that share a non-empty intersection with the \mathbf{cr}_1 are called the 2-petals(\mathbf{pt}_2). The union of all the 2-petals(\mathbf{pt}_2) is called the 2-corolla(\mathbf{cr}_2). We can extend the concept of k -petals and k -corolla to this image as we did to the image of the boat.

4.2. Čech complex using hole based keypoints

After discussing the results of approximating the objects in an image using Čech complexes based on SIFT based keypoints, we develop a new type of key points. These types of keypoints are based on the notion of a hole. The notion of a hole is a vital one in topology (Alexandroff, 1965).

Let us consider an extension of this concept to a digital image. We use the notion of a **descriptive hole**, defined as below.

Definition 31. A descriptive hole is a finite bounded sub-region of a plane with a matching description. The description is obtained by the probe function, $\phi : 2^X \rightarrow \mathbb{R}$.

Algorithm 2: Hole based Keypoints

Input : digital image img , Horizontal filter radius r_x , Vertical filter radius r_y , Hole threshold t , Number of holes n_{hole}

Output: Hole locations \mathcal{K}_{holes}

```

1  $f_G(x, y) := \frac{1}{2\pi r_x r_y} \exp(-(\frac{x^2}{2r_x^2} + \frac{y^2}{2r_y^2}));$ 
2  $img_{filt} \leftarrow f_G(x, y) * img;$ 
3  $g := empty\ matrix;$ 
4 foreach  $pixel \in img$  do
5    $J := empty\ matrix;$ 
6   foreach  $channel \in pixel$  do
7      $J(channel, :) := Grad(pixel);$ 
8    $(i, j) \leftarrow location\ of\ pixel;$ 
9    $g(i, j) \leftarrow \sqrt{\lambda_{max}(J^T J)};$ 
10  $g := set\ all\ values\ of\ g < t\ to\ 1\ and\ rest\ to\ 0;$ 
11  $g \mapsto connected\ components;$ 
12  $connected\ components \mapsto size\ in\ terms\ of\ pixels;$ 
13  $/*\ arrange\ in\ descending\ order\ w.r.t.\ size\ in\ terms\ of\ pixels\ */;$ 
14  $connected\ components \mapsto arranged\ connected\ components;$ 
15  $hole \leftarrow first\ n_{hole}\ arranged\ connected\ components;$ 
16  $hole \mapsto centroids;$ 
17  $\mathcal{K}_{holes} \leftarrow centroids;$ 

```

In this paper we consider the description to be the pixel intensity, which for coloured images is a vector of values in domain \mathbb{R}^n . Where n is the number of channels in the image. For a classical coloured image, the RGB color image, this is 3. A digital image is represented as a matrix (size $m \times n$) for computation, and for the RGB image this becomes a collection of three matrices or a multi-way array of size $m \times n \times 3$. For the purpose of detecting holes, we first convolve the image with a normalized Gaussian kernel to remove noise. The 2D normalized Gaussian is defined as:

$$f_G(x, y) = \frac{1}{2\pi r_x r_y} e^{-\left(\frac{x^2}{2r_x^2} + \frac{y^2}{2r_y^2}\right)},$$

where r_x and r_y define the standard deviation of the Gaussian in the x and y direction respectively. The standard deviation dictates the radius of the smoothing filter. This smoothed image

is then used to calculate the derivative for each pixel. The derivative is calculated using the a traditional derivative filter such as the sobel operator. In an image, for each pixel there are two derivative, one in the x and one in the y direction. We combine the derivatives in a Jacobian matrix(J).

To illustrate this concept, let us consider the RGB image to be a map(img) that assigns an intensity(\mathbb{R}^3) to each pixel location(\mathbb{R}^2). This is represented as $img : \mathbb{R}^2 \rightarrow \mathbb{R}^3$. The Jacobian matrix(J) for each pixel would thus be a matrix of size 2×3 , defined as:

$$J(img(x,y)) = \begin{bmatrix} \frac{\partial img(1)}{\partial x} & \frac{\partial img(1)}{\partial y} \\ \frac{\partial img(2)}{\partial x} & \frac{\partial img(2)}{\partial y} \\ \frac{\partial img(3)}{\partial x} & \frac{\partial img(3)}{\partial y} \end{bmatrix},$$

where $img(i)$ represents the i th channel of the image. The gradient magnitude for each pixel $g(x,y)$ is calculated as:

$$g(x,y) = \sqrt{\lambda_{max}(J(img(x,y))^T J(img(x,y)))},$$

where $\lambda_{max}(A)$ is the largest eigen value of matrix A .

Once, we have the gradient magnitude for each pixel in the image we can then threshold to yield the location of regions with relatively constant pixel intensities. The value of the threshold decides the amount of variation that we are willing to allow in the description of a hole. Since, we are interested in the areas with a gradient close to 0, we set all the pixels with gradient values less than the threshold to be 1 and the rest to be 0. In this fashion we mark all the regions of interest(or holes as we refer to them) as 1. We can extract these regions using connected component analysis, and then calculate the size of the region in terms of the number of pixels and its centroid. The holes are arranged in the descending order with respect to their size and the hole based keypoints(\mathcal{K}_{hole}) are the centroids of these holes. The size of the hole is a determinant of its importance in the image. This method is summarized in Alg. 2.

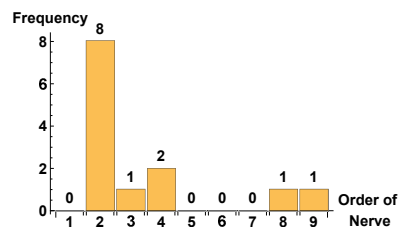
Once, we have the hole-based keypoints, \mathcal{K}_{hole} , they are input to the Alg. 1 to build the Čech complex so that features of the objects in the image can be extracted. This process is similar to the one performed for the SIFT keypoints in § 4.1. Let us discuss the application of this new class of keypoints to the images used in § 4.1. Let us first illustrate the location of the new form of keypoints on the image of the boat, as shown in Fig. 11.1. The difference between the objects extracted from the digital images using the two different types of keypoints lies in the location of the keypoints. Let us compare the location of the SIFT and hole-based keypoints(\mathcal{K}_{hole}), shown in Fig. 9.3 and Fig. 11.1 respectively. We can see that due to the specific construction of \mathcal{K}_{hole} , these exist on a region with a constant intensity. It can be seen that in the SIFT based keypoints are located along edges of the main body of the boat, while one of the \mathcal{K}_{hole} lies near the center of the body of the boat. The rest of the keypoints are mostly concentrated near the bottom edge of the boat on the shadow, and the life tube. These objects are very narrow, so the SIFT and the \mathcal{K}_{hole} are at almost similar locations. It is easy to see that for a narrow objects the centroids are close to the



11.1: Hole-based keypoints



11.2: Čech nerve of different order



11.3: Čech Spectrum



11.4: Maximal Čech nerve

Figure 11. Fig. 11.1 displays the hole-based keypoints on the original image of the boat. Čech nerves of different order are shown in Fig. 11.2. Fig.11.3 displays the Čech spectrum for the hole-based keypoints. The maximal Čech nerve or the 1-corolla($\mathbf{cr1}_1$) is displayed in Fig. 11.4.

edges.

Let us now look at the Čech nerves of different order imposed on the image. It is shown in Fig. 11.2 and the nerves are color coded with respect to order (the number of geometric balls in the nerve). The radius used to generate this result is the same as that for the SIFT based keypoints shown in Fig. 9.4. It can be seen that the Čech nerve generated using \mathcal{H}_{holes} covers more area of the boat, than the Čech nerve generated using the SIFT keypoints.

The Čech spectrum or the number of unique Čech nerves of a specific order in the Čech complex ($\check{C}ech_r(K)$) is defined as Def. 20. For the Čech complex generated using the hole-based keypoints, denoted as $\check{C}ech_r(\mathcal{H}_{holes})$, the Čech spectrum (\mathcal{C}) is displayed in Fig. 11.3. Comparing this with the result for the SIFT keypoints (Fig. 9.5), it can be seen that the Čech spectrum is quite different. It can be seen that the order of the maximal Čech nerve for \mathcal{H}_{hole} is 9, while for the SIFT based case is 11.

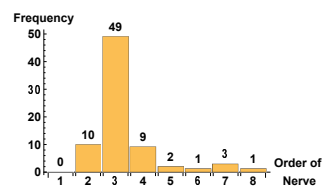
One commonality between the Čech complexes for both the SIFT and \mathcal{H}_{holes} is the location of the maximal Čech nerve or the 1-corolla ($\mathbf{cr1}_1$). Other topological structures can be extracted for the \mathcal{H}_{holes} based $\check{C}ech_r(\mathcal{H}_{holes})$ in the same fashion as for the SIFT based case explained with



12.1: Hole-based keypoints



12.2: Čech nerve of different order



12.3: Čech Spectrum



12.4: Maximal Čech nerve

Figure 12. Fig. 12.1 displays the hole-based keypoints on the original image of the car. Čech nerves of different order are shown in Fig. 12.2. Fig. 12.3 displays the Čech spectrum for the hole-based keypoints. The maximal Čech nerve or the 1-corolla(\mathbf{crl}_1) is displayed in Fig. 12.4.

detail in § 4.1.

Let us move on to the case of extracting the shape of the car using \mathcal{K}_{holes} . We consider the location of the SIFT keypoints(Fig.10.3) and compare them with the location of the \mathcal{K}_{holes} (Fig. 12.1). For this image it can be seen that the SIFT based keypoints for the image of the car are located in the center of the object on the front door. While, the \mathcal{K}_{holes} for this image are located towards the upper contour of the car. The SIFT points are located on the front door due to the text. There are lot of edges on the front door thus the SIFT based keypoints lie on these edges. Moreover, the location of \mathcal{K}_{holes} are on the rims, the mirros and the bumper of the car on the back. The locations of the keypoints for the SIFT are on the on the text, bottom fender of the car on the back and on the arm of the man in the car. Due to the centrality of the text on the car the SIFT keypoints for this image are concentrated on the center of the car.

This is the main reason for the $\check{C}ech_r(K)$ generated using the SIFT keypoints(Fig. 10.4) cover the car better than the \mathcal{K}_{holes} (Fig. 12.2). The nerves of different order are color coded for the Čech complex generated using \mathcal{K}_{holes} are displayed in Fig. 12.2. The Čech complex for the \mathcal{K}_{holes} conforms very well to the top contour of the car and the tires, but the bottom part of the front door remains uncovered. The Čech spectrum of the $\check{C}ech_r(K)$ usig the \mathcal{K}_{holes} is shown in Fig. 12.3. It can be seen that this Čech spectrum is different from the case of the SIFT keypoints shown in Fig. 10.5. The maximal Čech nerve for the \mathcal{K}_{holes} is 8 while for the SIFT keypoints is 9. Moreover the case of the car, the location of the maximal nerver or the 1-corolla(\mathbf{crl}_1) is also different for the SIFT keypoints and the \mathcal{K}_{holes} . For the SIFT the maximal nerve is on the front door while for the \mathcal{K}_{holes} it is on the top of the rear end of the car. This difference is fundamentally due to the difference in the spatial distribution of the specific types of keypoints.

4.3. Applications of the proposed framework

In this section we will present a few possible applications of the proposed framework. The idea is to formulate a proof of concept, that the current framework can be used in object extraction and recognition tasks in computer vision. We will begin with the idea of extracting the shapes from an image using the topological notion of cover.

4.3.1. Persistence of Čech Shapes

One of the major themes in this article is to develop a topological framework for covers of an object in a digital image. The idea of using simple geometrical objects to cover a topological space, so as to extract topological and geometrical information about it dates back to Poincaré (Poincaré, 1895). The objects used here are disks (here called Čech balls), parameterized by the location of centers and radii. An important question that arises here is related to the choice of these parameters.

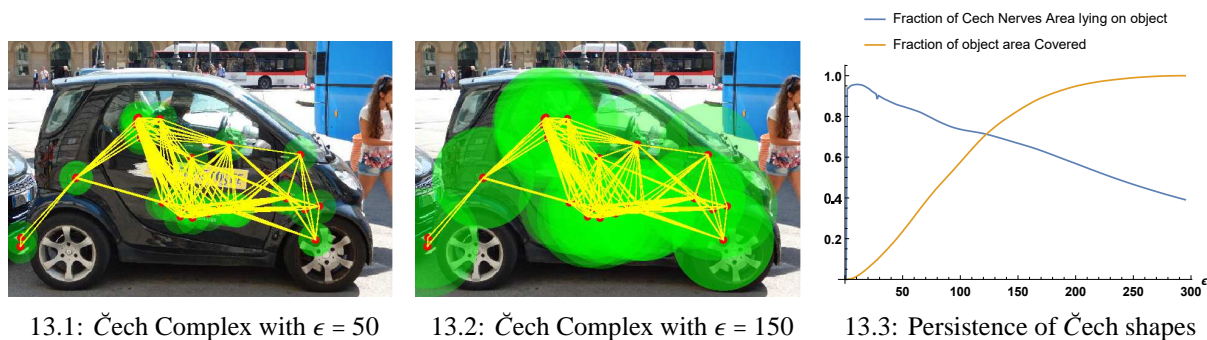
We choose the centroids to be the keypoints contributed by either SIFT or by hole based keypoints of Alg. 2. As to the choice of radii goes, we will use a recently developed technique, called persistent topology (Edelsbrunner & Harer, 2010), which is aimed at filtering out noise. The idea is that as we increase the radius of the Čech balls we get a new Čech complex which is a superset of the previous one. This can be written as: *for all* $r, s \in \mathbb{R}, r < s \Rightarrow cx_r \subseteq cx_s$. Here cx_r is a Čech complex yielded by Čech balls of radius r . This means that we get a filtration of Čech complexes indexed by the radius. Since, the question under investigation is the quality of a cover of the objects in digital images, we will employ appropriate measures in this regard.

The two measures that we use are the fraction of the area of the object covered by the Čech complex and the fraction of the area of Čech complex that lies on the object in the image. We want to cover the maximum area of the object while reducing the area of the background (parts of an image that are not the object), in the resulting cover. This is a trade off, which can be seen from Figs. 13.1 and 13.2. In Fig. 13.1 all of the Čech complex lies on the object, but it only covers a small area of the object. In Fig. 13.2, we are covering a significant portion of the object area, but the spillage into the background is also present. We need to strike a balance between the two parameters to attain a cover of the object that is well suited to approximating its geometrical and topological features. Hence, we plot the normalized values (on the range $[0, 1]$) of the measures for the whole filtration in the plot shown in Fig. 13.3.

From this plot we can decide upon the appropriate value of the radii for the Čech balls, which will suite our application. If the objective is to maximize the area of the object covered we would like the radius to be in the range of 170 – 220, but the spillage of the complex would be significant. Another possible choice could be the intersection point of the two curves, which would yield a value of 120 for the radius. This perspective gives us a view of the topological and geometrical properties of the object, indexed by scale. Other possible extensions of this method could be, to consider indexing by the number of keypoints or to index by both the number of keypoints and the radius.

4.3.2. Shapes of Bird Species

Let us consider another application of the concepts developed in this article. We will consider the identification of different species of birds, based on their shape. The aim like the previous

13.1: Čech Complex with $\epsilon = 50$ 13.2: Čech Complex with $\epsilon = 150$

13.3: Persistence of Čech shapes

Figure 13. This figure illustrates the change in the area of the object covered by the Čech complexes of varying diameters, ϵ . The plots illustrate the fraction of the object area covered by the Čech complexes and the fraction of the complex area that lies on the object of interest.

application is to demonstrate the viability of the theoretical framework. For this purpose we select the database used in (Lazebnik *et al.*, 2005). We select three images of birds belonging to three different species. On the images we perform an analysis similar to the one detailed in Sec. 4.2.

We use the holebased keypoints as they take into account the description of the image (in terms of the locations of constant pixel intensity regions). This will be a crucial feature in the classification of different patterns. The patterns inside an object and the shape of its contour are some of the most important features when it comes to classification in computer vision. This can be understood by looking at the pictures of the three birds we aim to classify, shown in Figs. 14.1, 14.4 and 14.7. These differ not only in terms of the shapes of their boundary contours but the patterns on them. For each of the birds, the Čech complexes formed by considering the holebased keypoints determined using Alg. 2, are shown in Figs. 14.2, 14.5 and 14.8. We can see that the 1-skeleton of the Čech complex (i.e. all the 1-simplices in it) conforms to the shape of the bird. An additional point to note here is that, it represents the proximity of the centroids of the regions with different descriptions. Hence, it is a more structural representation of the bird in terms of the regions of matching description in it.

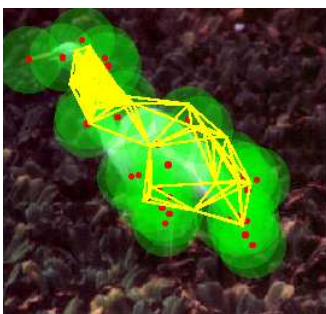
For this example we resort to looking at the Čech spectra (Def. 20) of each of the images. This is the number of Čech nerves of different order in the image. Each of the Čech Nerves is itself a hyper-connected space, as it is an intersection of varying number of Čech balls. The order of the hyper-proximity is the same as the cardinality of the nerve. Hence, the Čech spectra can be equivalently thought of as the number of hyper-connected subspaces of different order in the image. The Čech spectra for the images of the three birds are shown in Figs. 14.3, 14.6 and 14.9. It can be seen that the Čech spectra for the three birds are significantly different, hence providing a possible feature for classification.

4.3.3. Shapes of Butterfly Species

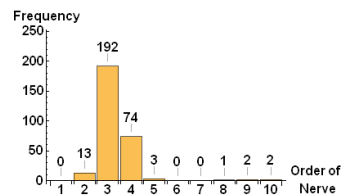
We present another application of the proposed framework to object detection for classification in computer vision. The aim is to provide a proof of concept, that the theoretical framework developed in this article can be used in practical applications. For this purpose, we aim to classify



14.1: Egret



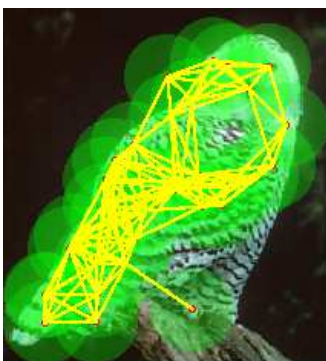
14.2: Čech Complex covering of Egret



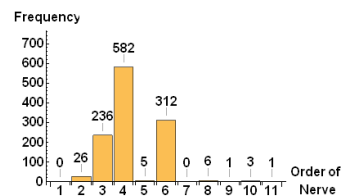
14.3: Čech Spectrum of Egret



14.4: Owl



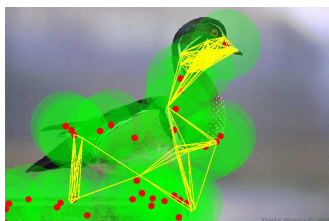
14.5: Čech Complex covering of Owl



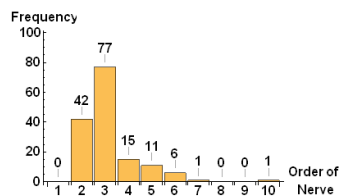
14.6: Čech Spectrum of Owl



14.7: Wood Duck



14.8: Čech Complex of Wood Duck



14.9: Čech Spectrum of Wood Duck

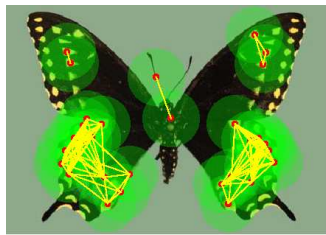
Figure 14. This figure illustrates the Čech complex coverings and the associated Čech spectra for images of three different birds taken from the database developed in (Lazebnik *et al.*, 2005).

butterflies based on their shapes. The dataset used in this task is taken from Lazebnik *et al.* (2004). We take three different images of butterflies belonging to different species. These images are shown in Figs. 15.1, 15.4 and 15.7. All these butterflies are different from one another based on their shape and patterns.

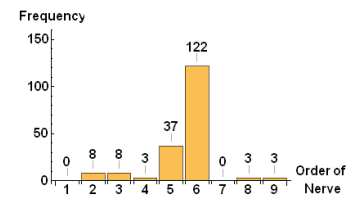
As we discussed in the case of classifying bird species, a classifier that incorporates the pixel intensity description would perform better than the one that considers boundary contours alone. This fact becomes even more evident when we compare Figs. 15.1 and 15.4, showing Black Snow-tail and Machaon butterflies respectively. Both the butterflies have almost identical boundary



15.1: Black Snowtail



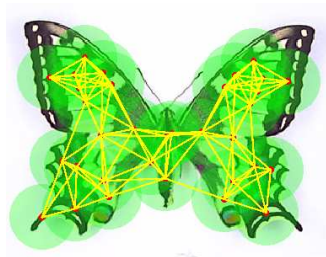
15.2: Čech Complex covering of Black Snowtail



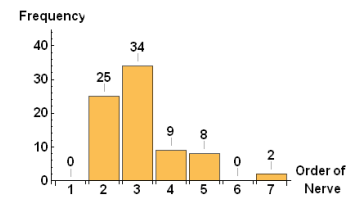
15.3: Čech Spectrum of Black Snowtail



15.4: Machaon



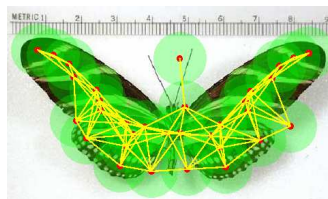
15.5: Čech Complex covering of Machaon



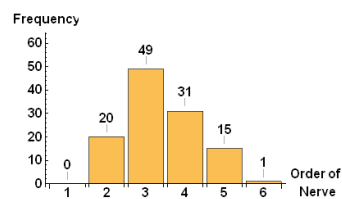
15.6: Čech Spectrum of Machaon



15.7: Zebra Butterfly



15.8: Čech Complex of Zebra Butterfly



15.9: Čech Spectrum of Zebra Butterfly

Figure 15. This figure illustrates the Čech complex coverings and the associated Čech spectra for images of three different butterflies taken from the database developed in (Lazebnik *et al.*, 2004).

contours but differ drastically in terms of patterns. Thus, we must use the holebased keypoints calculated using Alg. 2, which are centroids of regions with matching description(in this case [pixel intensities]). The methodology used here is similar to the Secs. 4.2 and 4.3.2. The Čech covers of the different butterflies are shown in Figs. 15.2,15.5 and 15.8. It can be seen that the difference of patterns dictates the locations of the holebased keypoints. It leads to different Čech complexes for both the Black Snowtail(Fig. 15.1) and the Machaon(Fig. 15.4). Moreover, the 1-skeletons(all the 1-simplices) of the Čech complexes conform to the structure of the shape.

Now to exploit this difference in the Čech complexes we use Čech spectrum(Def. 20) as a measure. The relationship between the Čech spectrum and hyper-connected subspaces of the image has been detailed in Sec. 4.3.2. We can see that Čech spectra for the different butterflies as shown in Figs.15.3,15.6 and 15.9 are substantially different. This difference elucidates the possibility of classification.

5. Conclusion

This paper uses proximal Čech complexes to approximate the shape of objects in digital images. Several topological structures with closed geometric balls as the primitive are formulated to study the geometrical and topological properties of objects. Moreover, instead of considering only the boundary contours of the object we also include the interior of the shape using descriptive proximity relations. The classical notion of proximity as a relation on two subsets has been extended to functions over arbitrary number of subsets. Moreover, the usual binary proximity is also extended to a continuous valued function also yielding the extent of nearness between objects. We define the concept of a descriptive hole in an image as a finite bounded region with a matching description. These are then used to formulate an algorithm to extract keypoints from an image. The distribution of the orders of the different Čech nerves in the image is used to define a signature for the shape of an object in digital images. The results for the computational experiments along with the algorithms used have also been presented.

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On Comultisets and Factor Multigroups

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Abstract

In this paper, we vividly study the concept of comultisets of a multigroup and obtain some related results. The notion of Lagrange theorem in multigroup setting is proposed. Also, this paper proposes the notion of factor multigroups as an extension of factor groups in classical group setting and deduces some results. Subsequently, some homomorphic properties of factor multigroups are presented.

Keywords: Comultisets, Multisets, Multigroups, Normal submultigroups, Factor multigroups.
2010 MSC: 03E72, 06D72, 11E57, 19A22.

1. Introduction

Many fields of modern mathematics emerged by violating a basic principle of a known theory or concept. For example, fuzzy set theory emerged by violating the notion of definite collection of objects in cantor set theory. Similarly, the theory of multisets (see (Knuth, 1981), (Singh *et al.*, 2007), (Syropoulos, 2001), (Wildberger, 2003) for details) has been defined by assuming that, for a given set X , an element x occurs a finite number of times. This violates the idea of distinct collection of objects.

The concept of (classical) groups is built on the foundation of cantor set theory. Since group is defined over a nonempty set hence, an algebraic study of multisets is an extension of group theory. The notion of multigroup was proposed in (Nazmul *et al.*, 2013) as an algebraic structure of multiset that generalized the concept of group. The notion is consistent with other non-classical groups in (Biswas, 1989), (Rosenfeld, 1971), (Shinoj *et al.*, 2015), (Shinoj & Sunil, 2015). The term *multigroups* has been earlier mentioned in (Barlotti & Strambach, 1991), (Dresher & Ore, 1938), (Mao, 2009), (Prenowitz, 1943), (Schein, 1987), (Tella & Daniel, 2013) as an extension of group theory (with each of the authors having a divergent view). Nonetheless, the

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idea of multigroups captured in (Nazmul *et al.*, 2013) is quite acceptable because it is in consonant with other non-classical groups and defined in the light of multiset.

A complete survey on the concept of multigroups from various authors were reviewed in (Ibrahim & Ejegwa, 2016). Further studies on the concept of multigroups in the light of multisets have been carried out. See (Awolola & Ejegwa, 2017), (Awolola & Ibrahim, 2016), (Ejegwa, 2017), (Ejegwa & Ibrahim, 2017b), (Ejegwa & Ibrahim, 2017c), (Ejegwa & Ibrahim, 2017a), (Ibrahim & Ejegwa, 2017a), (Ibrahim & Ejegwa, 2017b) for details.

In this paper, we explore more on the concept of comultisets of a multigroup studied in (Ejegwa & Ibrahim, 2017b), (Nazmul *et al.*, 2013), deduce some results, and propose Lagrange theorem in multigroups context. Also, we extend the idea of factor or quotient groups to multigroups and obtain some related results. Finally, the notion of homomorphism of factor multigroups is explored.

The paper is organized as follows. Section 2 gives some basic definitions and existing results on multisets and multigroups, respectively, for the sake of reference. Section 3 presents more results on comultisets of a multigroups and proposes the analogous of Lagrange theorem in multigroup setting. Details on the notion of factor multigroups as an extension of factor groups and its homomorphic properties are explicated in Section 4. Section 5 concludes the paper.

2. Preliminaries

Definition 2.1. (Singh *et al.*, 2007) Let $X = \{x_1, x_2, \dots, x_n, \dots\}$ be a set. A multiset A over X is a cardinal-valued function,

$$C_A : X \rightarrow N = \{0, 1, \dots\}$$

such that for $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$ denoted the number of times an object x occur in A , that is, a counting function of A (where $C_A(x) = 0$, implies $x \notin Dom(A)$).

The set X is called the ground or generic set of the class of all multisets containing objects from X . The set of all multisets over X is depicted by $MS(X)$.

Definition 2.2. (Wildberger, 2003) Let A and B be two multisets over X . Then A is called a submultiset of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. A multiset is called the parent in relation to its submultiset.

Definition 2.3. (Syropoulos, 2001) Let A and B be two multisets over X . Then the intersection and union of A and B , denoted by $A \cap B$ and $A \cup B$, respectively, are defined by the rules that for any object $x \in X$,

- (i) $C_{A \cap B}(x) = C_A(x) \wedge C_B(x)$,
- (ii) $C_{A \cup B}(x) = C_A(x) \vee C_B(x)$,

where \wedge and \vee represent minimum and maximum, respectively.

Definition 2.4. (Nazmul et al., 2013) Let X be a group. A multiset G is called a multigroup of X if the count function of G ,

$$C_G : X \rightarrow N = \{0, 1, \dots\}$$

satisfies the following conditions:

- (i) $C_G(xy) \geq C_G(x) \wedge C_G(y) \forall x, y \in X$,
- (ii) $C_G(x^{-1}) = C_G(x) \forall x \in X$.

By implication, a multiset G is a multigroup of a group X if $\forall x, y \in X$,

$$C_G(xy^{-1}) \geq C_G(x) \wedge C_G(y).$$

It follows immediately that,

$$C_G(e) \geq C_G(x) \forall x \in X$$

where e is the identity element in X . A multigroup G is regular if

$$C_G(x) = C_G(y) \forall x, y \in X.$$

We denote the set of all multigroups of X by $MG(X)$.

The count of an element in G is the number of occurrence of the element in G , and denoted by C_G . The order of G is the sum of the count of each of the elements in G , and is given by

$$|G| = \sum_{i=1}^n C_G(x_i) \forall x_i \in X.$$

Remark. (Ejegwa, 2017) Every multigroup is a multiset but the converse is not necessarily true.

Theorem 2.1. (Nazmul et al., 2013) Let $A, B \in MG(X)$. Then $A \cap B \in MG(X)$.

Definition 2.5. (Nazmul et al., 2013) Let $A \in MG(X)$. Then A is said to be abelian or commutative if for all $x, y \in X$, $C_A(xy) = C_A(yx)$.

Definition 2.6. (Nazmul et al., 2013) Let $A \in MG(X)$. Then the sets A_* and A^* are defined as

$$A_* = \{x \in X \mid C_A(x) > 0\}$$

and

$$A^* = \{x \in X \mid C_A(x) = C_A(e)\}.$$

Proposition 2.1. (Nazmul et al., 2013) Let $A \in MG(X)$. Then A_* and A^* are subgroups of X .

Definition 2.7. (Ejegwa, 2017) Let $A \in MG(X)$. A nonempty submultiset B of A is called a submultigroup of A denoted by $B \sqsubseteq A$ if B form a multigroup. A submultigroup B of A is a proper submultigroup denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

A submultigroup B of A is complete if $B_* = A_*$ and incomplete otherwise. A submultigroup B of A is also a multigroup of X itself.

Definition 2.8. (Ejegwa & Ibrahim, 2017b) Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is called a normal submultigroup of B if for all $x, y \in X$, it satisfies $C_A(xy x^{-1}) \geq C_A(y)$.

Definition 2.9. (Nazmul et al., 2013) Let A and B be multigroups of X . Then A^{-1} and $A \circ B$ are defined by

$$C_{A^{-1}}(x) = C_A(x^{-1}) \forall x \in X$$

and

$$C_{A \circ B}(x) = \begin{cases} \bigvee_{x=yz} (C_A(y) \wedge C_B(z)), & \text{if } \exists y, z \in X \text{ such that } x = yz \\ 0, & \text{otherwise,} \end{cases}$$

respectively.

Definition 2.10. (Ejegwa & Ibrahim, 2017b) Let X be a group. For any submultigroup A of a multigroup G of X , the submultiset yA of G for $y \in G$ defined by

$$C_{yA}(x) = C_A(y^{-1}x) \forall x \in A_*$$

is called the left comultiset of A . Similarly, for right comultiset of A . It follows that, $xA = yA$, $Ay \circ Az = Ay z$ and $yA \circ zA = y z A \forall x, y, z \in X$.

Definition 2.11. (Ejegwa & Ibrahim, 2017c) Let X and Y be groups and let $f : X \rightarrow Y$ be a homomorphism. Suppose A and B are multigroups of X and Y , respectively. Then f induces a homomorphism from A to B which satisfies

- (i) $C_A(f^{-1}(y_1 y_2)) \geq C_A(f^{-1}(y_1)) \wedge C_A(f^{-1}(y_2)) \forall y_1, y_2 \in Y$,
- (ii) $C_B(f(x_1 x_2)) \geq C_B(f(x_1)) \wedge C_B(f(x_2)) \forall x_1, x_2 \in X$,

where

- (i) the image of A under f , denoted by $f(A)$, is a multiset of Y defined by

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$ and

- (ii) the inverse image of B under f , denoted by $f^{-1}(B)$, is a multiset of X defined by

$$C_{f^{-1}(B)}(x) = C_B(f(x)) \forall x \in X.$$

Definition 2.12. (Ejegwa & Ibrahim, 2017c) Let X and Y be groups and let $A \in MG(X)$ and $B \in MG(Y)$, respectively.

- (i) A homomorphism f from X to Y is called a weak homomorphism from A to B if $f(A) \subseteq B$. If f is a weak homomorphism from A to B , then we say that, A is weakly homomorphic to B denoted by $A \sim B$.

- (ii) An isomorphism f from X to Y is called a weak isomorphism from A to B if $f(A) \subseteq B$. If f is a weak isomorphism from A to B , then we say that, A is weakly isomorphic to B denoted by $A \simeq B$.
- (iii) A homomorphism f from X to Y is called a homomorphism from A to B if $f(A) = B$. If f is a homomorphism from A to B , then A is homomorphic to B denoted by $A \approx B$.
- (iv) An isomorphism f from X to Y is called an isomorphism from A to B if $f(A) = B$. If f is an isomorphism from A to B , then A is isomorphic to B denoted by $A \cong B$.

Lemma 2.1. (Ejegwa & Ibrahim, 2017c) Let $f : X \rightarrow Y$ be a homomorphism of groups, $A \in MG(X)$ and $B \in MG(Y)$, respectively.

- (i) If f is an epimorphism, then $f(f^{-1}(B)) = B$.
- (ii) If $\ker f = \{e\}$, then $f^{-1}(f(A)) = A$.

The kernel of f is defined by $\ker f = \{x \in X \mid C_A(x) = C_B(e'), f(e) = e'\}$, where e and e' are the identities of X and Y , respectively. The kernel of f is a normal subgroup of X , and always contains the identity element of X . It reduces to the identity element if and only if f is one to one.

Theorem 2.2. (Ejegwa & Ibrahim, 2017c) Let X and Y be groups and $f : X \rightarrow Y$ be an isomorphism. Then $A \in MG(X) \Leftrightarrow f(A) \in MG(Y)$ and $B \in MG(Y) \Leftrightarrow f^{-1}(B) \in MG(X)$.

Definition 2.13. (Ejegwa & Ibrahim, 2017b) Suppose $A \in MG(X)$. Then the normalizer of A is the set given by

$$N(A) = \{g \in X \mid C_A(gy) = C_A(yg) \forall y \in X\}.$$

Theorem 2.3. (Awolola & Ibrahim, 2016) Let $A \in MG(X)$ with identity $e \in X$. Then $\forall x, y \in X$, $C_A(x) = C_A(y)$ if $C_A(xy^{-1}) = C_A(e)$.

3. Some results on comultisets of a multigroup

We assume that if G is a multigroup of a group X , then $G_* = X$ (except otherwise stated). That is, every element of X is in G with its multiplicity or count. In this section, we present some result on comultisets of a multigroups.

Recall that, for any submultigroup A of a multigroup G of a group X , the submultiset yA of G for $y \in X$ defined by

$$C_{yA}(x) = C_A(y^{-1}x) \forall x \in A_*$$

is called the left comultiset of A . Similarly, the submultiset Ay of G for $y \in X$ defined by

$$C_{Ay}(x) = C_A(xy^{-1}) \forall x \in A_*$$

is called the right comultiset of A .

Example 3.1. Let $X = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ be a permutation group on a set $S = \{1, 2, 3\}$ such that

$$\rho_0 = (1), \rho_1 = (123), \rho_2 = (132), \rho_3 = (23), \rho_4 = (13), \rho_5 = (12)$$

and $G = [\rho_0^7, \rho_1^5, \rho_2^5, \rho_3^3, \rho_4^3, \rho_5^3]$ be a multigroup of X . Then $H = [\rho_0^6, \rho_1^4, \rho_2^4]$ is an incomplete submultigroup of G .

Now, we find the left comultisets of H by pre-multiplying each element of G by H .

$$\begin{aligned} \rho_0 H &= [\rho_0^6, \rho_1^4, \rho_2^4] \\ \rho_1 H &= [\rho_2^4, \rho_0^6, \rho_1^4] \\ \rho_2 H &= [\rho_1^4, \rho_2^4, \rho_0^6] \\ \rho_3 H &= [\rho_3^0, \rho_5^0, \rho_4^0] = \emptyset \\ \rho_4 H &= [\rho_4^0, \rho_3^0, \rho_5^0] = \emptyset \\ \rho_5 H &= [\rho_5^0, \rho_4^0, \rho_3^0] = \emptyset. \end{aligned}$$

Similarly, the right comultisets of H are thus.

$$\begin{aligned} H\rho_0 &= [\rho_0^6, \rho_1^4, \rho_2^4] \\ H\rho_1 &= [\rho_2^4, \rho_0^6, \rho_1^4] \\ H\rho_2 &= [\rho_1^4, \rho_2^4, \rho_0^6] \\ H\rho_3 &= [\rho_3^0, \rho_4^0, \rho_5^0] = \emptyset \\ H\rho_4 &= [\rho_4^0, \rho_5^0, \rho_3^0] = \emptyset \\ H\rho_5 &= [\rho_5^0, \rho_3^0, \rho_4^0] = \emptyset. \end{aligned}$$

Suppose $H = [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2]$, that is, a complete submultigroup of G . The left comultisets of H are thus listed.

$$\begin{aligned} \rho_0 H &= [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2] \\ \rho_1 H &= [\rho_2^3, \rho_0^6, \rho_1^3, \rho_5^2, \rho_3^2, \rho_4^2] \\ \rho_2 H &= [\rho_1^3, \rho_2^3, \rho_0^6, \rho_4^2, \rho_5^2, \rho_3^2] \\ \rho_3 H &= [\rho_3^2, \rho_5^2, \rho_4^2, \rho_0^6, \rho_2^3, \rho_1^3] \\ \rho_4 H &= [\rho_4^2, \rho_3^2, \rho_5^2, \rho_1^3, \rho_0^6, \rho_2^3] \\ \rho_5 H &= [\rho_5^2, \rho_4^2, \rho_3^2, \rho_2^3, \rho_1^3, \rho_0^6]. \end{aligned}$$

The right comultisets of H are below.

$$\begin{aligned} H\rho_0 &= [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2] \\ H\rho_1 &= [\rho_2^3, \rho_0^6, \rho_1^3, \rho_4^2, \rho_5^2, \rho_3^2] \\ H\rho_2 &= [\rho_1^3, \rho_2^3, \rho_0^6, \rho_5^2, \rho_3^2, \rho_4^2] \\ H\rho_3 &= [\rho_3^2, \rho_4^2, \rho_5^2, \rho_0^6, \rho_1^3, \rho_2^3] \\ H\rho_4 &= [\rho_4^2, \rho_5^2, \rho_3^2, \rho_2^3, \rho_0^6, \rho_1^3] \\ H\rho_5 &= [\rho_5^2, \rho_3^2, \rho_4^2, \rho_1^3, \rho_2^3, \rho_0^6]. \end{aligned}$$

Remark. Let H be a submultigroup of $G \in MG(X)$. We notice that

- (i) H and its comultisets are equal because a multiset is an unordered collection. Consequently, $xH = yH \forall x, y \in X$.

- (ii) the number of comultisets of H equals the cardinality of H_* , and the union and intersection of the comultisets of H are comparable to H .
- (iii) there is a one-to-one correspondence between the right comultisets and the left comultisets of H .

Proposition 3.1. Let $A, B \in MG(X)$ such that $A \subseteq B$. If $xA = yA$, then $C_A(x) = C_A(y) \forall x, y \in X$.

Proof. Let X be a group and $x \in X$. Suppose $xA = yA$, then we have

$$C_{xA}(x) = C_{yA}(x) \Rightarrow C_A(x^{-1}x) = C_A(y^{-1}x) \Rightarrow C_A(e) = C_A(y^{-1}x).$$

Then, it follows that $C_A(y) = C_A(x) \forall x, y \in X$ by Theorem 2.3. □

Proposition 3.2. Let $B \in MG(X)$ and A be a submultigroup of B . If $(Ay \circ Az)^{-1} = Ay \circ Az$ and $Ay \circ Az = Ayz$, then $(Ay \circ Az)^{-1} = Ayz$.

Proof. Straightforward from Definition 2.10. □

Theorem 3.1. Let X be a finite group and A be a submultigroup of $B \in MG(X)$. Define

$$\begin{aligned} H &= \{g \in X \mid C_A(g) = C_A(e)\}, \\ K &= \{x \in X \mid C_{Ax}(y) = C_{Ae}(y)\}, \end{aligned}$$

where e denotes the identity element of X . Then H and K are subgroups of X . Again $H = K$.

Proof. Let $g, h \in H$. Then

$$\begin{aligned} C_A(gh) &\geq C_A(g) \wedge C_A(h) \\ &= C_A(e) \wedge C_A(e) \\ &= C_A(e) \end{aligned}$$

$$\Rightarrow C_A(gh) \geq C_A(e).$$

But, $C_A(gh) \leq C_A(e)$ from Definition 2.4. Thus, $C_A(gh) = C_A(e)$, implying that $gh \in H$. Since X is finite, it follows that H is a subgroup of X .

Now, we show that $H = K$. Let $k \in K$. Then for $y \in X$ we get

$$C_{Ak}(y) = C_{Ae}(y) \Rightarrow C_A(yk^{-1}) = C_A(y).$$

Choosing $y = e$, we obtain

$$C_A(k^{-1}) = C_A(e) \Rightarrow k^{-1} \in H,$$

and so $k \in H$ since H is a subgroup of X . Thus, $K \subseteq H$.

Again, let $h \in H$. Then $C_A(h) = C_A(e)$. Also,

$$C_{Ah}(y) = C_A(yh^{-1}) \forall y \in X$$

and

$$C_{Ae}(y) = C_A(y) \forall y \in X.$$

Thus to show that $h \in K$, it suffices to prove that

$$C_A(yh^{-1}) = C_A(y) \forall y \in X.$$

Now,

$$\begin{aligned} C_A(yh^{-1}) &\geq C_A(y) \wedge C_A(h^{-1}) \\ &= C_A(y) \wedge C_A(h) \\ &= C_A(y) \wedge C_A(e) \\ &= C_A(y). \end{aligned}$$

Again,

$$\begin{aligned} C_A(y) &= C_A(yh^{-1}h) \\ &\geq C_A(yh^{-1}) \wedge C_A(h) \\ &= C_A(yh^{-1}) \wedge C_A(e) \\ &= C_A(yh^{-1}) \end{aligned}$$

$\Rightarrow C_A(yh^{-1}) = C_A(y)$, thus, $H \subseteq K$. Hence, $H = K$. □

Corollary 3.1. *With the same notation as in Theorem 3.1, H is a normal subgroup of X if A is a normal submultigroup of B .*

Proof. Let $y \in X$ and $x \in H$. Then

$$\begin{aligned} C_A(yxy^{-1}) &= C_A(yy^{-1}x) \text{ since } A \text{ is normal in } B \\ &= C_A(x) = C_A(e). \end{aligned}$$

Thus, $yxy^{-1} \in H$. Hence, H is normal in X . □

Theorem 3.2. *Let X be a finite group and A be a submultigroup of $B \in MG(X)$. Define*

$$H = \{g \in X \mid C_A(g) = C_A(e)\}.$$

Then for $x, y \in X$, we get $Hx = Hy \Leftrightarrow Ax = Ay$. Similarly, $xH = yH \Leftrightarrow xA = yA$.

Proof. This result gives a relationship between comultisets of a submultigroup of a multigroup and the cosets of a subgroup of a given group.

By Theorem 3.1, we recall that H is a subgroup of X and

$$H = \{x \in X \mid C_{Ax}(z) = C_{Ae}(z)\}.$$

Now, suppose that $Hx = Hy$. Then $xy^{-1} \in H$. So

$$C_{Axy^{-1}}(z) = C_{Ae}(z)$$

and

$$C_A(zyx^{-1}) = C_A(z) \forall z \in X.$$

Replacing z by zy^{-1} , which is also an arbitrary element of X , we get

$$C_A(zx^{-1}) = C_A(zy^{-1}) \forall z \in X,$$

implying that $Ax = Ay$.

Conversely, suppose that $Ax = Ay$. This implies that

$$C_A(zx^{-1}) = C_A(zy^{-1}) \forall z \in X.$$

Put $z = y$, we get

$$C_A(yx^{-1}) = C_A(e).$$

So $yx^{-1} \in H$ and therefore, $Hx = Hy$. □

Corollary 3.2. *Let X be a group. If A is a submultigroup of a multigroup B of X and $x, y \in X$. Then $xA = yA$ and $Ax = Ay \Leftrightarrow C_A(y^{-1}x) = C_A(yx^{-1}) = C_A(e)$.*

Proof. Let $x, y \in X$, and recall that $H = \{x \in X \mid C_A(x) = C_A(e)\}$. Suppose $xA = yA$ and $Ax = Ay$. Then, $y^{-1}x, yx^{-1} \in H$ as in Theorem 3.2. So, $C_A(y^{-1}x) = C_A(e) = C_A(yx^{-1})$.

Conversely, assume $C_A(y^{-1}x) = C_A(e) = C_A(yx^{-1})$. This implies that, $C_A(y^{-1}x) = C_A(x^{-1}x)$ and $C_A(yx^{-1}) = C_A(yy^{-1}) \Rightarrow C_{yA}(x) = C_{xA}(x)$ and $C_{Ax}(y) = C_{Ay}(y) \forall x, y \in X$. Hence, $xA = yA$ and $Ax = Ay$. □

Lemma 3.1. *Let X be a group. If B is a submultigroup of a finite multigroup $A \in MG(X)$, then $|B| = |xB| \forall x \in X$.*

Proof. Let $A \in MG(X)$. Since A is finite and $B \sqsubseteq A$, it follows that $|A| = n$ and $|B| = m$ such that $m \leq n$. Then the order of each comultisets xB of B for all $x \in X$ must be m . Hence $|B| = |xB|$ for all $x \in X$. □

Now, we state and prove the analogous of Lagrange theorem in multigroup setting.

Theorem 3.3. *Let G be a finite multigroup and let H be a complete submultigroup of G wherein the count of every element in H is a factor of the count of the corresponding element in G . Then the order of H divides the order of G .*

Proof. Let $|G| = n$ and $|H| = m$, then $m \leq n$ by Lemma 3.1. That is, since G is finite and H is a complete submultigroup of G , it follows that H is also finite and $G_* = H_*$. We prove that m is a factor of n . Because $H \sqsubseteq G$ wherein the count of every element in H is a factor of the count of the corresponding element in G , it then follows that m divides n . This completes the proof. □

Remark. Let X be a finite group and G be a regular multigroup of X , then the order of X divides the order of G .

4. Concept of factor multigroups and its homomorphic properties

In this section, we define factor multigroup as an extension of factor group and obtain some results. Unless otherwise stated, the multigroups in this section are complete.

Definition 4.1. Let A be a multigroup of a group X and B a normal submultigroup of A . Then the set of right/left comultisets of B with the property $C_{xB \circ yB}(z) = C_{xyB}(z) \forall x, y, z \in X$ form a multigroup called factor or quotient multigroup of A determined by B , denoted as A/B .

Remark. Let A be a multigroup of a group X , and C a normal submultigroup of A . Then

- (i) if B is a submultigroup of A such that $C \subseteq B \subseteq A$, then B/C is a submultigroup of A/C .
- (ii) every submultigroup of A/C is of the form B/C , for some submultigroup B of A such that $C \subseteq B \subseteq A$.
- (iii) $|A/C| = n|C|$, where n is the number of comultisets of C in A . This is unlike in classical group where, suppose X is a group and Y a subgroup of X , then $|X/Y| = \frac{|X|}{|Y|}$.

Theorem 4.1. Let A be a normal submultigroup of $B \in MG(X)$. Then A is commutative if and only if B/A is commutative.

Proof. Let $x, y \in X$. Suppose A is commutative, then

$$C_A(xy x^{-1} y^{-1}) = C_A(e),$$

and hence,

$$C_A(xy) = C_A(yx).$$

Consequently, A is normal.

Now, since

$$C_A(xy(yx)^{-1}) = C_A(xy x^{-1} y^{-1}) = C_A(e),$$

we have

$$\begin{aligned} C_A(xy(yx)^{-1}) = C_A(e) &\Rightarrow C_A(xy(yx)^{-1}) = C_A(xy(xy)^{-1}) \\ &\Rightarrow C_{Ayx}(xy) = C_{Axy}(xy). \end{aligned}$$

Thus, $Axy = Ayx$. It follows that, $Ax \circ Ay = Ay \circ Ax$ since $Ax \circ Ay = Axy$ and $Ay \circ Ax = Ayx$ by Definition 2.10. Hence, B/A is commutative.

Conversely, if B/A is commutative, then

$$Ax \circ Ay = Ay \circ Ax \Rightarrow Axy = Ayx.$$

Thus,

$$C_A(xy(yx)^{-1}) = C_A(e) \Rightarrow C_A(xy) = C_A(yx),$$

completes the proof. □

Theorem 4.2. Let $A, B, C \in MG(X)$ such that A and B are normal submultigroups of C and $A \subseteq B$, then B/A is a normal submultigroup of C/A .

Proof. Let $x \in X$. Then $C_{B/A}(x) \leq C_{C/A}(x) \forall x \in X$ since $A \subseteq B$ and A and B are normal submultigroups of C . So B/A is a submultigroup of C/A . For all $x, y \in X$,

$$C_{B/A}(yxy^{-1}) \geq C_{B/A}(x).$$

Hence, B/A is a normal submultigroup of C/A by Definition 2.8. □

Remark. Let C be a multigroup of a group X , and B a normal submultigroup of C . Then every normal submultigroup of C/A is of the form B/A , for some normal submultigroup A of C such that $A \subseteq B \subseteq C$.

Theorem 4.3. Let $A, B \in MG(X)$ and A a normal submultigroup of B . Then $A \cap B/A_*$ is a normal submultigroup of A .

Proof. By Proposition 2.1, A_* is a subgroup of X and $A \cap B \in MG(X)$ by Theorem 2.1. So, $A \cap B/A_*$ is a multigroup of X . Since A is a normal submultigroup of B , then $A \cap B$ is a submultigroup of B and $A \cap B/A_*$ is a submultigroup of A . We show that $A \cap B/A_*$ is a normal submultigroup of A . Let $x, y \in A_*$. Then $xyx^{-1} \in A_*$ since

$$C_A(xyx^{-1}) = C_A(y) > 0$$

by Definition of A_* . This proves that A_* is normal. Also $A \cap B$ is normal since

$$C_{A \cap B}(xyx^{-1}) = C_A(xyx^{-1}) \wedge C_B(xyx^{-1}) \geq C_A(y) \wedge C_B(y) = C_{A \cap B}(y).$$

Hence, $A \cap B/A_*$ is a normal submultigroup of A . □

Theorem 4.4. Let $A \in MG(X)$ and $N(A)$ be a normalizer of $x \in X$. Then $N(A)$ is a subgroup of X and $A/N(A)$ is a normal submultigroup of A .

Proof. Clearly, $e \in N(A)$. Let $x, y \in N(A)$. Then for any $z \in X$, we have

$$\begin{aligned} C_A((xy^{-1})z) &= C_A(x(y^{-1}z)) = C_A((y^{-1}z)x) \\ &= C_A(y^{-1}(zx)) = C_A(y(zx)^{-1}) \\ &= C_A(y(x^{-1}z^{-1})) = C_A(z(xy^{-1})). \end{aligned}$$

Hence, $xy^{-1} \in N(A)$. Therefore, $N(A)$ is a subgroup of X . By Definition 4.1, it follows that $A/N(A) \in MG(N(A))$ and clearly, $A/N(A)$ is a submultigroup of A . Since $C_{A/N(A)}(xyx^{-1}) = C_{A/N(A)}(y) \forall x, y \in X$, it implies that $A/N(A)$ is a normal submultigroup of A . □

Theorem 4.5. Let A be a commutative multigroup of X and B a normal submultigroup of A . Then there exists a natural homomorphism $f : A \rightarrow A/B$ defined by $C_{f(A)}(y) = C_B(x^{-1}y) \forall x, y \in X$.

Proof. Let $f : A \rightarrow A/B$ be a mapping defined by

$$C_{f(A)}(y) = C_B(x^{-1}y) \forall x, y \in X.$$

That is, $C_{f(A)}(y) = C_{xB}(y) \Rightarrow f(A) = xB$ (consequently, $f(A_*) = xB_*$). Since $f : A \rightarrow A/B$ is derived from $f : A_* \rightarrow A_*/B_*$ such that B_* is a normal subgroup of A_* , then to prove that f is a homomorphism, we show that

$$C_{xyB}(z) = C_{xB \circ yB}(z) \forall z \in X \Rightarrow f(xy) = f(x)f(y).$$

Since B is commutative, then

$$C_B(xz) = C_B(zx) \Rightarrow C_B(z^{-1}xz) = C_B(x) \forall z \in X.$$

We know that,

$$C_{xB}(z) = C_B(x^{-1}z) \text{ and } C_{yB}(z) = C_B(y^{-1}z).$$

Then

$$C_{xyB}(z) = C_B((xy)^{-1}z).$$

Now,

$$\begin{aligned} C_{xB \circ yB}(z) &= \bigvee_{z=rs} (C_{xB}(r) \wedge C_{yB}(s)) \\ &= \bigvee_{z=rs} (C_B(x^{-1}r) \wedge C_B(y^{-1}s)). \end{aligned}$$

And

$$\begin{aligned} C_{xyB}(z) = C_B((xy)^{-1}z) &= C_B(y^{-1}x^{-1}z) \\ &\geq \bigvee_{z=rs} (C_B(x^{-1}r) \wedge C_B(y^{-1}s)). \end{aligned}$$

Suppose by hypothesis,

$$C_B(y^{-1}x^{-1}z) = \bigvee_{z=rs} (C_B(x^{-1}r) \wedge C_B(y^{-1}s)),$$

then it follows that $C_{xyB}(z) = C_{xB \circ yB}(z) \forall z \in X$. Consequently, we have $f(xy) = f(x)f(y) \forall x, y \in X$. Therefore, f is a homomorphism. \square

Corollary 4.1. Let $A, B \in MG(X)$ such that $C_A(x) = C_A(y) \forall x, y \in X$ and $C_A(e) \geq C_B(x) \forall x \in X$. If $f : A \rightarrow A/B$ is a natural homomorphism defined by $C_{f(A)}(y) = C_B(x^{-1}y) \forall x, y \in X$, then $f^{-1}(f(B)) = A \circ B$.

Proof. Let $x \in X$. To proof the result, we assume that $f(x) = f(y) \forall x, y \in X$. Thus,

$$\begin{aligned} C_{f^{-1}(f(B))}(x) &= \bigvee_{x \in X} (C_{f(B)}(f(x))), f(x) = f(y) \\ &= \bigvee_{x \in X} (C_B(f^{-1}(f(y)))) \forall y \in X \\ &= C_B(y). \end{aligned}$$

Again,

$$\begin{aligned} C_{A \circ B}(x) &= \bigvee_{x=zy} (C_A(z) \wedge C_B(y)) \forall y, z \in X \\ &= \bigvee_{x \in X} (C_A(xy^{-1}) \wedge C_B(y)), z = xy^{-1} \forall y, z \in X \\ &= \bigvee_{x \in X} (C_A(e) \wedge C_B(y)) \forall y \in X \\ &= C_B(y). \end{aligned}$$

$$\Rightarrow f^{-1}(f(B)) = A \circ B. \quad \square$$

Remark. We assume there is a bijective correspondence between every (normal) submultigroup of A that contains B and the (normal) submultigroups of A/B ; if C is a (normal) submultigroup of A containing B , then the corresponding (normal) submultigroup of A/B is $f(C)$.

Theorem 4.6. Let $f : X \rightarrow Y$ be an isomorphism of groups and A a normal submultigroup of $B \in MG(X)$ such that $C_B(x) = C_B(y) \forall x, y \in X$ with $\ker f = \{e\}$. Then $B/A \cong f(B)/f(A)$.

Proof. By Theorem 2.2 and Definition 4.1, B/A and $f(B)/f(A)$ are multigroups. Let

$$h : B/A \rightarrow f(B)/f(A)$$

be defined as

$$h(Ax) = f(A)(f(x)) \forall x \in X.$$

If $Ax = Ay$, then $C_A(xy^{-1}) = C_A(e)$. Since $\ker f = \{e\}$ meaning $\ker f \subseteq A^*$, then $f^{-1}(f(A)) = A$ by Lemma 2.1. Thus,

$$C_{f^{-1}(f(A))}(xy^{-1}) = C_{f^{-1}(f(A))}(e),$$

that is,

$$C_{f(A)}(f(xy^{-1})) = C_{f(A)}(f(e)),$$

then

$$C_{f(A)}(f(x)(f(y))^{-1}) = C_{f(A)}(f(e)),$$

so

$$C_{f(A)}(f(x)) = C_{f(A)}(f(y)e') \text{ (where } f(e) = e').$$

Hence,

$$C_{f(A)}(f(x)) = C_{f(A)}(f(y)) \Rightarrow f(A)(f(x)) = f(A)(f(y)).$$

Hence, h is well-defined. It is also a homomorphism because

$$\begin{aligned} h(AxAy) = h(Axy) &= f(A)(f(xy)) \\ &= f(A)(f(x)f(y)) \\ &= f(A)(f(x))f(A)(f(y)) \\ &= h(Ax)h(Ay). \end{aligned}$$

Suppose f is an epimorphism, then $\exists x \in X$ such that $f(x) = y$. So,

$$h(Ax) = f(A)(f(x)) = f(A)(y).$$

Moreover,

$$\begin{aligned} f(A)(f(x)) = f(A)(f(y)) &\Rightarrow C_{f(A)}(f(x)(f(y))^{-1}) = C_{f(A)}(e') \Rightarrow \\ C_{f(A)}(f(xy^{-1})) = C_{f(A)}(f(e)) &\Rightarrow C_{f^{-1}(f(A))}(xy^{-1}) = C_{f^{-1}(f(A))}(e) \end{aligned}$$

implies $C_A(xy^{-1}) = C_A(e) \Rightarrow Ax = Ay$, which proves that h is an isomorphism. Hence, the result follows. \square

Corollary 4.2. *Let $f : X \rightarrow Y$ be an isomorphism of groups and B a normal submultigroup of $A \in MG(Y)$ such that $C_A(x) = C_A(y) \forall x, y \in Y$. Then $f(A)/f(B) \cong A/B$.*

Proof. By Theorem 2.2, $f(A), f(B) \in MG(X)$ and $f(A)/f(B)$ and A/B are multigroups by Definition 4.1. Again, since $B \in MG(Y)$, then $f(f^{-1}(B)) = B$ by Lemma 2.1. If $x \in \ker f$, then $f(x) = e' = f(e)$, and so

$$C_B(f(x)) = C_B(f(e)),$$

that is,

$$C_{f^{-1}(B)}(x) = C_{f^{-1}(B)}(e).$$

Hence, $x \in f^{-1}(B)$, that is, $\ker f \subseteq f^{-1}(B)$. The proof is completed following the same process as in Theorem 4.6. \square

Theorem 4.7. *Let $A, B \in MG(X)$ and A a normal submultigroup of B . Then $B/B_* \approx B/A$.*

Proof. Let f be a natural homomorphism from B_* onto B_*/A_* defined by $f(xA_*) = xA_* \forall x \in B_*$. Then we have

$$C_{f(B/B_*)}(xA_*) = \vee(C_{B/B_*}(z)), \forall z \in B_*, f(z) = xA_*$$

Since B/B_* and B are bijective correspondence to each other and $z = f^{-1}(xA_*) = xA_*$, it follows that

$$\begin{aligned} C_{f(B/B_*)}(xA_*) &= \vee(C_{B/B_*}(z)), \forall z \in B_*, f(z) = xA_* \\ &= \vee(C_B(y)), \forall y \in xA_* \\ &= C_{B/A}(xA_*) \forall x \in B_*, \end{aligned}$$

because B/A and B are bijective correspondence to each other. Therefore, $B/B_* \approx B/A$. \square

Lemma 4.1. *If $f : X \rightarrow Y$ and $A \in MG(X)$, then $(f(A))_* = f(A_*)$.*

Proof. Straightforward. \square

Theorem 4.8. *Let $B \in MG(X)$. Suppose Y is a group and $C \in MG(Y)$ such that $B \approx C$. Then there exists a normal submultigroup A of B such that $B/A \cong C/C_*$.*

Proof. Since $B \approx C$, \exists an epimorphism f of X onto Y such that $f(B) = C$. Define $A \in MG(X)$ as follows: $\forall x \in X$,

$$C_A(x) = \begin{cases} C_B(x) & \text{if } x \in \ker f \\ 0, & \text{otherwise} \end{cases}$$

Clearly, $A \subseteq B$. If $x \in \ker f$, then $yxy^{-1} \in \ker f \forall y \in X$, and so

$$C_A(yxy^{-1}) = C_B(yxy^{-1}) \geq C_B(x) = C_A(x) \forall y \in X.$$

If $x \notin \ker f$, then $C_A(x) = 0$ and so

$$C_A(yxy^{-1}) \geq C_A(x) = 0 \forall y \in X.$$

Hence, A is a normal submultigroup of B . Also, $B \approx C \Rightarrow f(B) = C$ which further implies $(f(B))_* = C_*$ and $f(B_*) = C_*$ by Lemma 4.1. Let $f = g$. Then g is a homomorphism of B_* onto C_* and $\ker g = A_*$. Thus, there exists an isomorphism h of B_*/A_* onto C_* such that $h(xA_*) = g(x) = f(x) \forall x \in B_*$. For such an h , we have

$$\begin{aligned} C_{h(B/A)}(z) &= \vee(C_{B/A}(xA_*)), \forall x \in B_*, h(xA_*) = z \\ &= \vee(\vee[C_B(y)], \forall y \in xA_*), \forall x \in B_*, g(x) = z \\ &= \vee(C_B(y)), \forall y \in B_*, g(y) = z \\ &= \vee(C_B(y)), \forall y \in X, f(y) = z \\ &= C_B(f^{-1}(z)) = C_{f(B)}(z) = C_C(z), \forall z \in C_*. \end{aligned}$$

Therefore, $B/A \cong C/C_*$. □

Theorem 4.9. Let $B \in MG(X)$ and A be a normal submultigroup of B . Then $B/A \cap B \simeq A \circ B/A$.

Proof. From Proposition 2.1, we infer that A_* is also a normal subgroup of X . By the Second Isomorphism Theorem for groups, we deduce

$$B_*/A_* \cap B_* \cong A_*B_*/A_*.$$

We know that

$$\begin{aligned} (A \cap B)_* &= A_* \cap B_*, \\ (A \circ B)_* &= A_*B_*. \end{aligned}$$

Consequently, we have

$$B_*/(A \cap B)_* \cong (A \circ B)_*/A_*,$$

where f is given by

$$f(x(A \cap B)_*) = xA_* \forall x \in B_*.$$

Thus,

$$\begin{aligned} C_{f(B/A \cap B)}(yA_*) &= C_{B/A \cap B}(y(A \cap B)_*) \\ &= \vee(C_B(z)), \forall z \in y(A \cap B)_* \\ &\leq \vee(C_{A \circ B}(z)), \forall z \in y(A_* \cap B_*) \\ &\leq \vee(C_{A \circ B}(z)), \forall z \in yA_* \\ &= C_{A \circ B/A}(yA_*), \forall y \in B_*. \end{aligned}$$

Hence, $f(B/A \cap B) \subseteq A \circ B/A$. Therefore, $B/A \cap B \simeq A \circ B/A$. □

Theorem 4.10. Let $A, B, C \in MG(X)$ such that $A \subseteq B$, and A and B are normal submultigroups of C . Then $(C/A)/(B/A) \cong C/B$.

Proof. If $A, B \in MG(X)$ and A is a normal submultigroup of B , obviously, A_* is a normal subgroup of B_* and both A_* and B_* are normal submultigroups of C_* . From the principle of Third Isomorphism Theorem for groups, we have

$$(C_*/A_*)/(B_*/A_*) \cong C_*/B_*,$$

where f is given by

$$f(xA_*(B_*/A_*)) = xB_* \quad \forall x \in C_*.$$

Then

$$\begin{aligned} C_{f((C/A)/(B/A))(xB_*)} &= C_{(C/A)/(B/A)}(xA_*(B_*/A_*)) \\ &= \vee(C_{C/A}(yA_*)), \forall y \in C_*, yA_* \in xA_*(B_*/A_*) \\ &= \vee(\vee[C_C(z)], \forall z \in yA_*, \forall y \in C_*, yA_* \in xA_*(B_*/A_*)) \\ &= \vee(C_C(z)), \forall z \in C_*, zA_* \in xA_*(B_*/A_*) \\ &= \vee(C_C(z)), \forall z \in xA_*(B_*/A_*) \\ &= \vee(C_C(z)), \forall z \in C_*, f(z) \in xB_* \\ &= \vee(C_C(z)), \forall z \in C_*, f(z) = z \\ &= C_{C/B}(xB_*) \end{aligned}$$

$\forall x \in C_*$, where the equalities hold since f is one-to-one. Hence, the result follows. □

5. Conclusion

An indepth work on comultisets had been carried out and some results were deduced. We have extended the notion of factor groups to multigroups and explicated some properties of factor multigroups. Finally, we explored some homomorphic properties of factor multigroups. Nonetheless, more results on comultisets and factor multigroups could be exploited.

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