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An Application of Fuzzy Sets to Veterinary Medicine

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Abstract

In this paper, firstly, the waves P and T in ECG of kittens and adult cats were converted to fuzzy sets. After, using to entropy definition for fuzzy sets, we have assigned an entropy to waves P and T for kittens and adult cats. Also, using to some new formulates, the graphical representation of waves P and T for normal or diseased heart of cats were given.

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1. Introduction

The theoretical and practical applications of fuzzy sets have increased considerably since Zadeh's paper, (see (Abdollahian *et al.*, 2010) ; (Bilgin, 2003); (Dhar, 2013); (Diamond & Kloeden, 1994); (Goetschel & Voxman, 1986); (Li *et al.*, 1995); (Iwamoto & et al, 2007); (Kosko, 1986); (Matloka, 1986); Tong *et al.* (2007); (Zadeh, 1965) and (Zararsız & Şengönül, 2013)). In medicine, cardiologists are try to predetermine some heart diseases from electrocardiographs and this processes is also valid for veterinary medicine. Some fine details may not be seen in graphical representation of the waves electrocardiographs of human or animals. It is a fact that, long time can be spent for interpreting electrocardiographs (shortly; ECG) and sometimes small but important details can be unnoticed or ECG's can be misleading for junior vet or cardiologists. In this paper, by using entropy concept, we have obtained numerical values for ECGs of kittens and adult cats. These numerical values are the best way to observe fine details in the waves such as P, PQR complex and T. The numerical values are also very clear and can be easily interpreted for any person according to graphical representation of ECG's. It will be seen that these computations are

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completely different than computations of (Czogala & Leski, 2000). Let us give some background information on fuzzy sets and entropy of the fuzzy sets.

Let $\mathscr{X}$ be nonempty set. According to Zadeh, a fuzzy subset of $\mathscr{X}$ is a nonempty subset $\{(x, u(x)) : x \in \mathscr{X}\}$ of $\mathscr{X} \times [0, 1]$ for some function $u : \mathscr{X} \to [0, 1]$, (Diamond & Kloeden, 1994). Consider a function $u : \mathbb{R} \to [0, 1]$ as a subset of a nonempty base space $\mathbb{R}$. The function u is called membership function of the fuzzy set u.

Furthermore, we know that shape similarity of the membership functions does not reflect the conception of itself, but it will be used for examining the context of the membership functions. Whether a particular shape is suitable or not can be determined only in the context of a particular application. However, that many applications are not overly sensitive to variations in the shape. In such cases, it is convenient to use a simple shape, such as the triangular shape of membership function. Let us define fuzzy set u on the set $\mathbb{R}$ with membership function as follows:

$$u(x) = \begin{cases} \frac{h_u}{u_1 - u_0}(x - u_0), & x \in [u_0, u_1) \\ \frac{-h_u}{u_2 - u_1}(x - u_1) + h_u, & x \in [u_1, u_2] \\ 0, & others \end{cases}, \tag{1.1}$$

where the notations h_u denotes height of the fuzzy sets u. For brief, we write triple $(u_0, u_1 : h_u, u_2)$ for fuzzy set u. Notation $\mathscr{F}$ be the set of the all fuzzy sets in the form $u = (u_0, u_1 : h_u, u_2)$ on the $\mathbb{R}$.

Define the function S as follows:

$$S : \mathscr{F} \times \mathscr{F} \to \mathbb{R}, \quad S(u, v) = \frac{\min\{h_u, h_v\}}{\max\{h_u, h_v\}}[1 - \frac{1}{3}\sum_{k=0}^{2}|u_k - v_k|]. \tag{1.2}$$

The function S is called similarity degree between the fuzzy sets u and v. If $S(u, v) = 1$ then we say that u is full similar to v or vice versa, we say that v is completely similar to u. If $0 < S(u, v) < 1$ then we say that the fuzzy set u is S- similar to the fuzzy set v (or the fuzzy set v is S- similar to the fuzzy set u), if $S(u, v) \le 0$ we say that, u is not similar to v. Similar definitions can be found in (Sridevi & Nadarajan, 2009) and (Yıldız & Şengönül, 2014).

If we capture numerous ECG for any human or animal, it can be considered as a finite sequence of ECG's. Therefore we will give some definitions and properties about sequences of fuzzy sets.

The set

$$w(\mathscr{F}) = \{(u_k) \mid u : \mathbb{N} \to \mathscr{F}, u(k) = (u^k) = ((u_0^k, u_1^k : h_{u^k}, u_2^k))\} \tag{1.3}$$

is called sequence of fuzzy sets. Any element of the set $w(\mathscr{F})$ is called sequences of fuzzy sets, where $u_0^k, u_1^k, u_2^k \in \mathbb{R}$, $u_0^k \le u_1^k \le u_2^k$ and the mean of notation $u_1^k : h_{u^k}$ is the k^{th} therm of the sequence (u^k) takes highest membership degree at u_1^k and this membership degree is equal to h_{u^k} . If for all $k \in \mathbb{N}$, $h_{u^k} = 1$ then the set $w(\mathscr{F})$ turns into sequence set of fuzzy numbers and if $u_0^k = u_1^k = u_2^k$ and $h_{u_1^k} = 1$ the set $w(\mathscr{F})$ turns in to ordinary sequence space of the real numbers, respectively.

An another important class of the sequence set of the fuzzy sets is defined by

$$\varphi(\mathscr{F}) = \{(u_k) \in w(\mathscr{F}) \mid \exists k_0 \in \mathbb{N}, \forall k \ge k_0 : u^k = 0\}. \tag{1.4}$$

Clearly, the sequences of fuzzy sets can obtain by fuzzification of the term by term of sequence of real numbers with a suitable method.

Definition 1.1. Let us define the function $\mathscr{S}$ as follows:

$$\mathscr{S} : w(\mathscr{F}) \times w(\mathscr{F}) \to \mathbb{R}, \ \ \mathscr{S}(u_n, v_n) = \frac{\inf\{h_{u_n}, h_{v_n}\}}{\sup\{h_{u_n}, h_{v_n}\}}[1 - \frac{1}{3}\lim_{n} \sum_{k=0}^{2} |u_k^n - v_k^n|] = \lambda. \tag{1.5}$$

The function $\mathscr{S}$ is called similarity degree between sequences of fuzzy sets (u_n) and (v_n). If $\mathscr{S}(u_n, v_n) = 1$ then we say that (u_n) is completely similar to the sequence (v_n), if $0 < \mathscr{S}(u_n, v_n) = \lambda < 1$ then we say that the sequence (u_n) is λ- similar to the sequence (v_n), if $\lambda \leq 0$ we say that, (u_n) is not similar to (v_n).

In the fuzzy set theory, the fuzziness of a fuzzy set is a important matter and there are many method to measure the fuzziness of a fuzzy set. At first, the fuzziness was thought to be the distance between fuzzy set and its nearest nonfuzzy set. Later, the entropy was used instead of of fuzziness (de Luca & Termini, 1972) and has received attention, recently (Wang & Chui, 2000). Well, then what is the entropy?

Definition 1.2. (Zimmermann, 1991) Let $u \in \mathscr{F}$ and $u(x)$ be the membership function of the fuzzy set u and consider the function $H : \mathscr{F} \to \mathbb{R}^+$. If the function H satisfies conditions below,

1. $H(u) = 0$ iff u is crisp set,
2. $H(u)$ has a unique maximum, if $u(x) = \frac{1}{2}$, for all $x \in \mathbb{R}$
3. For $u, v \in \mathscr{F}$, if $v(x) \leq u(x)$ for $u(x) \leq \frac{1}{2}$ and $u(x) \leq v(x)$ for $u(x) \geq \frac{1}{2}$ then $H(u) \geq H(v)$,
4. $H(u^c) = H(u)$, where u^c is the complement of the fuzzy set u

then the $H(u)$ is called entropy of the fuzzy set u.

Let suppose that $u = u(x)$ be membership function of the fuzzy set u and the function $h : [0, 1] \to [0, 1]$ satisfies the following properties:

1. Monotonically increasing at $[0, \frac{1}{2}]$ and decreasing $[\frac{1}{2}, 1]$,
2. $h(x) = 0$ if $x = 0$ and $h(x) = 1$ if $x = \frac{1}{2}$.

The function h is called entropy function and the equality $H(u(x)) = h(u(x))$ holds for $x \in \mathbb{R}$. Some well known entropy functions are given as follows:

$h_1(x) = 4x(1 - x)$, $h_2(x) = -x \ln x - (1 - x)\ln(1 - x)$, $h_3(x) = \min\{2x, 2 - 2x\}$ and

$$h_4(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}] \\ 2(1 - x), & x \in [\frac{1}{2}, 1] \end{cases} .$$

Note that the function h_1 is the logistic function, h_2 is called Shannon function and h_3 is the tent function.

Let $\mathscr{X}$ be a continuous universal set. The total entropy of the fuzzy set u on the $\mathscr{X}$ is defined

$$e(u) = \int_{x \in \mathscr{X}} h(u(x))p(x)dx \tag{1.6}$$

where $p(x)$ is the probability density function of the available data in $\mathscr{X}$ (Pedrycz, 1994), (Pedrycz & Gomide, 2007). If we take $p(x) = 1$ in the (1.6) then the $e(u)$ is called entropy of the fuzzy set

u. It is known that the value of $e(u)$ is depend on support of the fuzzy set u. Let u be fuzzy set on the set $\mathbb{R}$ with membership function (1.1), then we see that the total entropy of fuzzy set u is equal to

$$e(u) = c(2h_u - \frac{4}{3}h_u^2)\ell(u) \tag{1.7}$$

for $p(x) = c$ and $h = h_1$, where $\ell(u) = \max\{x - y : x, y \in \overline{\{x \in \mathbb{R} : u(x) > 0\}}\}$. We know that each fuzzy set or a fuzzy number correspond to the fuzzy thoughts in the idea of user. So, any sequence of the fuzzy sets can be seen as sequence of thoughts or sequence fuzzy information. This sequence of fuzzy information may contain an useful information or not contain an useful information. But we can use terms of this sequence to obtain meaningful information from this sequence.

Definition 1.3. Let h be an entropy function, (u^k) be a sequence of fuzzy sets (or fuzzy thought) and $p_k(x)$ be probability density function of the available data in $\mathbb{R}$ for every $k \in \mathbb{N}$. Then sequence

$$e(u^k) = \int_{x\in\mathbb{R}} h(u^k(x))p_k(x)dx \tag{1.8}$$

is called total entropy sequence of the fuzzy sets (u^k). If the probability density function $p_k(x) = 1$ is fix, for all $k \in \mathbb{N}$, then the (1.8) is called entropy sequence of the fuzzy sets $u = (u^k)$.

If we take $u = (u^k) \in w(\mathscr{F})$, $p_k(x) = c_k \in (0, 1]$ and $h(u) = h_1(u)$ then from (1.8) we have

$$e(u^k) = (c_k(2h_{u^k} - \frac{4}{3}h_{u^k}^2)\ell(u^k)), \tag{1.9}$$

here and other places in the text, the notation $2h_{u^k}^2$ denotes second power of the h_{u^k}. If we choose the probability density functions $p_k(x) = c \in (0, 1]$ for all $k \in \mathbb{N}$ and $h_{u^k} = 1$ for all $k \in \mathbb{N}$ in the (1.9) then we see that $e(u^k) = \frac{2}{3}c\ell(u^k)$.

Let us suppose that $u = (u^k)$ be sequences of the fuzzy numbers (that is $h_{u^k} = 1$), $h(u) = h_1(u)$ and $p_k(x) = c_k = 1 \in (0, 1]$ for all $k \in \mathbb{N}$. Then the entropy $e(u^k)$ of the sequence of fuzzy numbers (u^k) is equal to

$$e(u^k) = \frac{2}{3}\ell(u^k). \tag{1.10}$$

Clearly, if $\ell(u^k) = 0$ for every $k \in \mathbb{N}$ then the sequence (u^k) returns to sequence of real numbers. In this case the entropy of the total entropy sequence is zero for sequences of real numbers. For example, let $u = (u^k)$ be $((1, 1 : 1, 1))$, then from (1.10) we obtain zeros sequence. Furthermore, the entropy sequence (e_k) can not be convergent but be bounded.

Definition 1.4. Let $\mathscr{A} = (a_{nk})$ be a lower triangular infinite matrix of real or complex numbers and

$$\sum_k a_{nk} \int_{x\in\mathbb{R}} h(u^k(x))p_k(x)dx \to E, \quad n \to \infty. \tag{1.11}$$

The real number E is called total $\mathscr{A}$-entropy of the sequence (u^k) of fuzzy sets, if it exists.

Definition 1.5. Let suppose that the $u = (u^k)$ be a sequence of fuzzy sets, $p_k(x) = c_k$, $(c_k \in (0,1])$ for all $k \in \mathbb{N}$ and

$$\lim_n \sum_k a_{nk} \int_{x\in\mathbb{R}} h(u^k(x))p_k(x)dx = \lim_n \sum_k a_{nk}c_k(2h_{u^k} - \frac{4}{3}h^2_{u^k})\ell(u^k) = E_1. \tag{1.12}$$

The real number E_1 is called total $\mathscr{A}$- entropy according to entropy function h and $p_k(x) = c_k$ is probability density functions of the sequence $u = (u^k)$ of fuzzy sets, and it is shown by $T_e^{\mathscr{A}}(u^k)$.

Let $n, k \in \mathbb{N}$, $\alpha > -1$, $p_k(x) = c_k$ and $\binom{n-k+\alpha-1}{n-k}$, $\binom{n+\alpha}{n}$ are binomial confidence. Let us define infinite matrices $A = (a_{nk})$ and $C^\alpha = (c^\alpha_{nk})$ as follows:

$$a_{nk} = \begin{cases} 1, & 0 \le k \le n \\ 0, & otherwise \end{cases} \quad \text{and} \quad c^\alpha_{nk} = \begin{cases} \frac{\binom{n-k+\alpha-1}{n-k}}{\binom{n+\alpha}{n}}, & 0 \le k \le n \\ 0, & otherwise \end{cases}.$$

If we write the matrices A and C^α instead of $\mathscr{A}$ in the expression (1.12) then we have

$$\lim_n \sum_{k=0}^{n} \int_{x\in\mathbb{R}} h(u^k(x))p_k(x)dx = T_e^A(u^k) \tag{1.13}$$

and

$$\lim_n \frac{1}{\binom{n+\alpha}{n}} \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} \int_{x\in\mathbb{R}} h(u^i(x))p_k(x)dx = T_e^{C^\alpha}(u^k), \tag{1.14}$$

respectively.

The expressions (1.13) and (1.14) are called A- total entropy and total Cesàro entropy of order α of the sequence $u = (u^k)$ of fuzzy sets, according to probability density functions $p_k(x)$, respectively. If we take $\alpha = 1$ and $p_k(x) = c_k$ from (1.14) we see that

$$T_e^{C^1}(u^k) = \lim_n \frac{1}{n+1} \sum_{k=0}^{n} c_k(2h_{u_1^k} - \frac{4}{3}h^2_{u_1^k})\ell(u^k) \tag{1.15}$$

which is called Cesàro normalized entropy of order 1 (shortly, Cesàro entropy) of the sequence $u = (u^k)$ of fuzzy sets.

It is easily prove that, if

$$T_e^{C^1}(u^k) = \lim_n \frac{1}{n+1} \sum_{k=0}^{n} c_k(2h_{u_1^k} - \frac{4}{3}h^2_{u_1^k})\ell(u^k) = a$$

then

$$T_e^{C^1}(u^k) = \lim_n \frac{s}{n+r} \sum_{k=0}^{n} c_k(2h_{u_1^k} - \frac{4}{3}h^2_{u_1^k})\ell(u^k) = a$$

where $r, s \in \mathbb{R}$. For example, the Cesàro entropy of sequence $(u^k) = ((\frac{k}{k+1} - t_1, \frac{k}{k+1} : 1, \frac{k}{k+1} + t_2))$ is

$$T_e^C(u^k) = \lim_n \frac{2(t_2 + t_1)}{3(n+1)} \sum_{k=0}^{n} c_k, \tag{1.16}$$

where we assume that $t_1 < t_2$ and $t_1, t_2 \in \mathbb{R}$ and $h_{u^k} = 1$ for all $k \in \mathbb{N}$. If the series $\sum_k c_k$ is convergent then the value $T_{e^{C^1}}(u^k)$ exists every time. As a comment of the (1.13) and (1.16), we point out that we can obtain an useful information from infinite fuzzy information by a suitable method. But, the total entropy and Cesàro entropy of the sequence v defined by $v = ((v_0^k, v_1^k, v_2^k)) = ((-k, 1 : 1, k+2))$ is infinite. This means that, the sequence v does not contain any useful information for us.

Since, every real number is also a fuzzy number then we can give following corollary:

Corollary 1.1. *Let the sequence $r = (r^k)$ be a convergent or divergent sequence of real numbers. Then the all entropies of the $r = (r^k)$ are zero.*

Corollary 1.1 can be interpreted as, in the any information sequence, if the elements of information sequence are crisp information then we obtain a crisp information from this sequence.

Proposition 1.1. If the fuzziness of the any sequence of fuzzy set is constantly increasing then the entropy is constantly grow and maybe is infinite. On the contrary if the fuzzyness of the any sequence of fuzzy set is constantly decreasing then the entropy is decreases and becomes 0.

It is calculated in (Chin, 2006) that the entropy of any fuzzy number is $\frac{2c(u_2-u_0)}{3}$. Therefore, in generally, if we take $h = h_1$ and $p_i(x) = c$, for every $i \in \mathbb{N}$, then entropy of the sequence of fuzzy numbers is given with (1.10).

In next section, we will investigate entropy of the electrocardiogram for cats and give some comments. We know that, an electrocardiogram is an important test for any relevant heart diseases of human or animals, the shortest way of identifying heart problems and you can detects cardiac (heart) abnormalities, as an example heart attacks, an enlarged heard or abnormal heart rhythms may cause heart failure, abnormal position of heart can be given, by measuring the electrical activity generated by the heart as it contacts, (for more, see (de Luna, 1987)).

2. The Applications to ECG's of the Idea Entropy and Some Comments

It is a fact that, the long time can be spent for interpreting electrocardiographs results by cardiologists or vet and sometimes small but important details can be unnoticed because of complexity of the *ECG*. The same situation is also valid for computerized electrocardiography. According to us, numerical values for *ECG* outputs can be more reliable for cardiologists and vet for interpreting *ECG* results. Furthermore, if the outputs are numerical then the consultation may be easy than consultation of the *ECG* papers. In this section we have proposed a new consultation method for cardiac problems of cats which will be based upon numerical value of *ECG*s, (see (Brady & Rosen, 2005); (Khan, 2003) for ECG).

Quite simply every heart beats can be considered as term of a sequence. Using to the waves P, QRS complex and T, we can construct the waves sequence $((P_k, (QRS)_k), T_k))$, where k is beat number or number of measurements and is finite. The graphical shapes of the waves P, QRS complex and T can imagine a membership functions a fuzzy set. With this idea, we can appoint an entropy value using to these membership functions which will be described below.

The entropy of the sequence $((P_k, (QRS)_k), T_k))$ can compute for finite or infinite many k and this computation gives to us a numerical value, not graphical. From numerical value, we can determine some cardiac problems. Namely, the sequence $((P_k, (QRS)_k), T_k))$ can divide three part for calculate entropy as follows:

1. The entropy of the sequence (P_k) waves,
2. The entropy of the sequence $((QRS)_k))$ complexes,
3. The entropy of the sequence (T_k) waves.

In this case, we can assume that the total entropy of the heart is equal to

$$\mathscr{E} = e(P_k) + e((QRS)_k) + e(T_k). \tag{2.1}$$

Now we will summarize some information about electrocardiographs without deepening the subject.

The electrocardiograph records the electrical activity of the heart muscle and displays this data as a trace on a screen or on paper and, later, this data is interpreted by a medical practitioner. ECG's from healthy hearts have a characteristic shape. Any irregularity in the heart rhythm or damage to the heart muscle can change the electrical activity of heart which leads to change in the shape of ECG's according to patients. Using this changes, we can investigate entropy of the heart rhythm or damage entropy of the heart muscle. It is known that, the QRS complex reflect the rapid depolarization of the right and left ventricles. The ventricles have a large muscle mass compared to the atria so the QRS complex usually has a much larger amplitude than the P- wave.

Furthermore, the heart movements are kept in check by various charges and pulses that change slightly on exertion, blood chemistry and strain. According to us, residence of skin and conductivity of blood are important for ECG, too. The conductivity and residence of the skin are vary according to some minerals in the blood plasma such as calcium, chloride, potassium or glucose concentration in a diabetic patients blood. So we have to consider the conductivity of blood in the calculations of transmitting electric current and therefore in the entropy calculations for a heart. For blood conductivity properties, you can read to (Hirsch & et al, 1950).

2.1. *The Entropy of The Waves Sequence (P_k) and Some Comments*

Primary wave of a heart in ECG, is called P wave and shortly denoted with P, have an entropy value and it can be compute as follows:

$$e(P) = \int_{x\in\mathbb{R}} h_1(P(x))r(x)dx, \tag{2.2}$$

where the function $P(x)$ is membership function of the fuzzy $\mathscr{P}$ set that we will correspond to wave P and the function $r(x)$ is conductivity function (generally the function r is fix) of the body .

Experimental measurements showed that to us for kittens, the wave P has maximal height about $0.12mV$, duration is shorter than 0.3 seconds but these values for adult cats are $0.2mV$ second and 0.04 (Lourenço & Ferreira, 2003).

Using the maximal height and duration of wave P as 0.12 second and 0.3 mV, respectively, the membership function $P_1(x)$ of the fuzzy $\mathscr{P}_1$ set which is correspond to wave P for kittens can write as follows:

$$P_1(x) = \begin{cases} 0.8x, & x \in [0, 0.15] \\ 0.24 - 0.8x, & x \in (0.15, 0.30] \\ 0, & \text{otherwise} \end{cases}. \tag{2.3}$$

Furthermore, the membership function $P_2(x)$ of the fuzzy $\mathscr{P}_2$ set which is correspond to wave P for adult cats is

$$P_2(x) = \begin{cases} 10x, & x \in [0, 0.02] \\ 0.4 - 10x, & x \in (0.02, 0.04] \\ 0, & \text{otherwise} \end{cases}. \tag{2.4}$$

It is clear that the support of the fuzzy set $\mathscr{P}_1$ is duration of the wave P and height is maximum height of wave P.

Let us take $supp\mathscr{P}_1 \approx]0, 0.30[$, $supp\mathscr{P}_2 \approx]0, 0.04[$ and closure of the $supp\mathscr{P}_1$ and $supp\mathscr{P}_2$ be $\overline{supp\mathscr{P}_1} = [0, 0.30]$ and $\overline{supp\mathscr{P}_2} = [0, 0.04]$ where the notations $supp\mathscr{P}_1$ and $supp\mathscr{P}_2$ denotes support of the $\mathscr{P}_1$ and $\mathscr{P}_2$.

In this case, we see that $h_1(P_1(x)) = \begin{cases} 3.2x - 2.56x^2, & x \in [0, 0.15] \\ 0.7296 - 1.664x - 2.56x^2, & x \in (0.15, 0.30] \\ 0, & \text{otherwise} \end{cases}$. Similarly to $h_1(P_1(x))$, we have $h_1(P_2(x)) = \begin{cases} 40x - 400x^2, & x \in [0, 0.02] \\ 0.96 - 8x - 400x^2, & x \in (0.02, 0.04] \\ 0, & \text{otherwise} \end{cases}$.

Let us denote P_1 and P_2 of wave P for kittens and adult cats, respectively. If we choose $r(x) = c$ in (2.2) then we see that the the entropy of wave P_1 is equal to

$$e(P_1) = 662.4 \times 10^{-4} c \tag{2.5}$$

for normal wave P for kittens. The P_2 wave entropy for adult cats is

$$e(P_2) = 138.667 \times 10^{-4} c. \tag{2.6}$$

If we compare (2.5) and (2.6) then we see that the P wave entropies of the kittens and adult cats are different.

Definition 2.1. The total Cesàro entropy of the sequence (P_k) is

$$T_e^{C^1}(P_k) = \frac{1}{k+1} \sum_{i=0}^{k} c_i a_2^i (2h_{a_1^i} - \frac{4}{3} h_{a_1^i}^2) S(P_k, P), \tag{2.7}$$

where c_i is resistance of the dry skin in the ith sample, k is number of sample of P wave and $S(P_k, P)$ is similarity degree between of the waves P_k and P.

Table 1. Non-clinical P waves data for adult cats

Gender: Male	Age:xx	Weight:xx	Height:xx							
Days	1	2	3	4	5	6	7	8	9	10
$m(h_{a_k^1})$	0.2	0.2	0.19	0.21	0.23	0.23	0.19	0.2	0.18	0.15
$m(a_2^k)$	0.04	0.04	0.03	0.05	0.05	0.05	0.045	0.044	0.043	0.043
$e(P_k)$	0,0138672	0,0138672	0,009956361	0,018060735	0,019474215	0,019474215	0,017526794	0,014602663	0,013622864	0,011610323
$S(P_k, P)$	1	1	0,94525	0,947619048	0,865217391	0,865217391	0,867391304	0,9481	0,89865	0,748875

Let the resistance of the dry skin be fix that is if c_i equal to c at the each every i. place then the (2.7) is turn to

$$T_e^{C^1}(P_k) = \frac{c}{k+1}\sum_{i=0}^{k} a_2^i(2h_{a_1^i} - \frac{4}{3}h_{a_1^i}^2)S(P_k, P). \tag{2.8}$$

Example 2.1. Let us suppose, the wave P values as height and width as given in Table 1 for any adult cat for 10 measurements with fix conductivity of blood and residence of the skin. Note that these data are not clinical measures. In this mean, the sequence (P_k) is in the set $\varphi(\mathscr{F})$. The notations $m(h_{a_k^1})$ and $m(a_2^k)$ in Table 1 denotes measured height and durations of the wave P in day. Then from (2.8), we see that the Cesàro total entropy of the wave P of adult cats according to Table 1 is

$$T_e^{C^1}(P_k) = 137.94345 \times 10^{-4}c \tag{2.9}$$

for 10 beats. If we compare (2.5) and (2.9), the P wave properties of the adult cat heart which given above example is very low than normal value. Using to (1.7), we can give a graphic for 10 sample of wave P which given in the Table 1 (see, Figure 2) .

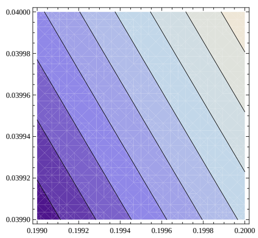

Figure 1
Graphical representation of $e(P_k)$ of the normal P wave for adult cats.

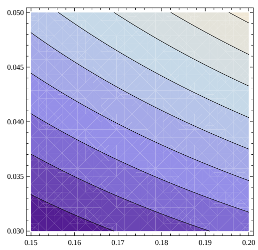

Figure 2
Graphical representation of $e(P_k)$ for Table 1 values for adult cats.

The Figure 1 is entropy graphic for the normal wave P of adult cats. If we compare the Figures 1 and 2 then we see that the height and duration of the P wave when changed with any effect, the all entropy zones are curl to upward at adult cats as in humans. It can be consider that the magnitude of the curl is P wave degenerations.

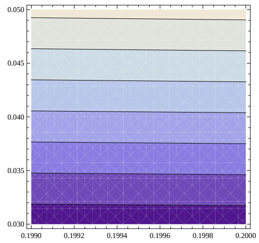

Figure 3
The values $h_{a_k^2}$ nearly fix but values a_2^k variable for adult cats.

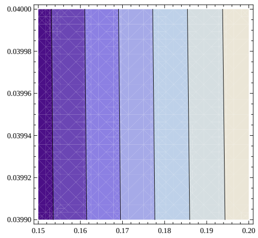

Figure 4
The values a_2^k nearly fix but the values $h_{a_k^2}$ variable for adult cats.

If the $h_{a_k^1}$ is fix but the value a_2^k be variable and conversely the $h_{a_k^1}$ is variable but the value a_2^k be fix then graphical representation of the entropy zones are shown as in Figure 3 and Figure 4, respectively.

As similar to (2.7), the A- entropy of the sequence wave P is

$$T_e^A(P_k) = 1379.43446 \times 10^{-4}c \tag{2.10}$$

from (1.13). But normal A-entropy value for 10 beats of adult cats should be $6624 \times 10^{-4}c$ and the P wave value in (2.10) very low than $6624 \times 10^{-4}c$. where c is resistance of the dry skin in the ith time.

Comment 1.

We know that the value of the $S(P_k, P)$ must be $0 \leq S(P_k, P) \leq 1$ for every $k \in \mathbb{N}$. After a certain place, if P_k waves is not exists, or the similarity values $S(P_k, P)$ nearly to the zero then the entropy of atrial depolarization of the heart, the $T_e^A(P_k)$ is near to zero. In this case we can say that this is a risk (for example, it can indicate hyperkalemia or hypokalemia or right atrial enlargement for this heart in the future as in human.

Comment 2.

Respectively, if the values $e(P_1)$ and $e(P_2)$ less than $662.4\times10^{-4}c$ and $138.667\times10^{-4}c$ for kitten and adult cats then, we can say that, there is a risk (for example, it can indicate hyperkalemia or hypokalemia or right atrial enlargement as in human for this heart in the future.

3. Comparison with the ECG

1. Long time can be spent for interpreting electrocardiographs results by cardiologists or vets and sometimes small but important details can be unnoticed because of the complexity of ECG.
2. Numerical values are more reliable than graphical representations.
3. If the outputs are numerical then the consultation may be easy than consultation of the ECG papers.

4. Weakness of This Model

The weakness of this model is that the data may be incomplete and not accurate enough because of the system that we use when we collect the data. Kittens adaptation to ECG machines is an important factor in the measurement phase since heart rates can change under stress and different circumstances. The numerical values may not reflect the reality if the information is not in the near proximity of real world assessment, shortly wrong inputs can produces misleading results.

5. Conclusions and Suggestions

The conclusions can be summarized as follows:

1. The entropy of the wave P for normal heart of the kitten should be $1379.43446 \times 10^{-4}c$ and should be $6624 \times 10^{-4}c$ for adult cats.
2. The graphical representation of the normal wave P of kittens should similar to Figure 1.
3. If the duration is fix but height is being altered by any reason then lines in graphical representation of the wave P becomes steeper.
4. The lines in the graphical representation of the wave T should be almost parallel to horizontal axis.

As a suggestion, clearly, one can define entropy value and graphical representations of QRS complex and wave T to similar entropy value wave P. So any numerical value can obtain for (2.1). If entropy value of the QRS complex and wave P are calculate then we can give a numerical entropy value for (2.1).

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On Certain Properties for Hadamard Product of Uniformly Univalent Meromorphic Functions with Positive Coefficients

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Abstract

In this paper we study some results concerning the Hadamard product of certain classes related to uniformly starlike and convex univalent meromorphic functions with positive coefficients.

Keywords: Univalent, meromorphic, starlike, convex, uniformly, Hadamard product.
2010 MSC: 30C45.

1. Introduction

Throughout this paper, let the functions of the form

$$\varphi(z) = c_1 z - \sum_{n=2}^{\infty} c_n z^n \qquad (c_1 > 0; c_n \geq 0), \tag{1.1}$$

and

$$\psi(z) = d_1 z - \sum_{n=2}^{\infty} d_n z^n \qquad (d_1 > 0; d_n \geq 0) \tag{1.2}$$

which are analytic and univalent in the unit disc

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\};$$

also, let

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^n \qquad (a_0 > 0; a_n \geq 0), \tag{1.3}$$

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$$f_i(z) = \frac{a_{0,i}}{z} + \sum_{n=1}^{\infty} a_{n,i} z^n \qquad (a_{0,i} > 0; a_{n,i} \geq 0), \tag{1.4}$$

$$g(z) = \frac{b_0}{z} + \sum_{n=1}^{\infty} b_n z^n \qquad (b_0 > 0; b_n \geq 0), \tag{1.5}$$

$$g_j(z) = \frac{b_{0,j}}{z} + \sum_{n=1}^{\infty} b_{n,j} z^n \qquad (b_{0,j} > 0; b_{n,j} \geq 0). \tag{1.6}$$

which are analytic and univalent in the punctured unit disc

$$U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}.$$

A function $f(z) \in \Sigma$ is meromorhpically starlike of order α if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \, (z \in U^*; 0 \leq \alpha < 1). \tag{1.7}$$

A function f of the form (1.3) is said to be in the class $U\Sigma S_0^*(\alpha,\beta)$ of meromorphic uniformly β -starlike functions of order α if it satisfies the condition:

$$-Re\left\{\frac{zf'(z)}{f(z)} + \alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)} + 1\right| (z \in U; 0 \leq \alpha < 1; \beta \geq 0). \tag{1.8}$$

Also, a function f of the form (1.3) is said to be in the class $U\Sigma C_0(\alpha,\beta)$ of meromorphic uniformly β -convex functions of order α if it satisfies the condition:

$$-Re\left\{1 + \frac{zf''(z)}{f'(z)} + \alpha\right\} > \beta\left|2 + \frac{zf''(z)}{f'(z)}\right| (z \in U; 0 \leq \alpha < 1; \beta \geq 0). \tag{1.9}$$

It follows from (1.8) and (1.9) that

$$f \in U\Sigma C_0(\alpha,\beta) \iff -zf' \in U\Sigma S_0^*(\alpha,\beta). \tag{1.10}$$

The classes $U\Sigma S_0^*(\alpha,\beta)$ and $U\Sigma C_0(\alpha,\beta)$ have been studied by (Aouf *et al.*, 2014), (Atshan & Kulkarni, 2007), and others. We note that
(i) $U\Sigma S_0^*(\alpha,0) = S_n^*(\alpha)$ and $U\Sigma C_0(\alpha,0) = C_n(\alpha)$ (see (Aouf & Silverman, 2008), *with* $n = 1$);
(ii) $U\Sigma S_0^*(\alpha,0) = \Sigma_p S_n^*(\alpha,\gamma)$ and $U\Sigma C_0(\alpha,0) = \Sigma_p C_n(\alpha,\gamma)$ (also see (R. M. El-Ashwah & Hassan, 2013), with $n = p = \gamma = 1$);
(iii) $U\Sigma S_0^*(\alpha,0) = \Sigma S_0^*(\alpha)$ and $U\Sigma C_0(\alpha,0) = \Sigma K_0(\alpha,\beta)$ (see (Mogra, 1991)).

Lemma 1.1. *Let the function f defined by (1.3). Then $f \in U\Sigma S_0^*(\alpha,\beta)$ if and only if*

$$\sum_{n=1}^{\infty}[n(1+\beta) + (\alpha+\beta)]a_n \leq (1-\alpha)a_0. \tag{1.11}$$

Lemma 1.2 (3). *Let the function f defined by (1.3). Then $f \in U\Sigma C_0(\alpha,\beta)$ if and only if*

$$\sum_{n=1}^{\infty} n[n(1+\beta)+(\alpha+\beta)]a_n \le (1-\alpha)a_0. \tag{1.12}$$

Definition 1.1. Let the function f defined by (1.3). Then $f \in U\Sigma S_m(\alpha,\beta)$ if and only if

$$\sum_{n=1}^{\infty} n^m[n(1+\beta)+(\alpha+\beta)]a_n \le (1-\alpha)a_0, \tag{1.13}$$

where $(0 \le \beta < \infty)$, $(0 \le \alpha < 1)$ and m any positive integer number.

We note that $U\Sigma S_1(\alpha,\beta) = U\Sigma C_0(\alpha,\beta)$ and $U\Sigma S_0(\alpha,\beta)$ is equivalent to $U\Sigma S_0^*(\alpha,\beta)$. Further, $U\Sigma S_m(\alpha,\beta) \subset U\Sigma S_r(\alpha,\beta)$ if $m > r \ge 0$, the containment beign proper. Whence, for any positive integer m, we have the inclusion relation

$$U\Sigma S_m(\alpha,\beta) \subset U\Sigma S_{m-1}(\alpha,\beta) \subset \ldots \subset U\Sigma S_2(\alpha,\beta) \subset U\Sigma C_0(\alpha,\beta)\, U\Sigma S_0^*(\alpha,\beta).$$

Also, we note that for nonnegative real number m the class $U\Sigma S_m(\alpha,\beta)$ is nonempty as the functions of the form

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} \frac{(1-\alpha)a_0}{n^m[n(1+\beta)+(\alpha+\beta)]}\lambda_n z^n,$$

where $a_0 > 0$, and $\sum_{n=1}^{\infty} \lambda_n \le 1$, satisfy the inequality (1.13). For the functions

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n \left(a_{n,j} \ge 0;\, j = 1, 2\right). \tag{1.14}$$

We denote by $(f_1 * f_2)(z)$ the Hadamard product (or convolution) of functions $f_1(z)$ and $f_2(z)$, that is

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n. \tag{1.15}$$

Similarly, we can define the Hadamard product of more than two functions. The quasi-Hadamard product of two or more functions $\varphi(z)$ and $\psi(z)$ given by (1.1) and (1.2), (see (Kumar, 1987)).

$$(\varphi * \psi)(z) = c_1 d_1 z - \sum_{n=2}^{\infty} c_n d_n z^n \tag{1.16}$$

In this paper, we can discuss certain results concerning the Hadamard product of functions in the classes $U\Sigma S_0^*(\alpha,\beta)$, $U\Sigma S_m(\alpha,\beta)$ and $U\Sigma C_0(\alpha,\beta)$.

2. Main results

Theorem 2.1. *Let the functions $f_i(z)$ defined by (1.4) be in the class $U\Sigma C_0(\alpha,\beta)$ for every $i = 1,2,...,m$, and suppose that the functions $g_j(z)$ defined by (1.6) be in the class $U\Sigma S_0^*(\alpha,\beta)$ for every $j = 1,2,...,q$. Then the Hadamard product $(f_1 * f_2 * ... * f_m * g_1 * g_2 * ...g_q)(z)$ belongs to the class $U\Sigma S_{2m+q-1}(\alpha,\beta)$.*

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty}\left\{n^{2m+q-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m}a_{n,i}\prod_{j=1}^{q}b_{n,j}\right]\right\}\leq(1-\alpha)\left[\prod_{i=1}^{m}a_{0,i}\prod_{j=1}^{q}b_{0,i}\right]. \tag{2.1}$$

Since $f_i(z)\in U\Sigma C_0(\alpha,\beta)$, we get

$$\sum_{n=1}^{\infty}n[n(1+\beta)+(\alpha+\beta)]a_{n,i}\leq(1-\alpha)a_{0,i}\quad(i=1,2,...,m). \tag{2.2}$$

Therefore,

$$a_{n,i}\leq\frac{(1-\alpha)}{n[n(1+\beta)+(\alpha+\beta)]}a_{0,i} \tag{2.3}$$

which implies that

$$a_{n,i}\leq n^{-2}a_{0,i}\quad(i=1,2,...,m). \tag{2.4}$$

Similarly, for $g_j(z)\in U\Sigma S_0^*(\alpha,\beta)$, we obtain

$$\sum_{n=1}^{\infty}[n(1+\beta)+(\alpha+\beta)]b_{n,j}\leq(1-\alpha)b_{0,j}, \tag{2.5}$$

for $j = 1,2,...,q$. Hence we have

$$b_{n,j}\leq n^{-1}b_{0,j}\quad(j=1,2,...,q). \tag{2.6}$$

Using (2.4) for $i = 1,2,...,m$, (2.6) for $j = 1,2,...,q-1$, and (2.5) for $j = q$, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\left\{n^{2m+q-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m}a_{n,i}\prod_{j=1}^{q}b_{n,j}\right]\right\}\\ \leq\;&\sum_{n=1}^{\infty}\left\{n^{2m+q-1}\{n(1+\beta)+(\alpha+\beta)\}\left[n^{-2m}n^{-(q-1)}\prod_{i=1}^{m}a_{0,i}\prod_{j=1}^{q-1}b_{0,j}\right]b_{n,q}\right\}\\ =\;&\left[\prod_{i=1}^{m-1}a_{0,i}\prod_{j=1}^{q-1}b_{0,j}\right]\sum_{n=1}^{\infty}\left[n\{n(1+\beta)+(\alpha+\beta)\}b_{n,q}\right]\leq(1-\alpha)\prod_{i=1}^{m}a_{0,i}\prod_{j=1}^{q}b_{0,j}.\end{aligned}$$

Hence $(f_1 * f_2 * ... * f_m * g_1 * g_2 * ...g_q)(z)\in U\Sigma S_{2m+q-1}(\alpha,\beta)$. The proof of Theorem 1 is completed. □

Theorem 2.2. *Let the functions $f_i(z)$ defined by (1.4) be in the class $U\Sigma C_0(\alpha,\beta)$ for every $i = 1,2,...,m$, then the Hadamard product $(f_1 * f_2 * ... * f_m)(z)$ belongs to the class $U\Sigma S_{2m-1}(\alpha,\beta)$.*

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty}\left\{n^{2m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m}a_{n,i}\right]\right\}\leq(1-\alpha)\left[\prod_{i=1}^{m}a_{0,i}\right]. \tag{2.7}$$

Since $f_i(z)\in U\Sigma C_0(\alpha,\beta)$, the inequalities (2.1) and (2.2) hold for every $i = 1,2,...,m$.

Using (2.2) for $i = 1,2,...,m-1$, and (2.1) for $i = 1,2,...,m$, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\left\{n^{2m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m}a_{n,i}\right]\right\}\\ \leq&\sum_{n=1}^{\infty}\left\{n^{2m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[n^{-2(m-1)}\prod_{i=1}^{m-1}a_{0,i}\right]a_{n,m}\right\}\\ =&\left[\prod_{i=1}^{m-1}a_{0,i}\right]\sum_{n=1}^{\infty}\left[n\{n(1+\beta)+(\alpha+\beta)\}a_{n,m}\right]\leq(1-\alpha)\prod_{i=1}^{m}a_{0,i}.\end{aligned}$$

Hence $(f_1 * f_2 * ... * f_m)(z)\in U\Sigma S_{2m-1}(\alpha,\beta)$. The proof of Theorem 2 is completed. □

Theorem 2.3. *Let the functions $f_i(z)$ defined by (1.4) be in the class $U\Sigma S_0^*(\alpha,\beta)$ for every $i = 1,2,...,m$, then the Hadamard product $(f_1 * f_2 * ... * f_m)(z)$ belongs to the class $U\Sigma S_{m-1}(\alpha,\beta)$.*

Proof. Since $f_i(z)\in U\Sigma S_0^*(\alpha,\beta)$, we have

$$\sum_{n=1}^{\infty}[n(1+\beta)+(\alpha+\beta)]a_{n,i}\leq(1-\alpha)a_{0,i}, \tag{2.8}$$

for every $i = 1,2,...,m$. Therefore, we obtain $a_{n,i}\leq\frac{(1-\alpha)}{n(1+\beta)+(\alpha+\beta)}a_{0,i}$ which implies that

$$a_{n,i}\leq n^{-1}a_{0,i}\quad(i=1,2,...,m). \tag{2.9}$$

Using (2.9) for $i = 1,2,...,m-1$, and (2.8) for $i = 1,2,...,m$, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\left\{n^{m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m}a_{n,i}\right]\right\}\\ \leq&\sum_{n=1}^{\infty}\left\{n^{m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[n^{-(m-1)}\prod_{i=1}^{m-1}a_{0,i}\right]a_{n,m}\right\}\\ =&\left[\prod_{i=1}^{m-1}a_{0,i}\right]\sum_{n=1}^{\infty}\left[\{n(1+\beta)+(\alpha+\beta)\}a_{n,m}\right]\leq(1-\alpha)\prod_{i=1}^{m}a_{0,i}.\end{aligned}$$

Hence $(f_1 * f_2 * ... * f_m)(z)\in U\Sigma S_{m-1}(\alpha,\beta)$, which completes the proof of Theorem 3. □

Remark. Taking $\beta = 0$ in our main results, we obtain the results obtained by Mogra (Mogra, 1991).

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On Uniform h-Stability of Evolution Operators in Banach Spaces

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Abstract

The paper treats the general concept of uniform h-stability, as a generalization of uniform exponential stability for evolution operators in Banach spaces.
The main aim is to give necessary and sufficient conditions of Datko-type and Barbashin-type for this property and also criterias for uniform h-stability using Lyapunov functions. As particular cases, we obtain the results for uniform exponential stability.

Keywords: uniform stability, growth rates, evolution operators.
2010 MSC: 34D05, 34D20, 34D23.

1. Preliminaries

One of the most important asymptotic properties studied for evolution operators is the uniform exponential stability. This concept was treated in a large number of papers and of the most important we recall (Coppel, 1965), (Lupa *et al.*, 2010), (Megan *et al.*, 2001), (van Neerven, 1995) and (Stoica & Megan, 2010).

In the last years, are considered more general concepts of stability, as h-stability (see (Megan, 1995)) or (h, k)-stability (see (Fenner & Pinto, 1997), (Megan & Cuc, 1997), (Minda & Megan, 2011)), where h and k are growth rates (i.e. nondecreasing functions with different properties).

In this paper is considered the concept of uniform h-stability, with $h : \mathbb{R}_+ \to [1, +\infty)$ a growth rate (more precisely a nondecreasing function with $\lim_{t\to+\infty} h(t) = +\infty$), for evolution operators in Banach spaces.
Are obtained necessary and sufficient conditions for this notion and as consequences, we emphasize the results for the case of uniform exponential stability.

In what follows, X represents a real or complex Banach space, X^* its topological dual and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. We will denote the norms on X, on X^*

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and on $\mathcal{B}(X)$ by $\|\cdot\|$.
Also, Δ is the set of all the pairs $(t,s) \in \mathbb{R}_+^2$ with $t \geq s$ and I represents the identity operator on X.

Definition 1.1. A mapping $\Phi : \Delta \to \mathcal{B}(X)$ is called *evolution operator* on X if

(eo_1) $\Phi(t,t) = I$, for every $t \geq 0$;
(eo_2) $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0)$, for all (t,s) and $(s,t_0) \in \Delta$.

We consider $\Phi : \Delta \to \mathcal{B}(X)$ an evolution operator and $h : \mathbb{R}_+ \to [1,+\infty)$ a growth rate.

Definition 1.2. We say that Φ has a *uniform h-growth* if there exists $N \geq 1$ such that for all $(t,s,x) \in \Delta \times X$:

$$h(s)\|\Phi(t,s)x\| \leq Nh(t)\|x\|.$$

If $h(t) = e^{\alpha t}$, with $\alpha > 0$, then we say that Φ has a *uniform exponential growth.*

Definition 1.3. The evolution operator Φ is called *uniformly h-stable* if there exists $S \geq 1$ such that for all $(t,s,x) \in \Delta \times X$:

$$h(t)\|\Phi(t,s)x\| \leq S\,h(s)\|x\|.$$

In particular, if $h(t) = e^{\alpha t}$, with $\alpha > 0$, then we recover the concept of *uniform exponential stability* and α is called *stability constant*.

Remark. If Φ is uniform h-stable, then it has a uniform h-growth. In general, the converse implication is not valid.

Example 1.1. Considering the evolution operator $\Phi : \Delta \to \mathcal{B}(X)$, defined by

$$\Phi(t,s) = \frac{h(t)}{h(s)}, \quad \text{for all} \quad (t,s) \in \Delta,$$

it is easy to observe that Φ has a uniform h-growth, but Φ is not uniformly h-stable.

Remark. The evolution operator Φ has a uniform h-growth if and only if there exists $N \geq 1$ with

$$h(s)\|\Phi(t,t_0)x_0\| \leq Nh(t)\|\Phi(s,t_0)x_0\|,$$

for all $(t,s),\ (s,t_0) \in \Delta,\ x_0 \in X$.

Remark. Φ is uniformly h-stable if and only if there is $S \geq 1$ such that

$$h(t)\|\Phi(t,t_0)x_0\| \leq S\,h(s)\|\Phi(s,t_0)x_0\|,$$

for all $(t,s),\ (s,t_0) \in \Delta,\ x_0 \in X$.

Definition 1.4. We say that $\Phi : \Delta \to \mathcal{B}(X)$ is

(i) *strongly measurable* if for all $(s,x) \in \mathbb{R}_+ \times X$ the mapping

$$t \mapsto \|\Phi(t,s)x\| \text{ is measurable on } [s,+\infty);$$

(ii) **-strongly measurable* if for all $(t,x^*) \in \mathbb{R}_+ \times X^*$ the mapping

$$s \mapsto \|\Phi(t,s)^*x^*\| \text{ is measurable on } [0,t].$$

2. Necessary conditions for uniform *h*-stability

In this section we will denote by $\mathcal{H}$ the set of the growth rates $h : \mathbb{R}_+ \to [1, +\infty)$ with the property that there is a constant $M \geq 1$ such that

$$\int_s^{+\infty} \frac{dt}{h(t)} \leq \frac{M}{h(s)}, \quad \text{for all } s \geq 0.$$

Also, $\mathcal{H}_1$ represents the set of the growth rates $h : \mathbb{R}_+ \to [1, +\infty)$ with the property that there exist a growth rate $h_1 : \mathbb{R}_+ \to [1, +\infty)$ and a constant $M_1 \geq 1$ with

$$\int_s^{+\infty} \frac{h_1(t)}{h(t)} dt \leq M_1 \frac{h_1(s)}{h(s)}, \quad \text{for all } s \geq 0.$$

Remark. Denoting by $\mathcal{E}$ the set of functions $h : \mathbb{R}_+ \to [1, +\infty)$, $h(t) = e^{\alpha t}$, with $\alpha > 0$, it results that $\mathcal{E} \subset \mathcal{H} \cap \mathcal{H}_1$.

Remark. The growth rate $h : \mathbb{R}_+ \to [1, +\infty)$ is in $\mathcal{H}_1$ if and only if there exists a growth rate $h_2 : \mathbb{R}_+ \to [1, +\infty)$, defined by $h_2(t) = \dfrac{h(t)}{h_1(t)}$, for all $t \geq 0$ such that $h_2 \in \mathcal{H}$.

A first result concerning the connections between the uniform exponential stability and uniform h-stability of an evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ is

Theorem 2.1. *Following statements are equivalent:*

(i) *Φ is uniformly exponentially stable;*
(ii) *there exists $h \in \mathcal{H}_1$ such that Φ is uniformly h-stable;*
(iii) *there exists $h \in \mathcal{H}$ such that Φ is uniformly h-stable.*

Proof. (1) ⇒ (2). It results for $h(t) = e^{\alpha t}$, with $\alpha > 0$.
(2) ⇒ (3). From the hypothesis, there is a growth rate $h_1 : \mathbb{R}_+ \to [1, +\infty)$ and $M_1 \geq 1$ with

$$\int_s^{+\infty} \frac{h_1(t)}{h(t)} dt \leq M_1 \frac{h_1(s)}{h(s)}, \quad \text{for all } s \geq 0$$

and using the second *Remark* from this section it follows that $h_2 \in \mathcal{H}$.

Thus, for all $(t, s, x) \in \Delta \times X$ we have

$$h_2(t)||\Phi(t, s)x|| = \frac{h(t)}{h_1(t)}||\Phi(t, s)x|| \leq$$

$$\leq S \frac{h(s)}{h_1(t)}||x|| \leq S h_2(s)||x||,$$

which shows that Φ is h_2-stable.
(3) ⇒ (1). It is immediate from the first *Remark* of this section. □

We consider $\Phi : \Delta \to \mathcal{B}(X)$ a strongly measurable evolution operator and a first necessary condition of Datko-type, due to R. Datko ((Datko, 1972)) is

Theorem 2.2. *If $\Phi : \Delta \to \mathcal{B}(X)$ is uniformly h-stable with $h \in \mathcal{H}_1$ then there are a growth rate $h_1 : \mathbb{R}_+ \to [1, +\infty)$ and a constant $D \geq 1$ such that*

$$\int_s^{+\infty} h_1(t)\|\Phi(t, t_0)x_0\|dt \leq Dh_1(s)\|\Phi(s, t_0)x_0\|,$$

for all $(t, s),\ (s, t_0) \in \Delta,\ x_0 \in X$.

Proof. It is immediate for $D = M_1 S$, where M_1 and h_1 are given by definition of $\mathcal{H}_1$ and S is given by Definition 1.3. □

Corollary 2.1. *If $\Phi : \Delta \to \mathcal{B}(X)$ is uniformly exponentially stable, then there are the constants $\beta > 0$ and $D \geq 1$ such that*

$$\int_s^{+\infty} e^{\beta t}\|\Phi(t, t_0)x_0\|dt \leq De^{\beta s}\|\Phi(s, t_0)x_0\|,$$

for all $(t, s),\ (s, t_0) \in \Delta,\ x_0 \in X$.

Proof. It is a particular case of Theorem 2.2. □

Definition 2.1. A mapping $L : \Delta \times X \to \mathbb{R}_+$ is said to be a *h-Lyapunov function for* Φ if

$$L(t, t_0, x_0) + \int_s^t h(\tau)\|\Phi(\tau, t_0)x_0\|d\tau \leq L(s, t_0, x_0),$$

for all $(t, s),\ (s, t_0) \in \Delta$, $x_0 \in X$.

In particular, if $h(t) = e^{\alpha t}$, with $\alpha > 0$, then the function L is called *exponential Lyapunov function*.
The importance of the Lyapunov functions in the study of the stability property is described for instance in (Barreira & Valls, 2008), (Barreira & Valls, 2013).

Another significant result for the uniform h-stability of an evolution operator is given by

Theorem 2.3. *If the evolution operator Φ is uniformly h-stable with $h \in \mathcal{H}_1$, then there exist a growth rate $h_1 : \mathbb{R}_+ \to [1, +\infty)$, a h_1-Lyapunov function for Φ and $D \geq 1$ such that*

$$L(s, s, x_0) \leq Dh_1(s)\|x_0\|,$$

for all $(s, x_0) \in \mathbb{R}_+ \times X$.

Proof. Let $L : \Delta \times X \to \mathbb{R}_+,\ L(t, s, x_0) = \int\limits_t^{+\infty} h_1(\tau)\|\Phi(\tau, s)x_0\|d\tau.$

Thus, L is a h_1-Lyapunov function for Φ and using Theorem 2.2 we obtain

$$L(s, s, x_0) = \int\limits_s^{+\infty} h_1(\tau)\|\Phi(\tau, s)x_0\|d\tau \leq Dh_1(s)\|x_0\|,$$

for all $(s, x_0) \in \mathbb{R}_+ \times X$. □

In particular, we obtain

Corollary 2.2. *If $\Phi : \Delta \to \mathcal{B}(X)$ is uniformly exponentially stable, then there are the constants $\beta > 0$, $D \geq 1$ and an exponential Lyapunov function L for Φ with*

$$L(s, s, x_0) \leq De^{\beta s}\|x_0\|,$$

for all $(s, x_0) \in \mathbb{R}_+ \times X$.

We consider now the set $\tilde{\mathcal{H}}$ of the growth rates $h : \mathbb{R}_+ \to [1, +\infty)$ with the property that there is a growth rate $h_1 : \mathbb{R}_+ \to [1, +\infty)$ and a constant $\tilde{M} \geq 1$ with

$$\int\limits_0^t \frac{h(\tau)}{h_1(\tau)}d\tau \leq \tilde{M}\frac{h(t)}{h_1(t)}, \quad \text{for all } t \geq 0.$$

Remark. It is easy to see that the functions $h \in \mathcal{E}$ (considered in Remark 2) are in $\tilde{\mathcal{H}}$.

Let $\Phi : \Delta \to \mathcal{B}(X)$ be a $*$-strongly measurable evolution operator. A first result for this type of evolution operators is proved by E. A. Barbashin ((Barbashin, 1967)) in the case of uniform exponential stability.
Concerning the uniform h-stability, we prove

Theorem 2.4. *If Φ is uniformly h-stable with $h \in \tilde{\mathcal{H}}$, then there is a growth rate $h_1 : \mathbb{R}_+ \to [1, +\infty)$ and $B \geq 1$ with*

$$\int\limits_0^t \frac{\|\Phi(t, \tau)^* x^*\|}{h_1(\tau)}d\tau \leq \frac{B}{h_1(t)}\|x^*\|,$$

for all $(t, x^*) \in \mathbb{R}_+ \times X^*$.

Proof. It results using Definition 1.3 and the definition of $\tilde{\mathcal{H}}$, for $B = S\tilde{M}$. □

As a consequence of the above result, we obtain

Corollary 2.3. *If Φ is uniformly exponentially stable, then there are the constants $\gamma > 0$ and $B \geq 1$ such that*

$$\int\limits_0^t e^{-\gamma\tau}\|\Phi(t, \tau)^* x^*\|d\tau \leq Be^{-\gamma t}\|x^*\|,$$

for all $(t, x^*) \in \mathbb{R}_+ \times X^*$.

3. Sufficient conditions for uniform *h*-stability

In what follows, we will denote by $\mathcal{H}_2$ the set of the functions $h : \mathbb{R}_+ \to [1, +\infty)$ with the property

$$\sup_{s\geq 0} \frac{h(s+1)}{h(s)} = M_2 < +\infty.$$

Remark. We observe that all the functions $h \in \mathcal{E}$ (defined in Remark 2) are in $\mathcal{H}_2$, i.e. $\mathcal{E} \subset \mathcal{H}_2$.

We consider $\Phi : \Delta \to \mathcal{B}(X)$ a strongly measurable evolution operator and a sufficient criteria of Datko-type is

Theorem 3.1. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform h-growth and $h \in \mathcal{H}_2$. If there is $D \geq 1$ such that*

$$\int_s^{+\infty} h(t)\|\Phi(t,t_0)x_0\|dt \leq Dh(s)\|\Phi(s,t_0)x_0\|,$$

for all $(t,s),\ (s,t_0) \in \Delta,\ x_0 \in X$, then Φ is uniformly h-stable.

Proof. Let $S = M_2^2 ND$.
Case 1. We consider $(t,s),\ (s,t_0) \in \Delta$ with $t \geq s+1$, $x_0 \in X$. Thus,

$$h(t)\|\Phi(t,t_0)x_0\| \leq \int_{t-1}^{t} h(t)\|\Phi(t,\tau)\| \cdot \|\Phi(\tau,t_0)x_0\|d\tau \leq$$

$$\leq N \int_{t-1}^{t} h(t)\frac{h(t)}{h(\tau)}\|\Phi(\tau,t_0)x_0\|d\tau \leq$$

$$\leq NM_2^2 \int_s^{+\infty} h(\tau)\|\Phi(\tau,t_0)x_0\|d\tau \leq S\,h(s)\|\Phi(s,t_0)x_0\|.$$

It results that

$$h(t)\|\Phi(t,t_0)x_0\| \leq S\,h(s)\|\Phi(s,t_0)x_0\|,$$

for all $(t,s),\ (s,t_0) \in \Delta$ with $t \geq s+1$, $x_0 \in X$.

Case 2. Let $(t,s),\ (s,t_0) \in \Delta$ with $t \in [s, s+1]$, $x_0 \in X$. We have

$$h(t)\|\Phi(t,t_0)x_0\| \leq h(t)\|\Phi(t,s)\| \cdot \|\Phi(s,t_0)x_0\| \leq$$

$$\leq N\frac{h^2(t)}{h^2(s)}h(s)\|\Phi(s,t_0)x_0\| \leq S\,h(s)\|\Phi(s,t_0)x_0\|.$$

In conclusion,

$$h(t)\|\Phi(t,t_0)x_0\| \leq S\,h(s)\|\Phi(s,t_0)x_0\|,$$

for all $(t,s),\ (s,t_0) \in \Delta$, $x_0 \in X$, which shows that Φ is uniformly h-stable. □

Corollary 3.1. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform exponential growth. If there is $D \geq 1$ such that*

$$\int_s^{+\infty} e^{\alpha t} \|\Phi(t, t_0)x_0\| dt \leq D e^{\alpha s} \|\Phi(s, t_0)x_0\|,$$

for all $(t, s),\ (s, t_0) \in \Delta,\ x_0 \in X$, then Φ is uniformly exponentially stable.

Proof. It results from Theorem 3.1. □

Theorem 3.2. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform h-growth and $h \in \mathcal{H}_2$. If there exist a h-Lyapunov function for Φ and $D \geq 1$ with*

$$L(s, s, x_0) \leq D h(s) \|x_0\|,$$

for all $(s, x_0) \in \mathbb{R}_+ \times X$, then Φ is uniformly h-stable.

Proof. From Definition 2.1, for $s = t_0$ we obtain

$$\int_s^t h(\tau) \|\Phi(\tau, s)x_0\| d\tau \leq L(s, s, x_0) \leq D h(s) \|x_0\|,$$

for all $(t, s, x_0) \in \Delta \times X$ and for $t \to +\infty$, it follows that Φ is uniformly h-stable. □

In particular, a sufficient condition for the uniform exponential stability is given by

Corollary 3.2. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform exponential growth. If there exist an exponential Lyapunov function for Φ and $D \geq 1$ such that*

$$L(s, s, x_0) \leq D e^{\alpha t} \|x_0\|,$$

for all $(s, x_0) \in \mathbb{R}_+ \times X$, then Φ is uniformly exponentially stable.

A sufficient condition of Barbashin-type for the uniform h-stability of a $*$-strongly measurable evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ is

Theorem 3.3. *We consider Φ an evolution operator with uniform h-growth and $h \in \mathcal{H}_2$. If there is $B \geq 1$ with*

$$\int_0^t \frac{\|\Phi(t, \tau)^* x^*\|}{h(\tau)} d\tau \leq \frac{B}{h(t)} \|x^*\|,$$

for all $(t, x^*) \in \mathbb{R}_+ \times X^*$, then Φ is uniformly h-stable.

Proof. We consider $S = N M_2^2 B$.

Let $(t, s) \in \Delta$, $t \geq s + 1$ and $(x, x^*) \in X \times X^*$. Then,

$$h(t) |< x^*, \Phi(t, s)x > | = \int_s^{s+1} h(t) |< \Phi(t, \tau)^* x^*, \Phi(\tau, s)x > | d\tau \leq$$

$$\leq h(t) \int_s^{s+1} \|\Phi(t,\tau)^* x^*\| \cdot \|\Phi(\tau, s)x\| d\tau \leq$$

$$\leq N h(t) \int_s^{s+1} \frac{\|\Phi(t,\tau)^* x^*\|}{h(\tau)} \frac{h^2(\tau)}{h^2(s)} h(s) d\tau \|x\| \leq$$

$$\leq S h(s) \|x\| \cdot \|x^*\|.$$

Considering the supremum relative to $\|x^*\| \leq 1$ it results that

$$h(t)\|\Phi(t,s)x\| \leq S h(s)\|x\|, \text{ for all } t \geq s+1, \ x \in X.$$

Let now $t \in [s, s+1], \ x \in X$. We obtain

$$h(t)\|\Phi(t,s)x\| \leq N \frac{h^2(t)}{h(s)} \|x\| \leq S h(s)\|x\|,$$

for all $t \in [s, s+1], \ x \in X$.
In conclusion, Φ is uniformly h-stable. □

As a particular case, we obtain

Corollary 3.3. *Let Φ be an evolution operator with uniform exponential growth. If there is $B \geq 1$ with*

$$\int_0^t e^{-\alpha\tau} \|\Phi(t,\tau)^* x^*\| d\tau \leq B e^{-\alpha t} \|x^*\|,$$

for all $(t, x^*) \in \mathbb{R}_+ \times X^*$, then Φ is uniformly exponentially stable.

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Quadratic Equations in Tropical Regions

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Dedicated to B. V. M.

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Abstract

In this note, the reader is invited to a walk through tropical semifields and the places where they border on "ordinary" algebra. Though mostly neglected in today's lectures on algebra, we point to the places where tropical structures inevitably pervade, and show that they frequently occur in ring theory and classical algebra, touching at least functional analysis, and algebraic geometry. Specifically, it is explained how valuation theory, which plays an essential part in classical commutative algebra and algebraic geometry, is essentially tropical. In particular, it is shown that Eisenstein's well-known irreducibility criterion and other more powerful criteria follow immediately by tropicalization. Some applications to algebraic equations in characteric 1, neat Bézout domains, and rings of continuous functions are given.

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1. Introduction

Mathematical ideas quite often originate from natural sciences where experiments help to understand what happens behind reality. In chemistry, the usual method to analyse a matter is by heating until the components begin to separate. "Tropical" mathematics did not quite emerge in that way, but at least one of its founders (Imre Simon) was working on it in the sunny regions of Brazil.

To illustrate the basic process, consider the function

$$a +_p b := (a^p + b^p)^{1/p}$$

for positive real numbers a, b. At "room temperature" ($p = 1$), the function $a +_1 b$ is just ordinary addition in $\mathbb{R}$. Now turn on the heating - proceed until $p \to \infty$ to get the real number system to

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melt. Recall that F. Riesz (Riesz, 1910) made such an experiment already in 1910, which led him to the invention of Lebesgue spaces $L^p(\mathbb{R})$. If p is replaced by the Planck constant $\hbar := \frac{1}{p}$, the limit process $\hbar \to 0$ is known as a *dequantization* (Litvinov, 2006). Indeed, the passage from L^1 to L^∞ bears a certain analogy to the correspondence principle in quantum mechanics (Bohr, 1920).

Now what remains after melting the real number system? For $p = \infty$, ordinary addition $a + b$ in $\mathbb{R}$ turns into $a \vee b := \max\{a, b\}$. The additive group of $\mathbb{R}$ becomes a semigroup, the field $\mathbb{R}$ of real numbers turns into the semifield $\mathbb{R}^+_{\max}$ of *tropical real numbers*, investigated in the 1987 thesis of Imre Simon (Simon, 1987). A remarkable feature of $\mathbb{R}^+_{\max}$ is that its addition is idempotent:

$$a \vee a = a.$$

Thus, if there would exist additive inverses, the whole system would collapse into the zero ring. So is there any reason to regard the elements of $\mathbb{R}^+_{\max}$ as numbers? Before taking up this question seriously, let us content ourselves for the moment with referring back to F. Riesz' early work on L^p-spaces. Here the connection between $p = 1$ and $p = \infty$ is very tight: $L^\infty(\mathbb{R})$ is just the Banach space dual of $L^1(\mathbb{R})$.

Hilbert once placed the number system between the three-dimensional space and the one-dimensional time, saying that numbers are 'two-dimensional'. Such a statement would still have shocked the mathematical community in the days of Euler who called imaginary numbers "impossible" (Euler, 1911). Nowadays, the two-dimensionality is firmly justified by analytical and algebraic reasons, the latter consisting in the algebraic closedness of $\mathbb{C}$. On the other hand, two-dimensionality would not make sense without reference to the base field $\mathbb{R}$ which is "really" fundamental.

In the tropical world, there is no such distinction: the semifield of tropical reals is "algebraically closed". Making this precise is a good exercise and an invitation to be more careful in stating the 'fundamental theorem of algebra'. To be sure, the latter does not mean that *every* complex polynomial has a root - the non-zero constants have to be excluded. This triviality becomes relevant in the wonderland of tropical algebra: there are tropical semifields where (non-constant) linear equations need not be solvable. Roots and solutions of polynomial equations fall apart, and quadratic equations need not be solvable by radicals. On the other hand, every algebraic equation can be reduced to quadratic ones.

In this paper, classical algebra is revisited with regard to tropical structures, and it is shown that they occur at various places. Apart from a revision of semifields of characteristic 1, we add new characterizations for their algebraic closedness (Theorem 6.1). A connection with neat Bézout domains is given in Corollary 2. As a second application, we show that if the semifield of characteristic 1 corresponding to an ℓ-group $\mathscr{C}(X)$ of contiuous functions on a completely regular space X is algebraically closed, the space X must be an F-space, that is, the corresponding ring $C(X)$ of continuous functions is a Bézout ring (Corollary 3).

Another motivation to study semifields of characteristic 1 comes from a recent, highly conjectural branch of arithmetic geometry. Since André Weil sketched his diagonal argument (Weil, 1940, 1941) to tackle the Riemann hypothesis, some research groups eagerly delve under the surface of $\mathbb{Z}$, searching for its "base field" to make $\mathbb{Z}$ (a ring of Krull dimension one) into an algebra over that field (see, e. g., (Connes & Consani, 2010, 2011; Deitmar, 2008; Soulé, 2011)). The way to this non-existing, mysterious, "field" of characteristic 1 inevitably leads through the tropical region. By Proposition 2.2, this hot region is nothing else than the vast and well-developed theory of lattice-ordered abelian groups.

2. The forgotten characteristic

To include the result of a dequantization, we are advised to consider semifields instead of fields. More generally, a *semiring* is an abelian monoid $(A; +, 0)$ with a multiplicative monoid structure $(A; \cdot, 1)$ satisfying the distributive laws and $a \cdot 0 = 0 \cdot a = 0$ for all $a \in A$. If the group of (multiplicatively) invertible elements, the *unit group* $A^\times$, coincides with $A \smallsetminus \{0\}$, we call A a *semi-skewfield*. If, in addition, the multiplicative monoid is commutative, A is said to be a *semi-field*. For example, the above mentioned $\mathbb{R}^+_{\max}$ is a semifield.

A *morphism* in the category of semirings is a map $f\colon A \to B$ which satisfies

$$f(a+b) = f(a) + f(b), \qquad f(0) = 0$$
$$f(a \cdot b) = f(a) \cdot f(b), \qquad f(1) = 1.$$

Like in the category of rings, there is an initial object, the semiring $\mathbb{N}$ of non-negative integers: For any semi-ring A there is a unique morphism $c\colon \mathbb{N} \to A$. The image of c is the intersection of all sub-semirings of A, the *prime semiring* of A. Similarly, every semi-skewfield A contains a smallest sub-semi-skewfield. If it coincides with A, we call A a *prime semi-skewfield*.

In general, the kernel $\operatorname{Ker} c := \{n \in \mathbb{N} \mid c(n) = 0\}$ is not of the form $\mathbb{N}p$ for some $p \in \mathbb{N}$. For example, $I := \mathbb{N} \smallsetminus \{1, 2, 4, 7\}$ is an ideal of the semiring $\mathbb{N}$ which occurs, e. g., as the grading of a simple curve singulatity (Greuel & Knörrer, 1985). Thus $\mathbb{N}/I$ is a finite semiring with $\operatorname{Ker}(c) = I$. On the other hand, there exist congruence relations on $\mathbb{N}$ which do not come from an ideal, even if A is a semifield. For example, let $\mathbb{B} := \{0, 1\}$ be the semifield with $1 + 1 = 1$. Then $c\colon \mathbb{N} \twoheadrightarrow \mathbb{B}$ satisfies $c(n) = 1$ for $n \neq 0$. So c has a trivial kernel, while it is far from being a monomorphism.

Note that $\mathbb{B}$ is the prime a sub-semifield of $\mathbb{R}^+_{\max}$. Therefore, we write $a \vee b$ for the addition in $\mathbb{B}$. So $\mathbb{B}$ is a Boolean algebra with $a \wedge b := ab$. The reader will notice that $\mathbb{B}$ can be derived from the prime field $\mathbb{F}_2$ via $a \vee b = a + b + ab$, but not vice versa.

Definition 2.1. We define the *characteristic* $\operatorname{char} A$ of a semiring A to be the smallest integer $p > 0$ with $c(n + p) = c(n)$ for some $n \in \mathbb{N}$. If such an integer p does not exist, we set $\operatorname{char} A := 0$.

In analogy to the theory of skew-fields, we have (cf. (Rump, 2015), Proposition 1)

Proposition 2.1. *Every prime semi-skewfield is a semifield. Up to isomorphism, the prime semi-fields are* $\mathbb{Q}^+$, $\mathbb{B}$, *and* $\mathbb{F}_p$ *for rational primes p. In particular, the prime semifields are determined by their characteristic.*

Proof. Let F be a prime semi-skewfield. Assume first that $\operatorname{char} F = 0$. Then $\mathbb{N}$ can be regarded as a sub-semiring of F. Every non-zero $n \in \mathbb{N}$ has an inverse $\frac{1}{n}$ in F which commutes with all elements of $\mathbb{N}$. Hence $\{\frac{m}{n} \mid m, n \in \mathbb{N}, n > 0\}$ is a sub-semifield isomorphic to the positive cone $\mathbb{Q}^+$ of $\mathbb{Q}$.

Now assume that $p := \operatorname{char} F \neq 0$. Then there is an integer $n \in \mathbb{N}$ with $c(n) + c(p) = c(n)$. As this equation holds for almost all n, we can assume that n is a multiple of p. Adding multiples of $c(p)$ on both sides, the equations obtained in this way imply that $c(n) + c(n) = c(n)$. If $c(n) = 0$, then $c(p) = 0$, and the usual argument shows that $c(\mathbb{N}) \cong \mathbb{F}_p$ for a prime p. Otherwise, we obtain $c(1) + c(1) = c(1)$, which yields $c(\mathbb{N}) \cong \mathbb{B}$. □

So the possible prime semifields are

$$\mathbb{Q}^+, \mathbb{B}, \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7, \ldots,$$

including the prime fields $\mathbb{F}_p$ and a natural sub-semifield of $\mathbb{Q}$. Note that formally, $\mathbb{Q}^+$ carries more information than $\mathbb{Q}$: The positive cone provides $\mathbb{Q}$ with its natural ordering. Thus, $\mathbb{Q}^+$ connects arithmetic (the semiring $\mathbb{N}$) with algebra and analysis (the ordered field $\mathbb{Q}$ and its completion $\mathbb{R}$), while the newcomer $\mathbb{B}$ bridges the gap between algebra and logic.

Every semifield contains one of the prime semifields according to its characteristic. For fields, this is a well-known piece of algebra. So the question arises how the "logical" semi-skewfields, those containing $\mathbb{B}$, look like. By Proposition 2.1, they are of characteristic 1, which means that they satisfy the equation $1+1 = 1$. Recall that a partially ordered group is said to be *lattice-ordered* or an *ℓ-group* if the partial order is a lattice. For the theory of ℓ-groups, the reader is referred to (Anderson & Feil, 1988; Bigard *et al.*, 1977; Darnel, 1995; Glass, 1999). The commutative case of the following result is due to Weinert and Wiegandt (Weinert & Wiegandt, 1940). Similar ideas have been developed independently by several authors (see (Castella, 2010; Lescot, 2009), and the literature cited there).

Proposition 2.2. *Up to isomorphism, there is a one-to-one correspondence between ℓ-groups and semi-skewfields of characteristic 1.*

Proof. Note first that a semi-skewfield F is of characteristic 1 if and only if $a + a = a$ holds for all $a \in F$. Then it easily checked that

$$a \leqslant b :\Longleftrightarrow a + b = b \tag{2.1}$$

makes F into a $\vee$-semilattice with $a \vee b := a + b$. Furthermore, the distributivity shows that $F^\times$ is an ℓ-group. Conversely, every ℓ-group G can be made into a semi-skewfield $\widetilde{G} := G \sqcup \{0\}$ by adjoining a smallest element 0 with $0a = a0 = 0$ for all $a \in \widetilde{G}$. Since $\widetilde{G}^\times = G$ and $\widetilde{F^\times} = F$, the correspondence is bijective. □

In particular, semifields of characteristic 1 are equivalent to abelian ℓ-groups, and our prime semifield $\mathbb{B}$ corresponds to the ℓ-group of order one. For those who would like to prove the Riemann hypothesis, we should add that $\mathbb{B}$ is not identical with the desperately sought field $\mathbb{F}_1$ - it is still "too big"!

3. Tropical semi-domains

To study field extensions, one has to understand polynomial rings first. Thus, in characteristic 1, we have to deal with polynomials over the semifield $\widetilde{G}$ of an abelian ℓ-group G. For an arbitrary field K, there are many integral domains with quotient field K. If K is an algebraic number field, there is a canonical subring $\mathcal{O}$ - the ring of integers - with quotient field K. Similarly, any semifield $\widetilde{G}$ of characteristic 1 has a canonical sub-semiring $\widetilde{G^-} := G^- \sqcup \{0\}$, where G^- is the negative cone of G. (Since 0 is the smallest element of $\widetilde{G}$, the cone that touches 0 is the negative one.)

Definition 3.1. We define a *semi-domain* to be a commutative semiring A satisfying $ac = bc \Rightarrow a = b$ for $a, b, c \in A$ with $c \neq 0$. We call A *tropical* if there exists an abelian ℓ-group G with $A = \widetilde{G^-}$.

In particular, a semi-domain has no zero-divisors. An intrinsic description of tropical semi-domains is obtained as follows. Recall that a *hoop* (Blok & Ferreirim, 2000) is a commutative monoid H with a binary operation $\rightarrow$ such that the following are satisfied for all $a, b, c \in H$:

$$a \rightarrow a = 1$$
$$ab \rightarrow c = a \rightarrow (b \rightarrow c)$$
$$(a \rightarrow b)a = (b \rightarrow a)b.$$

Every hoop is a $\wedge$-semilattice with respect to the *natural* partial order

$$a \leqslant b \;:\Longleftrightarrow\; \exists c \in H\colon a = cb \;\Longleftrightarrow\; a \rightarrow b = 1.$$

A hoop is called *self-similar* (Rump, 2008) if it is cancellative. (For an explanation of the terminology and equational characterizations, see (Rump, 2008), Proposition 5.) Every self-similar hoop H has a group of fractions, the *structure group* $G(H)$ of H, which consists of the fractions $a^{-1}b$ with $a, b \in H$.

Proposition 3.1. *Up to isomorphism, there is a one-to-one correspondence between*
(a) *semifields of charatceristic 1,*
(b) *tropical semi-domains,*
(c) *abelian ℓ-groups, and*
(d) *self-similar hoops.*

Proof. The equivalence between (a) and (c) follows by Proposition 2.2, while the equivalence between (b) and (c) is obvious. For an abelian ℓ-group G, we define

$$a \rightarrow b := ba^{-1} \wedge 1$$

for $a, b \in G^-$. By (Rump, 2008), Section 5, this makes G^- into a self-similar hoop with structure group G. Conversely, the structure group $G(H)$ of a self-similar hoop H is an abelian ℓ-group with $G(H)^- = H$ by (Rump, 2008), Proposition 19. □

Note that Proposition 3.1 implies that a self-similar hoop H is a lattice. Explicitly, the join is given by the formula

$$a \vee b = (a \rightarrow b) \rightarrow b$$

which is well known from the theory of BCK algebras (Iséki & Tanaka, 1978).

The concept of Grothendieck group (Lang, 1965) extends to semirings as follows.

Definition 3.2. Let A be a commutative semiring. We define an *ideal* of A to be an additive submonoid I which satisfies

$$a \in A,\ b \in I \implies ab \in I. \tag{3.1}$$

We say that an ideal P is *prime* if $A \setminus P$ is a submonoid of A.

Let I be an ideal of a commutative semiring. Then

$$a \sim b \ :\Longleftrightarrow\ \exists\, c \in I\colon a + c = b + c$$

is an equivalence relation, and it is easily checked that it is a congruence relation. So the equivalence classes form a commutative semiring A/I, the *factor semiring* modulo I. There is also a concept of localization.

Proposition 3.2. *Let P be a prime ideal of a commutative semiring A. There exists a morphism $q\colon A \to A_P$ of semirings with $q(A \setminus P) \subset A_P^\times$ such that every morphism $f\colon A \to B$ of semirings with $f(A \setminus P) \subset B^\times$ factors uniquely through q.*

Proof. Define an equivalence relation on the multiplicative monoid $A \times (A \setminus P)$:

$$(a, b) \sim (c, d) \ :\Longleftrightarrow\ \exists\, s \in A \setminus P\colon ads = bcs. \tag{3.2}$$

Then $x \sim y$ implies $xz \sim yz$ for all $x, y, z \in A \times (A \setminus P)$. So $\sim$ is a congruence relation on $A \times (A \setminus P)$. As usual, we write $\frac{a}{b}$ for the equivalence class of (a, b). So the equivalence classes form a commutative monoid A_P with a morphism $q\colon A \to A_P$ given by $q(a) := \frac{a}{1}$. Moreover, $q(A \setminus P) \subset A_P^\times$. Furthermore, it is easily checked that

$$\frac{a}{b} + \frac{c}{d} := \frac{ad + bc}{bd}$$

is well defined and makes A_P into a commutative semiring such that q becomes a morphism of semirings. Now the universal property is straightforward. □

We call A_P the *localization* of A at P. If the zero ideal is prime, the localization at 0 yields the *quotient semifield* $K(A)$ of A.

Note that there are semirings A where 0 is prime, but A is not a semi-domain. For example, let K be a semifield. We define a *(formal) polynomial* to be an expression

$$f = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

with $a_i \in K$. If $f \neq 0$, say, $a_n \neq 0$, we call $\deg f := n$ the *degree* of f. Thus, with the usual operations, the formal polynomials make up a semiring $K\langle x \rangle$, and 0 is a prime ideal. To see that $K\langle x \rangle$ need not be a semidomain, consider the case $\operatorname{char} K = 1$, that is, $K = \widetilde{G}$ for an abelian ℓ-group G. Consider two elements $a, b \in G$ with $a \nleqslant b$. Then the two formal polynomials $a \vee bx \vee x^2$ and $a \vee (a \vee b)x \vee x^2$ are distinct. However,

$$(a^2 \vee bx \vee x^2)(a \vee x) = (a^2 \vee (a \vee b)x \vee x^2)(a \vee x),$$

which shows that $\widetilde{G}\langle x\rangle$ fails to be a semi-domain! That is the reason why we speak of *formal* polynomials.

If A is a semidomain, the equivalence (3.2) simplifies to

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc,$$

which implies that all localizations A_P can be regarded as sub-semidomains of $K(A)$.

Example. Let A be a semidomain of characteristic 1. The quotient semifield $K(A)$ is of the form $K(A) = \widetilde{G}$ with an abelian ℓ-group G, and the monoid $A \smallsetminus \{0\} = A \cap G$ is a $\vee$-sub-semilattice. However, $A \cap G$ need not be the negative cone of G. Indeed, this happens if and only if A is tropical. Assume this from now on. By Definition 3.2, an ideal of A is the same as a $\vee$-sub-semilattice which is a downset. So the complement $Q := A \smallsetminus P$ of a prime ideal P of A is a convex submonoid of G^- with the property

$$a \vee b \in Q \implies a \in Q \ \text{ or } \ b \in Q,$$

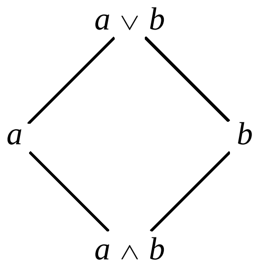
that is, Q is the negative cone of a prime ℓ-ideal in G (see (Darnel, 1995), Definitions 8.1 and 9.1). In other words, there is a one-to-one correspondence between prime ideals of A and prime ℓ-ideals of G. According to (Darnel, 1995), Proposition 14.3, the prime ideals of A can be identified with the prime filters of the negative cone G^- (with the reverse ordering). Note that the zero ideal of A corresponds to G, the "trivial" prime ℓ-ideal of G, which should not be excluded from the prime spectrum of G.

Definition 3.3. Let K be a semifield. The elements of the quotient semifield $K(x)$ of $K\langle x\rangle$ will be called *rational functions* in x. We write $K[x]$ for the image of the natural map $K\langle x\rangle \to K(x)$ and call the elements of $K[x]$ *polynomials* in x.

4. Divisors in characteristic 1

In classical algebraic geometry, divisors are intimately connected with line bundles, invertible sheaves, linear systems, and embeddings into projective spaces. Therefore, they play a decisive rôle. Here we shall study their behaviour in characteristic 1.

Thus, let G be an abelian ℓ-group. As a lattice, G is distributive. So the elements of G can be regarded as functions on a set. Let us take the simplest case where G satisfies the ascending chain condition. By a theorem of Birkhoff (Birkhoff, 1942), this implies that G is a cardinal sum $G = \boxplus_{p \in P} \mathbb{Z}$ with basis P. (Such ℓ-groups naturally arise as groups of fractional ideals of a Dedekind domain.) So each interval $[a, b] := \{c \in G \mid a \leqslant c \leqslant b\}$ has a composition series $a = c_0 < c_1 < \cdots < c_n = b$ with atomic intervals $[c_i, c_{i+1}] = \{c_i, c_{i+1}\}$. For a diagram



(4.1)

with $a, b \in G$, the intervals $[a \wedge b, a]$ and $[b, a \vee b]$ are said to be *isomorphic*, in analogy with the isomorphism theorem in group theory. *Isomorphism* between intervals is then defined by finite sequences of elementary isomorphisms (4.1). So each pair $a, b \in G$ can be connected by a finite chain $a = c_0, c_1, \ldots, c_n = b$ in G, with atomic intervals $[c_i, c_{i+1}]$ or $[c_{i+1}, c_i]$. If we attach a factor -1 to the intervals of the second type, the total count of isomorphism classes of atomic intervals on such a connecting path merely depends on the pair of endpoints a, b. Regarding the isomorphism classes of atomic intervals as "points", every element $a \in G$ is completely determined by the formal $\mathbb{Z}$-linear combination of points encountered on a path between 0 and a which is independent of the chosen path. For algebraic curves, a formal $\mathbb{Z}$-linear combination of points is called a *divisor*.

In general, there are no atomic intervals. So we have to watch out for a substitute. This naturally leads to the following

Definition 4.1. Let G be a (multiplicative) abelian ℓ-group, and let D be the subgroup of the free abelian group $\mathbb{Z}^{(G)}$ generated by the elements

$$(a \vee b) + (a \wedge b) - a - b$$

with $a, b \in G$. The factor group $\operatorname{Div}(G) := \mathbb{Z}^{(G)}/D$ will be called the group of *divisors* of G. The natural map $G \to \operatorname{Div}(G)$ will be denoted by $a \mapsto [a]$.

In the special case of a noetherian group G, it is clear that the homomorphism $G \to \operatorname{Div}(G)$ is injective. In general, this follows since $G \to \operatorname{Div}(G)$ admits a retraction $\operatorname{Div}(G) \to G$, given by the map

$$n_1[a_1] + \cdots + n_r[a_r] \mapsto a_1^{n_1} \cdots a_r^{n_r}.$$

The retraction is well defined by virtue of the equation

$$(a \vee b)(a \wedge b) = ab,$$

which holds in every abelian ℓ-group. However, even for $G = \mathbb{Z}$, the embedding

$$G \hookrightarrow \operatorname{Div}(G)$$

is far from being surjective. Instead, the group $\operatorname{Div}(G)$ tells us much about the polynomial semidomain $\widetilde{G}[x]$.

Let $G(x) := \widetilde{G}(x)^\times$ be the abelian ℓ-group which is freely generated by G and a single indeterminate x. Similarly, we set $G[x] := \widetilde{G}[x] \cap G(x)$. The degree of non-zero polynomials extends to a homomorphism

$$\deg\colon G(x) \to \mathbb{Z}.$$

of abelian ℓ-groups. For ordinary fields K, the degree function $\deg\colon K(x)^\times \to \mathbb{Z}$ is also important, but it is not a homomorphism of rings. So the degree of a polynomial or rational function in classical algebra signalizes a tropical structure!

The reader may check that

$$\big(x \vee (a \vee b)\big)\big(x \vee (a \wedge b)\big) = (x \vee a)(x \vee b)$$

holds for all $a, b \in G$. To generalize this fact, recall that an abelian ℓ-group G is *divisible* if every $a \in G$ admits an n-th root for each positive integer n, or equivalently, the pure equation

$$x^n = a$$

is solvable for any $a \in G$. (If G is written additively, this just means that G can be regarded as a $\mathbb{Q}$-vector space.) Now we have ((Rump, 2015), Theorem 1):

Fundamental theorem for abelian ℓ-groups. *Let G be a divisible abelian ℓ-group, and let $K := \widetilde{G}$ be the corresponding tropical semifield. Every non-zero polynomial $f \in K[x]$ has a unique factorization*

$$f = a(x \vee d_1)(x \vee d_2) \cdots (x \vee d_n) \tag{4.2}$$

with $a \in G$ and $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ in K.

For $K = \mathbb{R}^+_{\max}$, this theorem is known as the “fundamental theorem of tropical algebra” (see, e. g., (Cuninghame-Green & Meijer, 1980)). Two things are remarkable. First, the *roots* $d_1 \leqslant \cdots \leqslant d_n$ have to be put into linear order - otherwise, they won’t be unique. The roots of a polynomial are in fact nothing else than its divisor. So in contrast to divisors of algebraic curves, tropical divisors are not unique as unordered point sets with multiplicities. For the divisor $[a]+[b]$, the equivalence to $[a \vee b] + [a \wedge b]$ can be seen from the basic relation of Definition 4.1.

Secondly, the roots $d_1 \leqslant \cdots \leqslant d_n$ are not the zeros, because no non-zero polynomial $f \in K[x]$ satisfies $f(a) = 0$ for any $a \in G$. Only equations $f(x) = g(x)$ for a pair of polynomials are sensible! So the question whether polynomial equations can be solved in G is not answered by the fundamental theorem. We will come back to this in Section 5.

By the fundamental theorem, there is a well-defined map

$$\mathrm{div}\colon G[x] \to \mathrm{Div}(G^d) \tag{4.3}$$

for any abelian ℓ-group G with divisible closure G^d, given by

$$\mathrm{div}(f) := [d_1] + [d_2] + \cdots + [d_n]$$

for a non-zero polynomial (4.2). Every rational function $f \in G(x)$ can be written as

$$f = ax^{n_0}(x \vee d_1)^{n_1}(x \vee d_2)^{n_2} \cdots (x \vee d_r)^{n_r} \tag{4.4}$$

with $a, d_1, \ldots, d_r \in G$, and $n_0, \ldots, n_r \in \mathbb{Z}$. In contrast to polynomials where $n_1, \ldots, n_r \in \mathbb{N}$, the d_i cannot be put into linear order, which means that they are not unique! However, a and n_0 are unique. So let $G(x)^0$ denote the subgroup of rational functions $f \in G(X)$ with $a = 1$ and $n_0 = 0$. Then (Rump, 2015), Theorem 2, yields

Theorem 4.1. *Let G be a divisible abelian ℓ-group. The map (4.3) extends uniquely to a group isomorphism*

$$\mathrm{div}\colon G(x)^0 \xrightarrow{\sim} \mathrm{Div}(G)$$

with inverse map $[a] \mapsto (x \vee a)$.

This gives a complete description of the divisor group $\mathrm{Div}(G)$ and its relationship to the unit group of $\widetilde{G}(x)$, namely,

$$G(x) \cong G \times \mathbb{Z} \times G(x)^0.$$

5. Dequantization of Prüfer and Bézout domains

Proposition 3.1 suggests a study of abelian ℓ-groups via semi-domains. A first step of this program has already been taken in Section 3, where a decomposition of polynomials into linear factors has been achieved. Now let us come "back to the roots". The good news is that they are most easily calculated from the coefficients. For an abelian ℓ-group G and a polynomial $f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \widetilde{G}[x]$ with $a_0a_n \neq 0$, it is not hard to show that all coefficients a_i can be assumed to be non-zero, that is, they belong to G. (This is of course not true for polynomials over a field, but note that in the tropical case, the zero element is the absolutely smallest one, smaller than every element of G.) By (Rump, 2015), Propositions 3 and 4, we have the following explicit formula for the roots: $d_i = b_{i-1}b_i^{-1}$, where

$$b_j := a_j \vee \bigvee_{i<j<k} (a_i^{k-j}a_k^{j-i})^{\frac{1}{k-i}}.$$

So the roots d_i of each polynomial are expressible in terms of k-th roots, where k does not exceed the degree of the polynomial f. Compared with the efforts of classical algebra up to the final stroke after Ruffini, Abel, and Galois - a quick victory!

However, as already mentioned, roots are not solutions. Nevertheless, the decomposition into linear factors indicates a close relationship to classical solutions. Indeed, here is a point where tropical algebra applies to the classical case.

Recall that a *fractional ideal* of an integral domain R with quotient field K is a non-zero R-submodule I of K such that $I \subset Ra$ for some $a \in K^\times$. A fractional ideal I is said to be *invertible* if there is a (necessarily unique) fractional ideal I^{-1} with $I^{-1}I = R$. Note that every invertible fractional ideal is finitely generated. An integral domain R is said to be a *Prüfer domain* (see (Gilmer, 1992), chap. IV) if the non-zero finitely generated ideals are invertible. If every non-zero finitely generated ideal of R is principal (hence invertible), R is called a *Bézout domain*.

The invertible fractional ideals of a Prüfer domain R form an abelian ℓ-group $G(R)$ with respect to inclusion. Note that

$$(I + J)(I \cap J) = IJ$$

holds for $I, J \in G(R)$, which shows that $G(R)$ is closed under finite intersection. In the special case that R is a Bézout domain, $(G(R); \supset)$ can be identified with $K^\times/R^\times$, the *group of divisibility* of R (see (Gilmer, 1992), section 16).

For a Prüfer domain R, the finitely generated ideals form a tropical semi-domain $A(R)^-$, the *dequantization* of R. By Proposition 3.2 and the Jaffard-Ohm correspondence (Jaffard, 1953; Ohm, 1966), every tropical semi-domain occurs as the dequantization of a Bézout domain. Thus, tropical algebra makes no difference between Prüfer domains and the more special Bézout domains. Since $A(R)^-$ is a semi-domain, we consider its quotient semifield $A(R)$, consisting of all finitely generated R-submodules of K. There is a natural map

$$t\colon K \to A(R) \tag{5.1}$$

from the quotient field K of R to $A(R)$, given by $t(a) := Ra$. Note that t is a monoid homomorphism, but not a morphism of semirings since $R(a + b)$ need not be equal to $Ra + Rb$.

This is by no means an anomaly. To the contrary, here is another point where tropical concepts enter the classical world. Recall that a *valuation* of a field K is a function $v\colon K \to \Gamma$ into a totally ordered abelian group Γ, augmented by an element ∞ with $\alpha + \infty = \infty$ for all $\alpha \in \Gamma \sqcup \{\infty\}$ such that the following are satisfied:

$$v(a) = \infty \iff a = 0 \tag{5.2}$$

$$v(ab) = v(a) + v(b) \tag{5.3}$$

$$v(a+b) \geqslant \min\{v(a), v(b)\}. \tag{5.4}$$

In a time where order-theoretic terms have been almost completely eliminated from the standard curriculum[1], such a function v which is not a morphism in any sense should sting in the eye! Let us rewrite (5.2)-(5.4) as follows. Endow Γ with the opposite order and write it multiplicatively. Then ∞ becomes 0 with $\alpha \cdot 0 = 0$ for all $\alpha \in \Gamma \sqcup \{0\}$, and the inequality (5.4) turns into

$$v(a+b) \leqslant v(a) \vee v(b).$$

So $\tilde{\Gamma} := \Gamma \sqcup \{0\}$ becomes a tropical semifield. The map (5.1) is characterized by the following universal property:

Proposition 5.1. *Let R be a Prüfer domain with quotient field K. Then every valuation $v\colon K \to \tilde{\Gamma}$ with $v(R) \leqslant 1$ factors uniquely through $t\colon K \to A(R)$*

$$\begin{array}{ccc} K & \xrightarrow{\;t\;} & A(R) \\ & {\scriptstyle v}\searrow & \vdots\, {\scriptstyle f} \\ & & \tilde{\Gamma} \end{array} \tag{5.5}$$

such that $f\colon A(R) \to \tilde{\Gamma}$ is a morphism of semifields.

Proof. Define $f\colon A(R) \to \tilde{\Gamma}$ by $f(I) := \bigvee\{v(a) \mid a \in I\}$. Since every $I \in A(R)$ is of the form $I = Ra_1 + \cdots + Ra_n$, every $a = r_1a_1 + \cdots + r_na_n \in I$ with $r_i \in R$ satisfies $v(a) \leqslant v(a_1) \vee \cdots \vee v(a_n)$, which shows that f is well defined and renders (5.5) into a commutative diagram. The uniqueness of f is obvious. □

For an abelian ℓ-group G, the pure polynomial $1 \vee x^n$ is "purely inseparable":

$$1 \vee x^n = (1 \vee x)^n.$$

Therefore, the Frobenius identity

$$(a \vee b)^n = a^n \vee b^n$$

[1]It seems that Grothendieck's aversion against valuations had its bearing on this. In a letter of October 26, 1961, Serre complained: "You are very harsh on Valuations! I persist nonetheless in keeping them, for several reasons ...". Grothendieck's unrepentant response (October 31, 1961): "Your argument in favor of valuations is pretty funny ..."

holds in G, and (Darnel, 1995), 47.11, implies that G is a subdirect product of linearly ordered abelian groups. Thus, for a Prüfer domain R, the diagram (5.5) can be expressed by a single map

$$K^\times \xrightarrow{t} A(R)^\times \hookrightarrow \prod \Gamma,$$

where Γ runs through the value groups of all valuations of R. Moreover, t is surjective if and only if R is a Bézout domain. Examples of Bézout domains abound. The most prominent examples are the ring of algebraic integers ((Kaplansky, 1974), Theorem 102) and the ring of entire functions (Helmer, 1940). The ring $\mathrm{Int}(\mathbb{Z})$ of integer-valued polynomials $f \in \mathbb{Q}[x]$ is an example of a Prüfer domain which is not a Bézout domain (Brizolis, 1979) (cf. (Narkiewicz, 1995), VII). In contrast to $\mathbb{Z}[x]$, which is not a Prüfer domain, $\mathrm{Int}(\mathbb{Z})$ has an uncountable number of maximal ideals, while both rings have Krull dimension 2.

The valuations $v\colon R \to \widetilde{\Gamma}$ or rather their extensions

$$v\colon K \to \widetilde{\Gamma}$$

to K are just the components of the tropicalization t. Thus, if V is a valuation domain with quotient field K, the corresponding valuation is just the tropicalization

$$t\colon K \to A(V),$$

and $A(V)^\times$ is the value group of V. There is a natural extension $t'\colon K[x] \to A(V)[x]$ via $t'(x) := x$. Explicitly:

$$t'(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = t(a_0) \vee t(a_1)x \vee t(a_2)x^2 \vee \cdots \vee t(a_n)x^n.$$

Note that $K[x]$ is even a principal ideal domain. We add a prime to make sure that t' cannot be confused with the restriction of $t\colon K(x) \to A(K[x])$ to $K[x]$.

For higher rank valuations, Hensel's lemma, which roughly states that coprime factorizations of polynomials modulo the maximal ideal can be lifted, is no longer valid (see (Engler & Prestel, 2005), Remark 2.4.6). What remains is that the topology of a field K with a complete valuation extends uniquely to fields L which are finite over K (Roquette, 1958). The proper substitute for complete valuation rings (where Hensel's lemma merely holds in rank 1) are the *Henselian* local rings, introduced by Azumaya (Azumaya, 1951) and developed by Nagata (Nagata, 1962), which satisfy Hensel's lemma by definition. For equivalent characterizations, see (Ribenboim, 1985). The most important characterization of Henselian local integral domains is that every integral extension is local ((Nagata, 1954), Theorem 7). For Henselian valuations of a field K, this means that they uniquely extend to the algebraic closure $\overline{K}$.

Proposition 5.2. *Let V be a Henselian valuation domain with quotient field K. Then $t\colon K \to A(V)$ extends uniquely to the algebraic closure $\overline{K}$ of K, which gives a commutative diagram*

$$\begin{array}{ccc} K & \xrightarrow{t} & A(V) \\ \downarrow & & \downarrow \\ \overline{K} & \xrightarrow{t} & A(V)^d. \end{array}$$

For every non-zero polynomial $f \in K[x]$ with roots $\alpha_1, \dots, \alpha_n \in \overline{K}$, the roots of $t'(f)$ are $t(\alpha_1), \dots, t(\alpha_n)$.

Proof. Since V is Henselian, the integral closure S of V in $\overline{K}$ is local, hence a valuation ring ((Bourbaki, 1972), VI.8.6, Proposition 6). Furthermore, $A(S)$ can be identified with the divisible closure of $A(V)$. If a is the leading coefficient of f, we have $f = a(x - \alpha_1) \cdots (x - \alpha_n)$ in $\overline{K}[x]$. Since t' is multiplicative, this implies that $t'(f) = t(a)(x \vee t(\alpha_1)) \cdots (x \vee t(\alpha_n))$. As $A(V)^d$ is linearly ordered, this proves the claim. □

Proposition 5.2 is the basis for Newton's method, which makes use of the following result. Its first part is essentially due to Ostrowski (Ostrowski, 1935).

Proposition 5.3. *Let V be a Henselian valuation domain with quotient field K, and let $f = a_0 + a_1x + \cdots + a_nx^n \in K[x]$ be a non-zero polynomial. If f is irreducible, $t'(f)$ has a single root in $A(V)^d$. Conversely, if $t'(f)$ has a single root in $A(V)^d$, and there is no divisor $d > 1$ of n such that $A(V)^\times$ contains a d-th root of $t(a_0a_n^{-1})$, then f is irreducible.*

Proof. Let S be the integral closure of V in the splitting field L of f. Every element σ of the Galois group $G(L|K)$ leaves S invariant: $\sigma(S) = S$. Hence, if f is irreducible, every zero α of f satisfies $t(\sigma(\alpha)) = t(\alpha)$ for all $\sigma \in G(L|K)$. So there is a single root $t(\alpha)$ of $t'(f)$ of multiplicity $\deg f$.

Conversely, assume that $t'(f)$ has a single root in $A(V)^d$, and that there is no divisor $d > 1$ of n such that $A(V)^\times$ contains a d-th root of $t(a_0a_n^{-1})$. Let g be a monic irreducible factor of f. Without loss of generality, we can assume that $a_n = 1$. Then the single root α of $t'(f)$ satisfies $t'(f) = (x \vee \alpha)^n$ and $\alpha^n = t(a_0)$. If g is of degree m, then $t'(g) = (x \vee \alpha)^m$. Let $d > 0$ be the greatest common divisor of m and n. Then $d = pm + qn$ for some integers $p, q \in \mathbb{Z}$. Hence $h := (x \vee \alpha)^d = t'(g)^p t'(f)^q \in A(V)[x]$, and $d|m$ implies that $t'(g) = h^{m/d}$. Furthermore, the absolute term $a := \alpha^d$ of h belongs to $A(V)^\times$, and $a^{n/d} = t(a_0)$. By assumption, this gives $d = n$. Whence $f = g = h$ is irreducible. □

Proposition 5.3 reduces irreducibility of polynomials over K almost completely to the tropical semifield $A(V)$, where the complete factorization is obtained by straightforward calculation. Contrary to a remark in (Khanduja & Saha, 1997), the condition of the criterion is not necessary, as the trivial example $1 + x + x^2 \in \widehat{\mathbb{Q}}_2[x]$ shows. (The mistake is caused by rewriting the special version of Popescu and Zaharescu (Popescu & Zaharescu, 1995) in a logically different way.) In particular, we have the following

Corollary. *Let V be a Henselian valuation domain with quotient field K, and let $f = a_0 + a_1x + \cdots + a_nx^n \in K[x]$ be a non-zero polynomial. If $t'(f)$ has m distinct roots, f splits into m relatively prime factors.*

Newton's method was applied already in the early days of valuation theory, invented by Hensel (Hensel, 1908), and developed by Kürschák (Kürschák, 1913*a*; Kürschák, 1913*b*), Ostrowski (Ostrowski, 1916, 1917, 1933), and Rychlík (Rump & Yang, 2008; Rychlík, 1924). Newton's method also appears in a paper of Rella (Rella, 1927), but in essence, it can even be traced back to Newton himself via Puiseux's theorem which states, in modern terms, that the field of Puiseux series over

$\overline{\mathbb{C}}$ is the algebraic closure of the field $\mathbb{C}((x))$ of formal Laurent polynomials, the quotient field of $\mathbb{C}[[x]]$.

Here the field $\mathbb{C}((x))$ not only builts a bridge between algebraic curves and complex analysis; in addition, it is maximally close to its tropical shadow: Every finite extension field of $\mathbb{C}((x))$ is isomorphic to $\mathbb{C}((x))$, the extension being just given be extracting some n-th root of x. So if S denotes the the integral closure S of $\mathbb{C}[[x]]$ in the algebraic closure of $\mathbb{C}((x))$, the tropical picture is encoded in the commutative diagram

$$\begin{array}{ccc} \mathbb{C}((x)) & \xrightarrow{t} & A(\mathbb{C}[[x]]) = \mathbb{Z} \\ \big\downarrow & & \big\downarrow \\ \overline{\mathbb{C}((x))} & \xrightarrow{t} & A(S) = \mathbb{Q}. \end{array}$$

A lot of irreducibility criteria can be derived from Proposition 5.3, which seems to be the "true metaphysics"[2] behind polynomial factorization. Eisenstein's criterion is just the first of a series of irreducibility criteria (e. g., (Dumas, 1906; Kürschák, 1923; Ore, 1923, 1924; Rella, 1927; MacLane, 1938; Azumaya, 1951)) which follow the same "tropical" pattern.

6. Algebraic equations in characteristic 1

Now we return to the problem that solutions of equations between tropical polynomials cannot just be read off from the roots. Let us start with linear equations

$$ax \vee b = cx \vee d \tag{6.1}$$

in a tropical semifield K. Looking quite innocent, they already bear a mild challenge. In contrast to classical algebra, such an equation is not always solvable. To avoid trivialities, assume that $a, b, c, d \in G := K^{\times}$. Then x cannot be zero, unless $b = d$. To solve Eq. (6.1), consider the map $p\colon G \to G$ given by

$$p(x) := \big((ad \vee bc)x \vee bd\big)\big(acx \vee (ad \vee bc)\big)^{-1}. \tag{6.2}$$

Note the expression $\Delta := ad \vee bc$ which looks like a determinant! The roots of the left- and right-hand side of Eq. (6.1) are respectively

$$\alpha := a^{-1}b, \qquad \beta := c^{-1}d.$$

Proposition 6.1. *The map (6.2) is idempotent and maps G onto the interval*

$$[\alpha \wedge \beta, \alpha \vee \beta]. \tag{6.3}$$

Every solution x of Eq. (6.1) is mapped into a solution $p(x)$.

[2] A common expression of the 18th century (see (Carnot, 1860); or (Speiser, 1956), Chapter 17, concerning Lagrange who considered groups as "la vraie métaphysique" of algebraic equations).

Proof. To verify that $p^2 = p$, note first that $\Delta^2 \geqslant abcd$. Now Eq. (6.2) can be written as

$$p(x) = \frac{\Delta x \vee bd}{acx \vee \Delta}.$$

So we have

$$\begin{aligned} p(p(x)) &= \frac{\Delta(\Delta x \vee bd)(acx \vee \Delta)^{-1} \vee bd}{ac(\Delta x \vee bd)(acx \vee \Delta)^{-1} \vee \Delta} = \frac{\Delta(\Delta x \vee bd) \vee bd(acx \vee \Delta)}{ac(\Delta x \vee bd) \vee \Delta(acx \vee \Delta)} \\ &= \frac{(\Delta^2 \vee abcd)x \vee \Delta bd}{ac\Delta x \vee (\Delta^2 \vee abcd)} = \frac{\Delta^2 x \vee \Delta bd}{ac\Delta x \vee \Delta^2} = \frac{\Delta x \vee bd}{acx \vee \Delta} = p(x). \end{aligned}$$

Furthermore,

$$\begin{aligned} p(x) &= (\Delta x \vee bd)(acx \vee \Delta)^{-1} = (\Delta x \vee bd)(a^{-1}c^{-1}x^{-1} \wedge \Delta^{-1}) \\ &= \Delta x(a^{-1}c^{-1}x^{-1} \wedge \Delta^{-1}) \vee bd(a^{-1}c^{-1}x^{-1} \wedge \Delta^{-1}) \\ &\leqslant \Delta a^{-1}c^{-1} \vee bd\Delta^{-1} = a^{-1}b \vee c^{-1}d, \end{aligned}$$

and similarly, $p(x) = (\Delta x \vee bd)a^{-1}c^{-1}x^{-1} \wedge (\Delta x \vee bd)\Delta^{-1} \geqslant \Delta a^{-1}c^{-1} \wedge bd\Delta^{-1} = a^{-1}b \wedge c^{-1}d$. Thus p maps into the interval $[\alpha \wedge \beta, \alpha \vee \beta]$. For $x \in [\alpha \wedge \beta, \alpha \vee \beta]$, we have $acx \leqslant ac(a^{-1}b \vee c^{-1}d) = \Delta$, and secondly, $bd \leqslant (ad \vee bc)(a^{-1}b \wedge c^{-1}d) \leqslant \Delta x$. Hence $p(x) = (\Delta x \vee bd)\Delta^{-1} = \Delta x \Delta^{-1} = x$.

Finally, if x is a solution of Eq. (6.1), then $(ap(x) \vee b)(acx \vee \Delta) = a(\Delta x \vee bd) \vee b(acx \vee \Delta) = a\Delta x \vee b\Delta = (cx \vee d)\Delta = (c\Delta \vee acd)x \vee d(bc \vee \Delta) = c(\Delta x \vee bd) \vee d(acx \vee \Delta) = (cp(x) \vee d)(acx \vee \Delta)$, which shows that $p(x)$ is a solution of Eq. (6.1). □

By Proposition 6.1, the solutions of Eq. (6.1) are the fibers of the solutions in the interval (6.3) under the projection p. So it remains to consider solutions in the interval (6.3). To solve the equation, we consider another map $s\colon G \to G$ with

$$s(x) := a^{-1}d(ax \vee b)(cx \vee d)^{-1}. \tag{6.4}$$

Proposition 6.2. *The map (6.4) satisfies $s^2 = p$. In particular, s is an involution on the interval (6.3).*

Proof. We have

$$\begin{aligned} s(s(x)) &= a^{-1}d \cdot \frac{a \cdot a^{-1}d(ax \vee b)(cx \vee d)^{-1} \vee b}{c \cdot a^{-1}d(ax \vee b)(cx \vee d)^{-1} \vee d} = a^{-1}d \cdot \frac{d(ax \vee b) \vee b(cx \vee d)}{ca^{-1}d(ax \vee b) \vee d(cx \vee d)} \\ &= \frac{d(ax \vee b) \vee b(cx \vee d)}{c(ax \vee b) \vee a(cx \vee d)} = \frac{\Delta x \vee bd}{acx \vee \Delta} = p(x). \end{aligned}$$

□

Corollary. *The following are equivalent.*

(a) *Eq. (6.1) is solvable.*

(b) *$ad \wedge bc \leqslant ab \leqslant ad \vee bc$.*

(c) $ad \wedge bc \leqslant cd \leqslant ad \vee bc$.

If Eq. (6.1) *is solvable, the unique solution in the interval* (6.3) *is* $x = (b \vee d)(a \vee c)^{-1}$.

Proof. The equivalence of (b) and (c) follows by symmetry. Condition (c) is equivalent to $a^{-1}d \in [\alpha \wedge \beta, \alpha \vee \beta]$. Furthermore, Eq. (6.4) shows that s maps every solution of Eq. (6.1) to $a^{-1}d$. Hence, if Eq. (6.1) is solvable, there is a solution $x \in [\alpha \wedge \beta, \alpha \vee \beta]$, which yields $a^{-1}d = s(x) = sp(x) = s^3(x) = ps(x) \in [\alpha \wedge \beta, \alpha \vee \beta]$. Thus (c) is necessary for the solvability of Eq. (6.1). Moreover, $x = p(x) = s^2(x) = s(a^{-1}d) = a^{-1}d(d \vee b)(ca^{-1}d \vee d)^{-1} = (d \vee b)(c \vee a)^{-1}$.

Conversely, if $a^{-1}d \in [\alpha \wedge \beta, \alpha \vee \beta]$, then $x := s(a^{-1}d)$ satisfies $s(x) = p(a^{-1}d) = a^{-1}d$. Hence Eq. (6.4) implies that x is a solution. □

Our discussion of linear equations already shows that solutions need not exist, even for non-trivial equations. Therefore, a concept of algebraically closed semifield has to take this into account. So we arrive at the following

Definition 6.1. A semifield K is said to be *algebraically closed* if every equation $f(x) = 1$ with $f \in K(x)$ which is solvable in some extension semifield of K admits a solution in K.

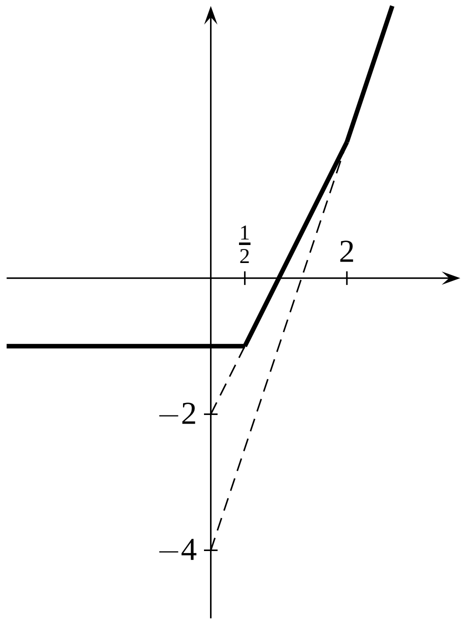
Note that an equation $f(x) = 1$ in $K(x)$ can also be written in the form

$$g(x) = h(x)$$

with polynomials $g, h \in K[x]$. We mention here that polynomials in $K[x]$ can be regarded as functions. Namely, for a non-trivial abelian ℓ-group G, Proposition 5 of (Rump, 2015) implies that $f \in \widetilde{G}[x]$ is uniquely determined by the corresponding function $f\colon G^d \to G^d$ on the divisible closure of G. For $\widetilde{G} = \mathbb{R}^+_{\max}$, it is convenient to write $\mathbb{R}^+$ additively via the logarithm. So G is turned into the additive group of $\mathbb{R}$, and 0 becomes $-\infty$. The graph of a polynomial is then piecewise linear, a classical Newton polygon. For example, the polynomial

$$-1 \vee (-2 + 2x) \vee (-4 + 3x)$$

looks as follows:



Here the coefficients are in $\mathbb{Z}$, while the roots are in $\frac{1}{2}\mathbb{Z}$, because the linear term is missing. The root $\frac{1}{2}$ is of multiplicity 2. Thus, if we add a linear term to get the polynomial into the normal form, the coefficient of x would be $-\frac{3}{2}$.

The corollary of Proposition 6.2 shows that in the tropical case, linear equations are not trivial, and that roots only play a certain rôle with respect to the solutions. In compensation for this initial difficulty of tropical equations, Theorem 4 of (Rump, 2015) states that we don't have to go beyond quadratic equations! Precisely, the theorem says that a tropical semifield K is algebraically closed if and only if the ℓ-group $G := K^\times$ is divisible, that is, the pure equations $x^n = a$ are solvable in G, and the quadratic equations

$$(a \vee 1)x^2 \vee (a^2 \vee b \vee 1)x \vee (a^2 \vee a) = ax^2 \vee a \tag{6.5}$$

are solvable for all $a, b \in G$. Note that the solvablity clause (in an extension semifield) of Definition 6.1 is missing. In fact, we have

Proposition 6.3. *The equations* (6.5) *are solvable in any totally ordered abelian group.*

Proof. For $a \geqslant 1$, Eq. (6.5) becomes $ax^2 \vee (a^2 \vee b)x \vee a^2 = ax^2 \vee a$. We show that this equation holds for all $x \geqslant a \vee a^{-1}b$. Indeed, the latter implies that $ax^2 \geqslant ax(a \vee a^{-1}b) = (a^2 \vee b)x \geqslant a^2$. So the equation (6.5) is solved. For $a \leqslant 1$, the equation becomes $x^2 \vee (b \vee 1)x \vee a = ax^2 \vee a$. Here we choose $x \leqslant a(b \vee 1)^{-1}$. Then $x \leqslant a$ and $(b \vee 1)x \leqslant a$. Hence $ax^2 \leqslant x \leqslant a$, which solves the equation. □

Corollary 1. *For any tropical semifield, there exists a (tropical) extension semifield where Eq.* (6.5) *is solvable.*

Proof. This follows since every abelian ℓ-group G is a subdirect product of totally ordered abelian groups ((Darnel, 1995), 47.11). □

Furthermore, Theorem 4 of (Rump, 2015) implies

Corollary 2. *Let G be a totally ordered abelian group. Then $\widetilde{G}$ is algebraically closed if and only if the pure equation $x^n = a$ is solvable for all positive integers n and $a \in G$.*

To analyse Eq. (6.5), consider an additive abelian ℓ-group G. The proof of Proposition 6.3 then tells us that in the totally ordered case, solutions of Eq. (6.5) exist, but depending on the sign of a, they must be either large enough if $a > 0$ or small enough if $a < 0$.

It is this point where geometry enters the scene. By the Jaffard-Ohm correspondence, every abelian ℓ-group G occurs as a tropicalized Bézout domain R. By (**?**), Proposition 7, the structure sheaf of R can be transferred to G, which yields a sheaf $\breve{G}$ on a spectral space X with totally ordered stalks such that $\Gamma(X, \breve{G}) \cong G$. In the archimedean case, $\breve{G}$ is a sheaf of germs of continuous functions. Therefore, the sensitivity of Eq. (6.5) against sign change of a is best illustrated by the following

Example. Let G be the ℓ-group $\mathscr{C}[-1,1]$ of continuous real functions on the closed interval $[-1,1]$. Multiplying Eq. (6.5) by $a^{-1}x^{-1}$, it takes the symmetric form

$$a^{-}x \vee c \vee a^{+}x^{-1} = |x| \tag{6.6}$$

with $a \in G$ and $c \geqslant |a|$, where $|a| := a \vee a^{-1}$ and

$$a^{+} := a \vee 1, \qquad a^{-} := a^{-1} \vee 1.$$

Writing Eq. (6.6) additively, it becomes

$$(a^{-} + x) \vee c \vee (a^{+} - x) = |x|.$$

Passing to $\mathscr{C}[-1,1]$, let c be the constant function $t \mapsto 1$, and let a be arbitrary with $|a| \leqslant c$. If $x(t) \geqslant 0$, this implies that $x(t) = |x|(t) \geqslant 1$, while $x(t) \leqslant 0$ gives $-x(t) \geqslant 1$, that is, $x(t) \leqslant -1$. Thus Eq. (6.5) cannot be solvable by a continuous function.

Recall that an element $u \geqslant 1$ of a (multiplicative) abelian ℓ-group G is said to be a *weak order unit* ((Darnel, 1995), 54.3) if $u \wedge a = 1$ implies that $a = 1$. For $a \in G^{+}$, we write $G(a)$ for the ℓ-ideal generated by a. It consists of the elements $x \in G$ with $|x| \leqslant a^n$ for some $n \in \mathbb{N}$.

Definition 6.2. (McGovern, 2005) An abelian ℓ-group G is said to be *weakly complemented* if for any pair $a, b \in G$ with $a \wedge b = 1$, there exist $a', b' \in G$ with $a \leqslant a'$ and $b \leqslant b'$ such that $a' \wedge b' = 1$ and $a'b'$ is a weak order unit of G. If $G(a)$ is weakly complemented for all $a \in G^{+}$, then G is called *locally weakly complemented*.

The following result shows that the solvability of Eq. (6.5) merely depends on the lattice structure of G. To state it, we need a very weak form of projectability. Recall that an abelian ℓ-group G is *strongly projectable* (Darnel, 1995) if the *polar*

$$I^{\perp} := \{a \in G \mid \forall\, b \in I \colon |a| \wedge |b| = 1\}$$

of any ℓ-ideal I is a cardinal summand: $G = I^{\perp} \boxplus I^{\perp\perp}$. If this holds for principal ℓ-ideals $I = G(a)$, then G is called *projectable*. More generally, G is said to be *semi-projectable*[3] (Bigard *et al.*, 1977) if

$$(a \wedge b)^{\perp} = a^{\perp} b^{\perp}$$

for $a, b \in G^{+}$. (For a geometric characterization, see (Rump, 2014), corollary of Theorem 1.) Still more generally, we call G *z-projectable* (Rump, 2014) if

$$(ab)^{\perp\perp} = a^{\perp\perp} b^{\perp\perp}$$

holds for $a, b \in G^{+}$. Thus

strongly projectable $\Longrightarrow$ projectable $\Longrightarrow$ semi-projectable $\Longrightarrow$ z-projectable

All these concepts are pairwise inequivalent. The line of implications could even be enlarged to seven types of projectability (Rump, 2014) which all have their particular relevance (cf. the hierarchy of T_n-spaces in general topology). Now we are ready to prove

Theorem 6.1. *Let G be an abelian ℓ-group. The following are equivalent.*

(a) *The quadratic equations (6.5) are solvable in G.*

(b) *For $a, b, c \in G$ with $a \wedge b = 1$ and $c \geqslant a \vee b$, there exist $a', b' \in G$ with $a' \geqslant a$ and $b' \geqslant b$ such that $a' \wedge b' = 1$ and $a' \vee b' = c$.*

(c) *G is semi-projectable and locally weakly complemented.*

(d) *G is z-projectable and locally weakly complemented.*

Proof. (a) $\Rightarrow$ (b): By assumption, there exists a solution $x \in G$ of Eq. (6.6) with ab^{-1} instead of a. Then $bx \leqslant x^{+}x^{-}$ and $ax^{-1} \leqslant x^{+}x^{-}$, which gives $(x^{-})^2 \geqslant b$ and $(x^{+})^2 \geqslant a$. Define $a' := (x^{+})^2 \wedge c$ and $b' := (x^{-})^2 \wedge c$. Then $a' \wedge b' = 1$ and $a' \vee b' = \big((x^{+})^2 \vee (x^{-})^2\big) \wedge c = c$.

(b) $\Rightarrow$ (c): Let $P \neq Q$ be minimal prime ℓ-ideals of G. Choose $a \in P \cap G^{+} \setminus Q$. Since P is minimal, $a \wedge b = 1$ for some $b \notin P$. For any $c \geqslant a \vee b$, the elements a', b' in (b) satisfy $a' \in b^{\perp}$ and $b' \in a^{\perp}$. Hence $a^{\perp}b^{\perp} = G$. Since $a^{\perp} \subset Q$ and $b^{\perp} \subset P$, we obtain $PQ = G$, which shows that G has stranded primes. By (Bigard *et al.*, 1977), Proposition 7.5.1, this implies that G is semi-projectable. Moreover, (b) implies that G is locally weakly complemented.

(c) $\Rightarrow$ (d): By (Rump, 2014), Proposition 4, every semi-projectable abelian ℓ-group is z-projectable.

(d) $\Rightarrow$ (a): Let $a, b, c \in G$ with $a \wedge b = 1$ and $c \geqslant a \vee b$ be given. By the equivalence of Eq. (6.5) and Eq. (6.6), it is enough to solve the equation

$$ax \vee c \vee bx^{-1} = |x|. \tag{6.7}$$

[3] Some authors replace this term by "having stranded primes", referring to an equivalent form proved in (Bigard *et al.*, 1977), Proposition 7.5.1. Darnel (Darnel, 1995) argues that "semi-projectable" does not come close to "projectable" (referring perhaps to the "projections" of a cardinal sum). Note, however, the equivalent version $a \wedge b = 1 \Rightarrow a^{\perp}b^{\perp} = G$, which gives half of a cardinal sum: "semi" $\times$ "projectable".

By assumption, there exist $a', b' \in G$ with $a' \geqslant a$ and $b' \geqslant b$ such that $a' \wedge b' = 1$ and $(a'b')^{\perp} \cap G(c) = \{1\}$. Since G is z-projectable, this yields $c \in (a'b')^{\perp\perp} = (a')^{\perp\perp}(b')^{\perp\perp}$. So there are $p \in (a')^{\perp\perp} \cap G^{+}$ and $q \in (b')^{\perp\perp} \cap G^{+}$ with $c = pq$. In particular, this implies that $p \wedge q = 1$. Hence $a = a \wedge (p \vee q) = (a \wedge p) \vee (a \wedge q) = a \wedge p$. So we have $a \leqslant p$, and similarly, $b \leqslant q$. Thus $x := qp^{-1}$ solves Eq. (6.7). □

By (Rump, 2015), Theorem 4, we obtain

Corollary 1. *Let G be an abelian ℓ-group. The tropical semifield $\widetilde{G}$ is algebraically closed if and only if G is divisible and its underlying lattice satisfies condition (b) of Theorem 6.1.*

Recall that a ring R is said to be *clean* (Nicholson, 1977) if every $a \in R$ is a sum of an idempotent and a unit. Nicholson (Nicholson, 1977) proved that a clean ring R satisfies the *exchange property* (Crawley & Jónsson, 1964; Warfield, 1972), which means that for every decomposition $M = R \oplus N = \bigoplus_{i \in I} M_i$ of modules, there are submodules $M_i' \subset M_i$ with $M = R \oplus \bigoplus_{i \in I} M_i'$. For example, commutative von Neumann regular rings, and semiperfect rings, are clean. For various characterizations, see (McGovern, 2005). If every non-isomorphic homomorphic image of R is clean, the ring R is called *neat* (McGovern, 2005).

Corollary 2. *A Bézout domain is neat if and only if its group of divisibility satisfies the equivalent conditions of Theorem 6.1.*

Proof. By (McGovern, 2005), Theorem 5.7, a Bézout domain is neat if and only if its group of divisibility is semi-projectable and locally weakly complemented. Thus Theorem 6.1 applies. □

Remark. Note that the underlying lattice of an abelian ℓ-group is self-dual via $x \mapsto x^{-1}$. Thus, for a Bézout domain R, Corollary 2 remains valid if the group of divisibility is replaced by the unit group $A(R)^{\times}$ of the tropical semifield $A(R)$. In particular, Corollary 2 gives a characterization of Bézout domains R with $A(R)$ algebraically closed.

Finally, we consider the abelian ℓ-group $\mathscr{C}(X)$ of continuous real valued functions on a topological space X. Note that $\mathscr{C}(X)$ is also a ring. To avoid confusion, let us denote this ring by $C(X)$. By (Gillman & Jerison, 1960), Theorem 3.9, there is no loss of generality if X is assumed to be completely regular. It is known that $C(X)$ is a Bézout ring (that is, every finitely generated ideal is principal) if and only if X is an *F-space*, which originally was just defined by this property (Gillman & Henriksen, 1956). For equivalent characterizations, see (Gillman & Jerison, 1960), Theorem 14.25. One of these characterizations states that for every $f \in C(X)$ there is an element $g \in C(X)$ with $f = g|f|$.

Corollary 3. *Let X be a completely regular space. If the tropical semifield $\widetilde{\mathscr{C}(X)}$ is algebraically closed, then X is an F-space.*

Proof. Let $f \in C(X)$ be given. Then $(f^{+} \wedge 1) \wedge (f^{-} \wedge 1) = 0$ and $(f^{+} \wedge 1) \vee (f^{-} \wedge 1) \leqslant 1$. Thus, by Corollary 1, there exist $g, h \in \mathscr{C}(X)$ with $f^{+} \wedge 1 \leqslant g$ and $f^{-} \wedge 1 \leqslant h$ such that $g \wedge h = 0$ and $g \vee h = 1$. We claim that $f = (g - h)|f|$. If $f(t) > 0$, then $0 < f^{+}(t) \wedge 1 \leqslant g(t)$. Hence $h(t) = 0$, and thus $(g - h)(t) = 1$. Similarly, $f(t) < 0$ implies that $0 < f^{-}(t) \wedge 1 \leqslant h(t)$, which yields $(g - h)(t) = -1$. Thus X is an F-space. □

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New Subclass of p - valent Harmonic Meromorphic Functions

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Abstract

In this paper, we have introduced a new subclass of p-valent harmonic meromorphic and orientation preserving functions in the exterior of the unit disc. Coefficient bounds, distortion bounds, extreme points, convolution conditions and convex combination for the functions belonging to this class are obtained.

Keywords: Harmonic functions, p-valent functions, meromorphic functions, convex combination, distortion bounds.
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1. Introduction

Let C be the field of complex numbers. A continuous function $f(z) = u + iv$ is a complex valued harmonic function in a domain $D \subseteq C$, if both u and v are real harmonic in D. Hengartner and Schober [5], among others, investigated the class of functions of the form $f(z) = h(z) + \overline{g(z)}$, which are harmonic, meromorphic, orientation preserving and univalent in $\widetilde{U} = \{z : |z| > 1\}$ so that $f(\infty) = \infty$. It is known that f(z) admits the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z| \tag{1.1}$$

where

$$h(z) = az + \sum_{n=0}^{\infty} a_n z^{-n}, \quad g(z) = \beta z + \sum_{n=0}^{\infty} b_n z^{-n} \tag{1.2}$$

For $0 \leq |\beta| \leq |\alpha|$ and $a(z) = \dfrac{\overline{f_{\bar{z}}}}{f_z}$ is analytic and satisfies $|a(z)| < 1$ for $z \in \widetilde{U}$. Since the affine transformation

$$\frac{\overline{\alpha} f - \overline{\beta f} - \overline{\alpha} a_0 + \overline{\beta a_0}}{|\alpha|^2 - |\beta|^2}$$

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is again in the class studied by Hengartner and Schober see (Hengartner & Schober, 1987). Recently, Jahangiri (Jahangiri, 2000) assumed $\alpha = 1, \beta = 0$ and removed the logarithmic singularity by letting $A = 0$ in (1.1) and focused on the study of the family of harmonic meromorphic functions.

For fixed positive integer p, consider the family $\Sigma_H(p)$ consisting of functions

$$f(z) = h(z) + \overline{g(z)} \tag{1.3}$$

which are $p-$valent harmonic meromorphic functions in $\widetilde{U}$, where

$$\begin{aligned} h(z) &= z^p + \sum_{n=1}^{\infty} a_{n+p-1} z^{-(n+p-1)}, \\ g(z) &= \sum_{n=1}^{\infty} b_{n+p-1} z^{-(n+p-1)}, \quad |b_p| < 1 \end{aligned} \tag{1.4}$$

we call $h(z)$ the analytic part and $g(z)$ is co-analytic part of $f(z)$. For $0 \leq \gamma < 1$, $k \geq 1$ and $0 \leq \alpha \leq 2\pi$, we define a new subclass as follows: Let $\Sigma_H(p, \gamma, k)$ consist of functions $f(z)$ satisfying the conditions

$$Re\left\{(1 + ke^{i\alpha})\frac{zf'(z)}{z'f(z)} - pke^{i\alpha}\right\} \geq p\gamma, \tag{1.5}$$

where $z' = \dfrac{\partial}{\partial\theta} z$ with $z = re^{i\theta}$, $r > 1$ and θ is real.

Further, let $\Sigma_{\overline{H}}(p, \gamma, k)$ denote the subclass of $\Sigma_H(p, \gamma, k)$ consisting of functions $f(z) = h(z) + \overline{g(z)}$ such that $h(z)$ and $g(z)$ are of the form

$$\begin{aligned} h(z) &= z^p + \sum_{n=1}^{\infty} |a_{n+p-1}| z^{-(n+p-1)}, \\ g(z) &= -\sum_{n=1}^{\infty} |b_{n+p-1}| z^{-(n+p-1)}, \quad |b_p| < 1 \end{aligned} \tag{1.6}$$

Note that, various other subclasses of harmonic $p-$valent meromorphic functions have been studied rather extensively by Ahuja and Jahangiri (Ahuja & Jahangiri, 2003) and Murugusundaramoorthy (Murugusundaramoorthy, 2003), we also note that, $\Sigma_H(1, \gamma, 1)$, the class of harmonic meromorphic functions, was studied by Rosy (T. Rosy & Jahangiri, 2001). Among other things, Ahuja and Jahangiri (Ahuja & Jahangiri, 2003), proved that if, $f(z) = h(z) + \overline{g(z)}$ is given by (1.4) and if,

$$\sum_{n=1}^{\infty} (n + p - 1)(|a_{n+p-1}| + |b_{n+p-1}|) \leq p, \tag{1.7}$$

then $f(z)$ is harmonic, sense -preserving and $p-$valent in $\widetilde{U}$ and $f \in \Sigma_H(p)$.

In the present paper, we have obtained coefficient bounds, extreme points, distortion bounds, convolution conditions and convex combinations for functions in the class $\Sigma_{\overline{H}}(p, \gamma, k)$.

2. Coefficient Bounds

First we state and prove the coefficient bound for the class $\Sigma_H(p,\gamma,k)$.

Theorem 2.1. *Let* $f(z) = h(z) + \overline{g(z)}$ *with* $h(z)$ *and* $g(z)$ *given by* (1.4) . *If*

$$\sum_{n=1}^{\infty} [(n+p-1)(1+k) + p(k+\gamma)] \left|a_{n+p-1}\right| \\ +[(n+p-1)(1+k) - p(k+\gamma)] \left|b_{n+p-1}\right| \leq p(1-\gamma), \tag{2.1}$$

then $f(z)$ *is harmonic, orientation preserving and* $p-$*valent in* $\widetilde{U}$ *and* $f \in \Sigma_H(p,\gamma,k)$.

Proof. Suppose that (2.1) holds. Then we have

$$Re\frac{(1+ke^{i\alpha})(zh'(z) - \overline{zg'(z)}) - pke^{i\alpha}(h(z) + \overline{g(z)})}{h(z) + \overline{g(z)}} = \frac{A(z)}{B(z)} \geq p\gamma, \tag{2.2}$$

where $z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \gamma < 1$, $k \geq 1$, $0 \leq \alpha \leq 2\pi$,
here, we let

$$A(z) = (1+ke^{i\alpha})(zh'(z) - \overline{zg'(z)}) - pke^{i\alpha}(h(z) + \overline{g(z)}) \tag{2.3}$$

and

$$B(z) = (h(z) + \overline{g(z)}). \tag{2.4}$$

Using the fact that $Rew \geq p\gamma$, if and only if $|p - \gamma + \omega| \geq |p + \gamma - \omega|$, it suffices to show that

$$|A(z) + p(1-\gamma)B(z)| - |A(z) - p(1+\gamma)B(z)| \geq 0. \tag{2.5}$$

Substituting the expressions for $A(z)$ and $B(z)$ in (2.5), we obtain

$$\begin{aligned} |A(z) + p(1-\gamma)B(z)| - |A(z) - p(1+\gamma)B(z)| &= \left|p(1-\gamma)h(z) + (1+ke^{i\alpha})zh'(z)) - pke^{i\alpha}h(z)\right| \\ &+ \left|\overline{p(1-\gamma)g(z) - (1+ke^{i\alpha})zg'(z) - pke^{i\alpha}g(z)}\right| - |\, p(1+\gamma)h(z) - (1+ke^{i\alpha})zh'(z) + pke^{i\alpha}h(z) \\ &\quad + \overline{p(1+\gamma)g(z) + (1+ke^{i\alpha})zg'h(z) + pke^{i\alpha}g(z)}\,| \\ &= \left|p(2-\gamma)z^p - \sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1) + p(ke^{i\alpha} - p - \gamma)]\right| a_{n+p-1}z^{-(n+p-1)} \\ &\quad - \sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1) + p(1-ke^{i\alpha} - p - \gamma)] \left|b_{n+p-1}\right| z^{-(n+p-1)}| \\ &\quad \left|\gamma p z^p - \sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1) + p(ke^{i\alpha} + 1 + \gamma)] \left|a_{n+p-1}\right| z^{-(n+p-1)}\right. \end{aligned}$$

$$\left. - \sum_{n=1}^{\infty} [(1+ke^{i\alpha})(n+p-1) - p(ke^{i\alpha}+1+\gamma)] \left|b_{n+p-1}\right| z^{-(n+p-1)} \right|$$

$$2p(1-\gamma)|z|^p - \sum_{n=1}^{\infty} 2(n+p-1)(1+k) + 2p(k+\gamma) \left|a_{n+p-1}\right| |z|^{-(n+p-1)}$$
$$- \sum_{n=1}^{\infty} 2(n+p-1)(1+k) - 2p(k+\gamma) \left|b_{n+p-1}\right| |z|^{-(n+p-1)}$$
$$2|z|^p \left\{ p(1-\gamma) - \sum_{n=1}^{\infty} (n+p-1)(1+k) + p(k+\gamma) \left|a_{n+p-1}\right| |z|^{-(n+p-2)} \right\}$$
$$+ \sum_{n=1}^{\infty} (n+p-1)(1+k) - p(k+\gamma) \left|b_{n+p-1}\right| |z|^{-(n+p-2)}$$

$$\geq 2\{p(1-\gamma) - \sum_{n=1}^{\infty} (n+p-1)(1+k) + p(k+\gamma) \left|a_{n+p-1}\right|$$
$$+ \sum_{n=1}^{\infty} (n+p-1)(1+k) - p(k+\gamma) \left|b_{n+p-1}\right|\} \geq 0,$$

by (2.1). □

Remark 2.2. *It is natural to ask if the condition* (2.1) *is also necessary for functions* $f \in \Sigma_H(p,\gamma,k)$.

In the next theorem we show that the answer to that question which is in affirmative.

Theorem 2.3. *Let* $f(z) = h(z) + \overline{g(z)}$ *be such that* $h(z)$ *and* $g(z)$ *given by* (1.6). *Then* $f(z) \in \sum_{\overline{H}}(p,\gamma,k)$, *if and only if the inequality* (2.1) *holds for the coefficients of* $f(z) = h(z) + \overline{g(z)}$.

Proof. In view of Theorem I, we only need to show that $f(z) \notin \sum_{\overline{H}}(p,\gamma,k)$, if the condition (2.1) does not hold. We note that for $f(z) \in \sum_H(p,\gamma,k)$, we have

$$Re\left\{ \frac{(1+ke^{i\alpha})(zh'(z) - \overline{zg'(z)}) - pke^{i\alpha}(h(z) + \overline{g(z)})}{h(z) + \overline{g(z)}} \right\} \geq p\gamma.$$

This is equivalent to

$$Re\left\{ \frac{(1+ke^{i\alpha})(zh'(z) - \overline{zg'(z)}) - pke^{i\alpha}(h(z) + \overline{g(z)})}{h(z) + \overline{g(z)}} \right\} - p\gamma =$$

$$Re\left\{ \frac{1}{z^p + \sum_{n=1}^{\infty} a_{n+p-1} z^{-(n+p-1)} - \sum_{n=1}^{\infty} b_{n+p-1} z^{-(n+p-1)}} \Big[p(1-\gamma) z^p \right.$$
$$- \sum_{n=1}^{\infty} [(1+ke^{i\alpha})(n+p-1) + pke^{i\alpha} + p\gamma] a_{n+p-1} z^{-(n+p-1)}$$
$$\left. - \sum_{n=1}^{\infty} [(1+ke^{i\alpha})(n+p-1) - pke^{i\alpha} - p\gamma] b_{n+p-1} z^{-(n+p-1)} \Big] \right\} \geq 0$$

The above condition must hold for all values of z such that $|z| = r < 1$. Upon choosing the values of z on the positive real axis, we must have

$$Re\left\{\frac{1}{1+\sum_{n=1}^{\infty}(a_{n+p-1}-b_{n+p-1})r^{-(n-1)}}\left[p(1-\gamma)-\sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1)+pke^{i\alpha}+p\gamma]a_{n+p-1}r^{-(n-1)}\right.\right.$$
$$\left.\left.-\sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1)-pke^{i\alpha}-p\gamma]b_{n+p-1}r^{-(n-1)}\right]\right\}\geq 0.$$

If the condition (2.1) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0 > 1$, for which the quotient in (2.5) is negative. This contradicts the conditions for $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$ and this completes the proof. □

3. Distortion Bounds and Extreme Points

The determination of the extreme points of a compact family of harmonic univalent functions enables us to solve many extremal problems for the family. The fundamental reason for considering extreme points for starlike and convex functions is to more easily categorize extremal properties under continuous linear functionals acting on these classes. In this section, we shall obtain distortion bounds for functions in $\Sigma_{\overline{H}}(p, \gamma, k)$ and also determine the extreme points for the class $\Sigma_{\overline{H}}(p, \gamma, k)$.

Theorem 3.1. *If* $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$ *then* $r^p - p(1-\gamma)r^{-p} \leq |f(z)| \leq r^p + p(1-\gamma)r^{-p}, |z| = r < 1.$

Proof. We only prove the inequality on the right. The argument for the inequality on the left is similar. Let $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$. Taking the absolute value of $f(z)$, we obtain

$$|f(z)| \leq \left|z^p + \sum_{n=1}^{\infty} a_{n+p-1}z^{-(n+p-1)} - \sum_{n=1}^{\infty} b_{n+p-1}z^{-(n+p-1)}\right| \leq r^p + \sum_{n=1}^{\infty}(a_{n+p-1}+b_{n+p-1})r^{-(n+p-1)}$$

$$\leq r^p + \sum_{n=1}^{\infty}(a_{n+p-1}+b_{n+p-1})r^{-p} \leq r^p + \sum_{n=1}^{\infty}(a_{n+p-1}+b_{n+p-1})r^{-(n+p-1)}$$

$$\leq r^p + \textstyle\sum_{n=1}^{\infty}[(n+p-1)(1+k)+p(k+\gamma)a_{n+p-1}] + \sum_{n=1}^{\infty}[(n+p-1)(1+k)-p(k+\sum)b_{n+p-1}]$$

$$\leq r^p + (p-\gamma)r^{-p}$$

by (2.1). Our next result shows how $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$ looks like. We precisely proved. □

Theorem 3.2. $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$ *, if and only if* $f(z)$ *can be expressed as*

$$f(z) = \sum_{n=1}^{\infty}(x_{n+p-1}h_{n+p-1} + y_{n+p-1}g_{n+p-1}) \tag{3.1}$$

where $z \in \widetilde{U}$,

$$h_{p-1}(z) = z^p,$$
$$h_{n+p-1}(z) = z^p + \frac{p(1-\gamma)}{(n+p-1)(1+k)+p(k+\gamma)} z^{(n+p-1)} \quad (n = 1, 2, 3, ...).$$

$$g_{p-1}(z) = z^p,$$
$$g_{n+p-1}(z) = z^p + \frac{p(1-\gamma)}{(n+p-1)(1+k)-p(k+\gamma)} z^{-(n+p-1)} \quad (n = 1, 2, 3, ...)$$

$$\sum_{n=1}^{\infty}(x_{n+p-1} + y_{n+p-1}) = 1, x_{n+p-1} \geq 0 \ \ and \ \ y_{n+p-1} \geq 0.$$

Proof. For the functions $f(z)$ given by (3.1), we may write,

$$f(z) = \sum_{n=1}^{\infty}\left(x_{n+p-1}h_{n+p-1} + y_{n+p-1}g_{n+p-1}\right)$$

$$= x_{p-1}h_{p-1} + y_{p-1}g_{p-1} + \sum_{n=1}^{\infty} x_{n+p-1}(z^p + \frac{p(1-\gamma)}{(n+p-1)(1+k)+p(k+\gamma)})z^{(n+p-1)}$$

$$+y_{n+p-1}z^p - \frac{p(1-\gamma)}{(n+p-1)(1+k)-p(k+\gamma)} z^{-(n+p-1)}.$$

Then,

$$= \sum_{n=1}^{\infty}\Bigg[((1+k)(n+p-1)+p(\gamma+k))\left(\frac{p(1-\gamma)}{(1+k)(n+p-1)+p(\gamma+k)}x_{n+p-1}\right)$$
$$+((1+k)(n+p-1)-p(\gamma+k))\left(\frac{p(1-\gamma)}{(1+k)(n+p-1)-p(\gamma+k)}y_{n+p-1}\right).$$
$$= p(1-\gamma)\sum_{n=1}^{\infty} x_{n+p-1} + y_{n+p-1} \leq p(1-\gamma),$$

and so $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$. Conversely, suppose that $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$. Set

$$x_n = \frac{(1+k)(n+p-1)+p(\gamma+k)}{p(1-\gamma)}|a_{n+p-1}|$$

and

$$y_n = \frac{(1+k)(n+p-1)+p(\gamma+k)}{p(1-\gamma)}|b_{n+p-1}|, (n = 1, 2, 3, ...)$$

Then note that by Theorem 2, $0 \le x_{p-1} \le 1$.

$$y_{p-1} = 1 - x_{p-1} - \sum_{n=1}^{\infty}(x_{n+p-1} + y_{n+p-1}),$$

we obtain

$$f(z) = \sum_{n=1}^{\infty}(x_{n+p-1}h_{n+p-1} + y_{n+p-1}g_{n+p-1})$$

as required. □

4. Convolution and Convex Linear Combination

In this section, we show that the class $\Sigma_{\overline{H}}(p, \gamma, k)$ is invariant under convolution and convex combinations of its members. For harmonic functions

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p-1}z^{-(n+p-1)} - \sum_{n=1}^{\infty} b_{n+p-1}(\overline{z})^{-(n+p-1)}$$

and

$$F(z) = z^p + \sum_{n=1}^{\infty} A_{n+p-1}z^{-(n+p-1)} - \sum_{n=1}^{\infty} B_{n+p-1}(\overline{z})^{-(n+p-1)}$$

we define the convolution of $f(z)$ and $F(z)$ as

$$(f * F)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p-1}A_{n+p-1}z^{-(n+p-1)} - \sum_{n=1}^{\infty} b_{n+p-1}B_{n+p-1}(\overline{z})^{-(n+p-1)} \tag{4.1}$$

Using this definition, we show in the next theorem that the class $\Sigma_{\overline{H}}(p, \gamma, k)$ is closed under convolution.

Theorem 4.1. *For* $0 \le \beta \le \gamma \le 1$, *let* $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$ *and* $F(z) \in \Sigma_{\overline{H}}(p, \beta, k)$. *Then*

$$f(z) * F(z) \in \Sigma_{\overline{H}}(p, \gamma, k) \subset \Sigma_{\overline{H}}(p, \beta, k). \tag{4.2}$$

Proof. Let

$$f(z) = z^p + \sum_{n=1}^{\infty} |a_{n+p-1}|z^{-(n+p-1)} - \sum_{n=1}^{\infty} |b_{n+p-1}|(\overline{z})^{-(n+p-1)}$$

$$F(z) = z^p + \sum_{n=1}^{\infty} A_{n+p-1}z^{-(n+p-1)} - \sum_{n=1}^{\infty} B_{n+p-1}(\overline{z})^{-(n+p-1)}$$

Note that $A_{n+p-1} \le 1$ and $B_{n+p-1} \le 1$. Obviously, the coefficients of f and F must satisfy conditions similar to the inequality (2.1). So for the coefficients of $f * F$ we can write,

$$\sum_{n=1}^{\infty}(1+k)(n+p-1)+p(\gamma+k)|a_{n+p-1}A_{n+p-1}|+(1+k)(n+p-1)-p(\gamma+k)|b_{n+p-1}B_{n+p-1}|$$

$$\leq (1+k)(n+p-1)+p(\gamma+k)|a_{n+p-1}|+(1+k)(n+p-1)-p(\gamma+k)|b_{n+p-1}|.$$

This right hand side of the above inequality is bounded by 2 because $f(z) \in \Sigma_{\overline{H}}(p,\gamma,k)$. By the same token, we can conclude that $f(z) * F(z) \in \Sigma_{\overline{H}}(p,\gamma,k) \subset \Sigma_{\overline{H}}(p,\beta,k)$. Our next result shows that $\Sigma_{\overline{H}}(p,\gamma,k)$ is closed under convex combination of its members. □

Theorem 4.2. *The family $\Sigma_{\overline{H}}(p,\gamma,k)$ is closed under convex combination*

Proof. For $i = 1,2,3,..,$ let $f_i(z) \in \Sigma_{\overline{H}}(p,\gamma,k)$ where $f_i(z)$ is given by

$$f_i(z) = z^p + \sum_{n=1}^{\infty}|a_{i,n+p-1}|(\overline{z})^{(n+p-1)} + \sum_{n=1}^{\infty}|b_{i,n+p-1}|(\overline{z})^{-(n+p-1)}.$$

Then by (2.1),

$$\sum_{n=1}^{\infty}(1+k)(n+p-1)+p(\gamma+k)|a_{i,n+p-1}|+(1+k)(n+p-1)-p(\gamma+k)|b_{i,n+p-1}| \leq p(1-\gamma) \quad (4.3)$$

for $\sum_{n=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of $f_i(z)$ may be written as

$$\sum_{n=1}^{\infty} t_i f_i(z) = z^p + \sum_{n=1}^{\infty}(t_i|a_{i,n+p-1}|)z^{-(n+p-1)} + \sum_{n=1}^{\infty}(t_i|b_{i,n+p-1}|)(\overline{z})^{-(n+p-1)}.$$

Then by (4.2),

$$\sum_{n=1}^{\infty}[(1+k)(n+p-1)+p(\gamma+k)|\sum_{n=1}^{\infty}(t_i|a_{i,n+p-1}|)|+(1+k)(n+p-1)$$
$$-p(\gamma+k)|\sum_{n=1}^{\infty}(t_i|b_{i,n+p-1}|)|$$

$$\sum_{n=1}^{\infty} t_i\left\{\sum_{n=1}^{\infty}(1+k)(n+p-1)+p(\gamma+k)a_{i,n+p-1}+(1+k)(n+p-1)\right.$$
$$\left.-p(\gamma+k)b_{i,n+p-1}\right\} \leq \sum_{n=1}^{\infty} t_i p(1-\gamma) = (1-\gamma).$$

Since this is the condition required by (2.1), we conclude that $\sum_{n=1}^{\infty} t_i f_i(z) \in \Sigma_{\overline{H}}(p,\gamma,k)$. This completes the proof of Theorem (2.1). □

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An Application for Certain Subclasses of p-Valent Meromorphic Functions Associated with the Generalized Hypergeometric Function

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Abstract

The main object of this paper is to give an application of a linear operator $H^{m,\mu}_{p,q,s}(\alpha_1)f(z)$ involving the generalized hypergeometric function. We define subclasses of the meromorphic function class $\Sigma_{p,m}$ by means of operator $H^{m,\mu}_{p,q,s}(\alpha_1)f(z)$.

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1. Introduction and definitions

Let $\Sigma_{p,m}$ denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U\backslash\{0\}$. We also denote $\Sigma_{p,1-p} = \Sigma_p$.
A function $f \in \Sigma_{p,m}$ is said to be in the class $\Sigma S^*_p(\alpha)$ of meromorphically p-valent starlike functions of order α in U if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (z \in U; 0 \leqslant \alpha < p). \tag{1.2}$$

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Also a function $f \in \Sigma_{p,m}$ is said to be in the class $\Sigma C_p(\alpha)$ of meromorphically p-valent convex of order α in U if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) < -\alpha \quad (z \in U; 0 \leqslant \alpha < p). \tag{1.3}$$

It is easy to observe from (1.2) and (1.3) that

$$f(z) \in \Sigma C_p(\alpha) \Leftrightarrow -\frac{zf'(z)}{p} \in \Sigma S_p^*(\alpha). \tag{1.4}$$

For a function $f \in \Sigma_{p,m}$, we say that $f \in \Sigma K_p(\beta,\alpha)$ if there exists a function $g \in \Sigma S_p^*(\alpha)$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) < -\beta \quad (z \in U; 0 \leqslant \alpha,\beta < p). \tag{1.5}$$

Functions in the class $\Sigma K_p(\beta,\alpha)$ are called meromorphically p-valent close-to-convex functions of order β and type α. We also say that a function $f \in \Sigma_{p,m}$ is in the class $\Sigma K_p^*(\beta,\alpha)$ of meromorphically quasi-convex functions of order β and type α if there exists a function $g \in \Sigma C_p(\alpha)$ such that

$$\Re\left(\frac{(zf'(z))'}{g'(z)}\right) < -\beta \quad (z \in U; 0 \leqslant \alpha,\beta < p). \tag{1.6}$$

It follows from (1.5) and (1.6) that

$$f(z) \in \Sigma K_p^*(\beta,\alpha) \Leftrightarrow -\frac{zf'(z)}{p} \in \Sigma K_p(\beta,\alpha),$$

where $\Sigma S_p^*(\alpha)$ and $\Sigma C_p(\alpha)$ are, respectively, the classes of meromorphically p-valent starlike functions of order α and meromorphically p-valent convex functions of order α $(0 \leqslant \alpha < p)$(see Aouf (Aouf, 2008) and Frasin (Frasin, 2012)).

For a function $f(z) \in \Sigma_{p,m}$, given by (1.1) and $g(z) \in \Sigma_{p,m}$ defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k,$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$f(z) * g(z) = (f * g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k = (g * f)(z) \quad (p \in \mathbb{N}).$$

For real or complex numbers

$$\alpha_1, ..., \alpha_q \text{ and } \beta_1, ..., \beta_s \quad (\beta_j \notin \mathbb{Z}_0^- = \{0,-1,-2,...\}; j = 1,2,...,s),$$

we consider the generalized hypergeometric function ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ by (see, for example, (Kiryakova, 2011, p.19))

$$ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1)_k ... (\beta_s)_k} . \frac{z^k}{k!} $$

$$ (q \leqslant s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U), $$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$ (\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C} \backslash \{0\} = \mathbb{C}^*), \\ \theta(\theta - 1)...(\theta + \nu - 1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases} $$

Corresponding to the function $\phi_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ given by

$$ \phi_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z^{-p} \ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) \ , $$

we introduce a function $\phi_{p,\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ defined by

$$ \phi_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * \phi_{p,\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \frac{1}{z^p(1 - z)^{\mu + p}} $$

$$ (\mu > -p; z \in U^*). $$

We now define a linear operator $H^{m,\mu}_{p,q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) : \Sigma_{p,m} \to \Sigma_{p,m}$ by

$$ H^{m,\mu}_{p,q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z) = \phi_{p,\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z) \tag{1.7} $$

$$ \left(\alpha_i, \beta_j \in \mathbb{C} \backslash \mathbb{Z}_0^- ; i = 1, ..., q, \ j = 1, ..., s \ ; \mu > -p, f \in \Sigma_{p,m}; z \in U^*\right) . $$

For; convenience, we write

$$ H^{m,\mu}_{p,q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) = H^{m,\mu}_{p,q,s}(\alpha_1) $$

and

$$ H^{1-p,\mu}_{p,q,s}(\alpha_1) = H^{\mu}_{p,q,s}(\alpha_1) \quad (\mu > -p). $$

If $f(z)$ is given by (1.1), then from (1.7), we deduce that

$$ H^{m,\mu}_{p,q,s}(\alpha_1) f(z) = z^{-p} + \sum_{k=m}^{\infty} \frac{(\mu + p)_{p+k} (\beta_1)_{p+k} ... (\beta_s)_{p+k}}{(\alpha_1)_{p+k} ... (\alpha_q)_{p+k}} a_k z^k \quad (\mu > -p; z \in U^*). \tag{1.8} $$

It is easily follows from (1.8) that

$$ z \left(H^{m,\mu}_{p,q,s}(\alpha_1) f(z)\right)' = (\mu + p) H^{m,\mu+1}_{p,q,s}(\alpha_1) f(z) - (\mu + 2p) H^{m,\mu}_{p,q,s}(\alpha_1) f(z). \tag{1.9} $$

From the identity (1.9), we readily have

$$z\left(H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)\right)' = (\mu+p-1)H^{m,\mu}_{p,q,s}(\alpha_1)f(z) - (\mu+2p-1)H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z) \quad (1.10)$$

and

$$z\left(H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)\right)' = (\mu+p+1)H^{m,\mu+2}_{p,q,s}(\alpha_1)f(z) - (\mu+2p+1)H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z). \quad (1.11)$$

The linear operator $H^{m,\mu}_{p,q,s}(\alpha_1)$ was introduced by Patel and Palit (Patel & Palit, 2009) .

We note that the linear operator $H^{m,\mu}_{p,q,s}(\alpha_1)$ is closely related to the Choi-Saigo-Srivastava operator (Choi *et al.*, 2002) for analytic functions and is essentially motivated by the operators defined and studied in (Cho & Noor, 2006) (see also, (Dziok & Srivastava, 1999), (Dziok & Srivastava, 2003), (Srivastava, 2007) and (Srivastava & Karlsson, 1985)).

Specializing the parameters $\mu, \alpha_i (i = 1, 2, ...q), \beta_j (j = 1, 2, ...s), q$ and s we obtain the following :

(i) $H^{m,0}_{p,2,1}(p, p; p)f(z) = H^{m,1}_{p,2,1}(p+1, p; p)f(z) = f(z)$;

(ii) $H^{m,1}_{p,2,1}(p, p; p)f(z) = \frac{2pf(z)+zf'(z)}{p}$;

(iii) $H^{m,2}_{p,2,1}(p+1, p; p)f(z) = \frac{(2p+1)f(z)+zf'(z)}{p+1}$;

(iv) $H^{m,1-p}_{p,1,1}(c+1, 1; c)f(z) = J_{c,p}(f)(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt\ (c > 0; z \in U^*)$, this integral operator is defined by

$$J_{c,p}(f)(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt \quad (c > 0; f \in \Sigma_{p,m}),$$

(v) $H^{m,0}_{p,2,2}(p+1, p; p)f(z) = \frac{p}{z^{2p}}\int_0^z t^{2p-1}f(t)dt\ ;\ (p \in \mathbb{N}; z \in U^*)$;

(vi) $H^{1-p,n}_{p,2,1}(a, 1; a)f(z) = \frac{1}{z^p(1-z)^{n+p}} = D^{n+p-1}f(z)\ (n > -p)$, the operator D^{n+p-1} studied by Ganigi and Uralegaddi (Ganigi & Uralegaddi, 1989), Yang (Yang, 1995), Aouf (Aouf, 1993), Aouf and Srivastava (Aouf & Srivastava, 1997) and Uralegaddi and Patil (Uralegaddi & Patil, 1989);

(vii) $H^{m,\mu}_{p,2,1}(c, p+\mu; a)f(z) = L_p(a, c)f(z)\ (a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \mu > -p)$ (see Liu (Liu, 2002));

(viii) $H^{o,\mu}_{1,2,1}(\mu+1, n+1; \mu)f(z) = I_{n,\mu}f(z)\ (\mu > 0; n > -1)$ (see Yuan et al. (Yuan *et al.*, 2008)).

We also observe that, for $m = 0$, $p = 1$ replacing μ by $\mu-1$, we have the operator $H^0_{1,\mu,q,s}(\alpha_1)f(z) = H_{\mu,q,s}(\alpha_1)f(z)$ defined by Cho and Kim (Cho & Kim, 2007).

The object of the present paper is to investigate some properties of meromorphic p- valent functions by the above operator $H^{m,\mu}_{p,q,s}(\alpha_1)f(z)$ given by (1.8).

Definition 1.1. *Let $\mathcal{H}$ the set of complex valued functions $h(r, s, t) : \mathbb{C}^3 \to \mathbb{C}$ such that*

$$h(r, s, t) \text{ is continuous in a domain } D \subset \mathbb{C}^3;$$

$$(1, 1, 1) \in D \quad \text{and} \quad |h(1, 1, 1)| < 1;$$

$$\left|h\left(e^{i\theta}, \frac{1}{\mu+p}+\frac{\mu+p-1}{\mu+p}e^{i\theta}+\frac{1}{\mu+p}\delta, \frac{2}{\mu+p+1}+\frac{\mu+p-1}{\mu+p+1}e^{i\theta}+\right.\right.$$

$$\left.\left.\frac{1}{\mu+p+1}\delta+\frac{(\mu+p-1)\delta e^{i\theta}+[\delta+\beta-\delta^2]}{(\mu+p+1)+(\mu+p-1)(\mu+p+1)e^{i\theta}+(\mu+p+1)\delta}\right)\right| \geqslant 1$$

whenever

$$\left(e^{i\theta}, \frac{1}{\mu+p}+\frac{\mu+p-1}{\mu+p}e^{i\theta}+\frac{1}{\mu+p}\delta, \frac{2}{\mu+p+1}+\frac{\mu+p-1}{\mu+p+1}e^{i\theta}+\right.$$

$$\left.\frac{1}{\mu+p+1}\delta+\frac{(\mu+p-1)\delta e^{i\theta}+[\delta+\beta-\delta^2]}{(\mu+p+1)+(\mu+p-1)(\mu+p+1)e^{i\theta}+(\mu+p+1)\delta}\right) \in D$$

with $\Re(\beta \geqslant \delta(\delta-1))$ *for real* θ, $\delta \geqslant 1$ *and* $\lambda > 0$.

2. The Main Result

In order to prove our main result, we recall the following lemma due to Miller and Mocanu (Miller & Mocanu, 1978).

Lemma 2.1. *Let* $w(z) = a + w_n z^n + ...$,*be analytic in* $U = \{z : z \in \mathbb{C}$ *and* $|z| < 1\}$ *with* $w(z) \neq a$ *and* $n \geqslant 1$. *If* $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ *and* $|w(z_0)| = \max_{|z| \leqslant r_0} |w(z)|$. *Then*

$$zw^{'}(z_0) = \delta w(z_0) \tag{2.1}$$

and

$$\Re(\left\{1+\frac{z_0 w^{''}(z_0)}{w^{'}(z_0)}\right\}) \geqslant \delta, \tag{2.2}$$

where δ *is a real number and*

$$\delta \geqslant n\frac{|w(z_0)-a|^2}{|w(z_0)|^2-|a|^2} \geqslant n\frac{|w(z_0)|-|a|}{|w(z_0)|+|a|}.$$

Theorem 2.1. *Let* $h(r, s, t) \in \mathcal{H}$ *and let* $f \in \Sigma_{p,m}$ *satisfies*

$$\left(\frac{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)}, \frac{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}, \frac{H^{m,\mu+2}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)}\right) \in D \subset \mathbb{C}^3 \tag{2.3}$$

and

$$\left| h\left(\frac{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)}, \frac{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}, \frac{H^{m,\mu+2}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)} \right) \right| < 1 \tag{2.4}$$

for all $z \in U$ and for some $m \in \mathbb{N}$.Then we have

$$\left| \frac{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)} \right| < 1 \qquad (z \in U; \mu > -p, 0 \leqslant \alpha < p; p \in \mathbb{N}).$$

Proof. Let

$$\frac{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)} = w(z). \tag{2.5}$$

Then it follows that $w(z)$ is either analytic or meromorphic in U, $w(0) = 1$ and $w(z) \neq 1$. Differentiating (2.5) logarithmically and multiply by z, we obtain

$$\frac{z\left(H^{m,\mu}_{p,q,s}(\alpha_1)f(z)\right)'}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)} - \frac{z\left(H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)\right)'}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)} = \frac{zw'(z)}{w(z)}.$$

Using the identities (1.6) and (1.10) , we have

$$\frac{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)} = \frac{1}{\mu+p} + \frac{\mu+p-1}{\mu+p}w(z) + \frac{1}{\mu+p}\frac{zw'(z)}{w(z)}. \tag{2.6}$$

Differentiating (2.6) logarithmically and multiply by z, we obtain

$$\frac{z\left(H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)\right)'}{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)} - \frac{z\left(H^{m,\mu}_{p,q,s}(\alpha_1)f(z)\right)'}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)} = \frac{z\left[\frac{1}{\mu+p} + \frac{\mu+p-1}{\mu+p}w(z) + \frac{1}{\mu+p}\frac{zw'(z)}{w(z)}\right]'}{\frac{1}{\mu+p} + \frac{\mu+p-1}{\mu+p}w(z) + \frac{1}{\mu+p}\frac{zw'(z)}{w(z)}} \tag{2.7}$$

$$= \frac{(\mu+p-1)zw'(z) + \left[\frac{zw'(z)}{w(z)} + \frac{z^2w''(z)}{w(z)} - \left(\frac{zw'(z)}{w(z)}\right)^2\right]}{1 + (\mu+p-1)w(z) + \frac{zw'(z)}{w(z)}}$$

Using the identities (1.9) and (1.11) , we have

$$\begin{aligned}(\mu+p+1)\frac{H_{p,q,s}^{m,\mu+2}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)} &= 1+(\mu+p)\frac{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)}+\\ &\quad \frac{(\mu+p-1)zw'(z)+\left[\frac{zw'(z)}{w(z)}+\frac{z^2w''(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2\right]}{1+(\mu+p-1)w(z)+\frac{zw'(z)}{w(z)}}\\ &= 1+\left[1+(\mu+p-1)w(z)+\frac{zw'(z)}{w(z)}\right]+\\ &\quad \frac{(\mu+p-1)zw'(z)+\left[\frac{zw'(z)}{w(z)}+\frac{z^2w''(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2\right]}{1+(\mu+p-1)w(z)+\frac{zw'(z)}{w(z)}}.\end{aligned}$$

We claim that $|w(z)| < 1$ for $z \in U$. Otherwise there exists a point $z_0 \in U$ such that $\max_{|z|\leqslant r_0} |w(z)| = |w(z_0)| = 1$. Letting $w(z_0) = e^{i\theta}$ and using Lemma 2.1 with $a = 1$ and $n = 1$, we have

$$\begin{aligned}\frac{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu-1}(\alpha_1)f(z)} &= e^{i\theta},\\ \frac{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)} &= \frac{1}{\mu+p}+\frac{\mu+p-1}{\mu+p}e^{i\theta}+\frac{1}{\mu+p}\delta,\\ \frac{H_{p,q,s}^{m,\mu+2}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)} &= \frac{2}{(\mu+p+1)}+\frac{(\mu+p-1)}{(\mu+p+1)}e^{i\theta}+\frac{1}{(\mu+p+1)}\delta\\ &\quad +\frac{(\mu+p-1)\delta e^{i\theta}+[\delta+\beta-\delta^2]}{(\mu+p+1)+(\mu+p+1)(\mu+p-1)e^{i\theta}+(\mu+p+1)\delta},\end{aligned}$$

where

$$\beta=\frac{z^2w''(z)}{w(z)} \quad \text{and} \quad \delta \geqslant 1.$$

Further, an application of (2.2) in Lemma 2.1 given $\Re(\beta \geqslant \delta(\delta-1)$. Since $h(r,s,t) \in \mathcal{H}$, we have

$$\left|h\left(\frac{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu-1}(\alpha_1)f(z)},\frac{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)},\frac{H_{p,q,s}^{m,\mu+2}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}\right)\right|$$

$$= \left|h\left(e^{i\theta},\frac{1}{\mu+p}+\frac{\mu+p-1}{\mu+p}e^{i\theta}+\frac{1}{\mu+p}\delta,\frac{2}{\mu+p+1}+\frac{\mu+p-1}{\mu+p+1}e^{i\theta}+\right.\right.$$
$$\left.\left.\frac{1}{\mu+p+1}\delta+\frac{(\mu+p-1)\delta e^{i\theta}+[\delta+\beta-\delta^2]}{(\mu+p+1)+(\mu+p-1)(\mu+p+1)e^{i\theta}+(\mu+p+1)\delta}\right)\right| \geqslant 1$$

which contradicts the condition (2.4) of Theorem 2.1. Therefore, we conclude that

$$\left| \frac{H_{p,q,s}^{m,\mu}(\alpha_1) f(z)}{H_{p,q,s}^{m,\mu-1}(\alpha_1) f(z)} \right| < 1 \quad (z \in U).$$

The proof is complete. □

Letting $\mu = 1, q = 2, s = 1, \alpha_1 = p + 1, \alpha_2 = p$ and $\beta_1 = p$ in Theorem 2.1, we have the following result.

Corollary 2.1. *Let $h(r, s, t) \in \mathcal{H}$ and let $f(z) \in \Sigma_{p,m}$ satisfies*

$$\left(\frac{2pf(z) + zf'(z)}{pf(z)}, \frac{p\left[(2p+1)f(z) + zf'(z)\right]}{(p+1)\left[2pf(z) + zf'(z)\right]}, \right.$$

$$\left. \frac{(2p+2)(2p+1)f(z) + 4(p+1)zf'(z) + z^2 f''(z)}{(p+2)(2p+1)f(z) + zf'(z)} \right) \in D \subset \mathbb{C}^3$$

and

$$\left| h\left(\frac{2pf(z) + zf'(z)}{pf(z)}, \frac{p\left[(2p+1)f(z) + zf'(z)\right]}{(p+1)\left[2pf(z) + zf'(z)\right]}, \right. \right.$$

$$\left. \left. \frac{(2p+2)(2p+1)f(z) + 4(p+1)zf'(z) + z^2 f''(z)}{(p+2)(2p+1)f(z) + zf'(z)} \right) \right| < 1$$

for all $z \in U$. Then we have

$$\left| \frac{2pf(z) + zf'(z)}{pf(z)} \right| < 1 \quad (z \in U).$$

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Some Concepts of Uniform Exponential Dichotomy for Skew-Evolution Semiflows in Banach Spaces

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Abstract

The exponential dichotomy is one of the most important asymptotic properties for the solutions of evolution equations, studied in the last years from various perspectives. In this paper we study some concepts of uniform exponential dichotomy for skew-evolution semiflows in Banach spaces. Several illustrative examples motivate the approach.

Keywords: Skew-evolution semiflow, invariant projection, uniform exponential dichotomy.
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1. Introduction

The property of exponential dichotomy is a mathematical domain with a substantial recent development as it plays an important role in describing several types of evolution equations. The literature dedicated to this asymptotic behavior begins with the results published in Perron (1930). The ideas were continued by in Massera & Schäffer (1966), with extensions in the infinite dimensional case accomplished in Daleckiĭ & Kreĭn (1974) and in Pazy (1983), respectively in Sacker & Sell (1994). Diverse and important concepts of dichotomy were introduced and studied, for example, in Appell *et al.* (1993), Babuţia & Megan (2015), Chow & Leiva (1995), Coppel (1978), Megan & Stoica (2010), Sasu & Sasu (2006) or Stoica & Borlea (2012).

The notion of skew-evolution semiflow that we sudy in this paper and which was introduced in Megan & Stoica (2008) generalizes the skew-product semiflows and the evolution operators. Several asymptotic properties for skew-evolution semiflows are defined and characterized see Viet Hai (2010), Viet Hai (2011), Stoica & Borlea (2014), Stoica & Megan (2010) or Yue *et al.* (2014).

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In this paper we intend to study some concepts of uniform exponential dichotomy for skew-evolution semiflows in Banach spaces. The definitions of various types of dichotomy are illustrated by examples. We also aim to give connections between them, emphasized by counterexamples.

2. Preliminaries

Let (X, d) be a metric space, V a Banach space and $\mathcal{B}(V)$ the space of all V-valued bounded operators defined on V. Denote $Y = X \times V$ and $T = \{(t, t_0) \in \mathbb{R}_+^2 : t \geqslant t_0\}$.

Definition 2.1. A mapping
$varphi : T \times X \to X$ is said to be *evolution semiflow* on X if the following properties are satisfied:

(es1) $\phi(t, t, x) = x$, $(\forall)(t, x) \in \mathbb{R}_+ \times X$;

(es2) $\phi(t, s, \phi(s, t_0, x)) = \phi(t, t_0, x)$, $(\forall)(t, s), (s, t_0) \in T, x \in X$.

Definition 2.2. A mapping $\Phi : T \times X \to \mathcal{B}(V)$ is called *evolution cocycle* over an evolution semiflow ϕ if:

(ec1) $\Phi(t, t, x) = I$, $(\forall)t \geqslant 0, x \in X$ (I - identity operator).

(ec2) $\Phi(t, s, \phi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x)$, $(\forall)(t, s), (s, t_0) \in T$, $(\forall)x \in X$.

Let Φ be an evolution cocycle over an evolution semiflow φ. The mapping $C = (\varphi, \Phi)$, defined by:

$$C : T \times Y \to Y,\ C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v)$$

is called *skew-evolution semiflow* on Y.

Example 2.1. We will denote $C = C(\mathbb{R}, \mathbb{R})$ the set of continous functions $x : \mathbb{R} \to \mathbb{R}$, endowed with uniform convergence topology on compact subsets of $\mathbb{R}$. The set C is metrizable with the metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)},\ \text{unde } d_n(x, y) = \sup_{t \in [-n, n]} |x(t) - y(t)|.$$

For every $n \in \mathbb{N}^*$ we consider a decreasing function

$$x_n : \mathbb{R}_+ \to \left(\frac{1}{2n+1}, \frac{1}{2n}\right),\ \lim_{t \to \infty} x_n(t) = \frac{1}{2n+1}.$$

We will denote

$$x_n^s(t) = x_n(t + s),\ \forall t, s \geqslant 0.$$

Let be X the closure in C of the set $\{x_n^s, n \in \mathbb{N}^*, s \in \mathbb{R}_+\}$. The application

$$\varphi : T \times X \to X,\ \varphi(t, s, x) = x_{t-s},\ \text{unde } x_{t-s}(\tau) = x(t - s + \tau),\ \forall \tau \geqslant 0,$$

is a evolution semiflow on X. Let consider the Banach space $V = \mathbb{R}^2$ with the norm $\|(v_1, v_2)\| = |v_1| + |v_2|$. Then, the application

$$\Phi : T \times X \to \mathcal{B}(V),\ \Phi(t, s, x)v = \left(e^{\alpha_1 \int_s^t x(\tau - s)d\tau} v_1, e^{\alpha_2 \int_s^t x(\tau - s)d\tau} v_2\right),$$

where $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ is fixed, is a cocycle aplication of evolution over the semiflow φ, and $C = (\varphi, \Phi)$ is a evolution cocycle on Y.

Let us remind the definition of an evolution operator, followed by examples that punctuate the fact that it is generalized by an skew-evolution semiflows.

Definition 2.3. A mapping $E : T \to \mathcal{B}(V)$ is called *evolution operator* on V if following properties hold:

(e_1) $E(t, t) = I,\ \forall t \in \mathbb{R}_+$;

(e_2) $E(t, s)E(s, t_0) = E(t, t_0),\ \forall (t, s), (s, t_0) \in T$.

Example 2.2. One can naturally associate to every evolution operator E the mapping

$$\Phi_E : T \times X \to \mathcal{B}(V),\ \Phi_E(t, s, x) = E(t, s),$$

which is an evolution cocycle on V over every evolution semiflow φ. Therefore, the evolution operators are particular cases of evolution cocycles.

Example 2.3. Let $X = \mathbb{R}_+$. The mapping

$$\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+,\ \varphi(t, s, x) = t - s + x$$

is an evolution semiflow on $\mathbb{R}_+$. For every evolution operator $E : T \to \mathcal{B}(V)$ we obtain that

$$\Phi_E : T \times \mathbb{R}_+ \to \mathcal{B}(V),\ \Phi_E(t, s, x) = E(t - s + x, x)$$

is an evolution cocycle on V over the evolution semiflow φ. It follows that an evolution operator on V is generating a skew-evolution semiflow on Y.

3. Sequences of Invariant Projections for a Cocycle

Definition 3.1. A continuous map $P : X \to \mathcal{B}(V)$ which satisfies the following relation:

$$P(x)P(x) = P(x), (\forall) x \in X$$

is called projection on V.

Definition 3.2. A projection P on V is called *invariant* for a skew-evolution semiflow $C = (\varphi, \Phi)$ if:

$$P(\varphi(t, s, x))\, \Phi(t, s, x) = \Phi(t, s, x)P(x),$$

for all $(t, s) \in T$ and $x \in X$.

Remark. If P is a projection on V, than the map

$$Q : X \to \mathcal{B}(V),\ Q(x) = I - P(x)$$

is also a projection on V, called complementary projection of P.

Remark. If the projection P is invariant for C then Q is also invariant for C.

Definition 3.3. We will name (C, P) a dichotomy pair where C is a skew-evolution semiflow and P is invariant or C.

4. Concepts of Uniform Exponential Dichotomy for Skew-Evolution Semiflows

Definition 4.1. Let (C, P) be a dichotomy pair. We say that (C, P) is *uniformly strongly exponentially dichotomic* (u.s.e.d) if there exist $N \geqslant 1$ and $\nu > 0$ such that:

(used1) $\|\Phi(t, s, x)P(x)\| \leqslant Ne^{-\nu(t-s)}$

(used2) $N\|\Phi(t, s, x)Q(x)\| \geqslant e^{\nu(t-s)}$

for all $(t, s) \in T$ and $x \in X$.

Definition 4.2. We say that (C, P) is *uniformly exponentially dichotomic* (u.e.d) if there exist $N \geqslant 1$ and $\nu > 0$ such that:

(ued1) $\|\Phi(t, s, x)P(x)v\| \leqslant Ne^{-\nu(t-s)}\|P(x)v\|$

(ued2) $N\|\Phi(t, s, x)Q(x)v\| \geqslant e^{\nu(t-s)}\|Q(x)v\|$

for all $(t, x) \in T \times X$ and for all $v \in V$.

Definition 4.3. We say that (C, P) is *uniformly weakly exponentially dichotomic* (u.w.e.d) if there exist $N \geqslant 1$ and $\nu > 0$ such that:

(uwed1) $\|\Phi(t, s, x)P(x)\| \leqslant Ne^{-\nu(t-s)}\|P(x)\|$

(uwed2) $N\|\Phi(t, s, x)Q(x)\| \geqslant e^{\nu(t-s)}\|Q(x)\|$

for all $(t, x) \in T_x X$ and for all $v \in V$.

Proposition 1. *If (C, P) is (s.u.e.d) then*

$$\sup_{x \in X} \|P(x)\| < +\infty. \tag{4.1}$$

Proof. Consider in (used1) $t = s$. Then we have

$$\|\Phi(t, t, x)P(x)\| = \|P(x)\| = \|P(x)\| \leqslant N \tag{4.2}$$

for all $x \in X$. □

Proposition 2. *If (C, P) is (u.s.e.d) then (C, P) is (u.w.e.d).*

Proof. If (C, P) is (u.s.e.d) then by (used1), for $x \in X$, we have that $\|P(x)\| \leqslant N$ and hence

$$\|Q(x)\| = \|I - P(x)\| \leqslant 1 + \|P(x)\| \leqslant 2N.$$

We have from (used1) and (used2) that:

$$\begin{aligned} \|\Phi(t,s,x)P(x)\| &\leqslant Ne^{-\nu(t-s)} \cdot 1 \leqslant Ne^{-\nu(t-s)}\|P(x)\| && (4.3)\\ &\leqslant 2N^2 e^{-\nu(t-s)}\|P(x)\|. && (4.4)\end{aligned}$$

$$2N^2\|\Phi(t,s,x)Q(x)\| \geqslant 2Ne^{\nu(t-s)} \geqslant e^{\nu(t-s)}\|Q(x)\|, \quad (4.5)$$

hence (C, P) is (u.w.e.d) □

Proposition 3. *If (C, P) is (u.e.d) then (C, P) is also (u.w.e.d)*

Proof. It follows immediately by taking the supremum over all $v \in V$ with $\|v\| = 1$. □

Definition 4.4. We say that C has a uniform exponential growth (u.e.g) if there exist $M \geqslant 1$, $\omega > 0$ such that

$$\|\Phi(t,s,x)\| \leqslant Me^{\omega(t-s)},$$

for all $(t, s) \in T$ and $x \in X$.

Theorem 4.1. *Assume that a dichotomy pair (C, P) is (u.w.e.d) and C has a uniform exponential growth. Then:*

$$\sup_{x \in X} \|P(x)\| < +\infty.$$

Proof. Let N, ν given by the (u.w.e.d) property of (C, P) and M, ω given by the (u.e.g) of C. Consider $s \geqslant 0$ fixed, $t \geqslant s$ and $x \in X$.

$$\begin{aligned} \left[\frac{1}{2N}e^{\nu(t-s)} - Ne^{-\nu(t-s)}\right]\|P(x)\| &\leqslant \frac{1}{N}e^{\nu(t-s)}\|Q(x)\| - Ne^{-\nu(t-s)}\|P(x)\|\\ &\leqslant \|\Phi(t,s,x)Q(x)\| - \| - \Phi(t,s,x)P(x)\|\\ &\leqslant \|\Phi(t,s,x)\| \leqslant Me^{\omega(t-s)}.\end{aligned}$$

Let $t_0 > 0$ be such that

$$\lambda_0 := \frac{1}{2N}e^{\nu t_0} - Ne^{-\nu t_0} > 0.$$

From the above estimation is follows that for $t = t_0 + s$,

$$\|P(x)\| \leqslant \frac{Me^{\omega t_0}}{\lambda_0}, \ (\forall) x \in X.$$

□

from where the conclusion follows.

Remark. In the following section we will see that for a dichotoomic pair (C, P):

1. (u.s.e.d) does not imply (u.e.d)
2. (u.e.d) does not imply (u.s.e.d)
3. (u.w.e.d) does not imply (u.e.d)
4. (u.w.e.d) does not imply (u.s.e.d)

5. Examples and Counterexamples

Example 5.1. Define, on $\mathbb{R}^3$, the family of projections

$$P(x)(v_1, v_2, v_3) = (v_1, 0, 0)$$

and the evolution cocycle on $\mathbb{R}^3$:

$$\Phi(t, s, x)(v_1, v_2, v_3) = \begin{cases} (v_1, v_2, v_3),\ t = s \\ (e^{s-t}v_1, e^{t-s}v_2, 0),\ t > s, \end{cases}$$

with the following norm:

$$\|x\| = |x_1| + |x_2| + |x_3|,\ x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We have that for all $(t, s) \in T, x \in X$ and $v \in \mathbb{R}^3$

$$\|\Phi(t, s, x)P(x)v\| = e^{s-t}v_1 = e^{s-t}\|P(x)v\|$$

from where we get that

$$\|\Phi(t, s, x)P(x)\| \leqslant e^{s-t}\|P(x)\|$$

and

$$\|\Phi(t, s, x)Q(x)v\| = \begin{cases} \|Q(x)v\|,\ t = s \\ \|(0, e^{t-s}v_2, 0)\|,\ t > s \end{cases} \leqslant e^{t-s}\|Q(x)v\|$$

hence

$$\|\Phi(t, s, x)Q(x)\| \leqslant \|Q(x)\|.$$

Choose $(0, 1, 0) \in \mathbb{R}^3$. Then

$$\|\Phi(t, s, x)Q(x)(0, 1, 0)\| = e^{t-s}\|Q(x)(0, 1, 0)\|$$

from where we finally obtain that:

$$\|\Phi(t, s, x)Q(x)\| = e^{t-s}\|Q(x)\|,$$

hence (C, P) is (u.w.e.d). Assume by a contradiction that (C, P) is (u.e.d). Then there exists, $N \geqslant 1$, $\nu > 0$ such that

$$N\|\Phi(t, s, x)Q(x)(v_1, v_2, v_3)\| \geqslant e^{\nu(t-s)}\|Q(x)(v_1, v_2, v_3)\|. \tag{5.1}$$

Put $t > s$ and $(v_1, v_2, v_3) = (0, 0, 1)$. Then $\|Q(x)(v_1, v_2, v_3)\| = 1$ and

$$e^{\nu(t-s)} \leqslant \|\Phi(t, s, x)(v_1, v_2, v_3)\| = \|\Phi(t, s, x)(0, 0, 1)\| = 0,$$

which is a contradiction.

Example 5.2 (u.e.d does not imply u.s.e.d)**.** On $V = \mathbb{R}^2$ and $(X, d) = (\mathbb{R}_+, d)$ endowed with the max - norm. Consider,

$$P(x) : \mathbb{R}^2 \to \mathbb{R}^2,\ P(x)(v_1, v_2) = (v_1 + xv_2, 0)$$

it follows that

$$\|P(x)\| = 1 + x, (\forall)x \geqslant 0 \tag{5.2}$$

Define the skew - evolutiv cocycle

$$\Phi(t, s, x) = e^{s-t}P(x) + e^{t-s}Q(x).$$

We have that

$$\begin{aligned} \|\Phi(t, s, x)P(x)\| = e^{s-t}\|P(x)\| \text{ and} \\ \|\Phi(t, s, x)Q(x)\| \geqslant e^{t-s}\|Q(x)\| \end{aligned} \tag{5.3}$$

Hence (C, P) is (u.e.d). It can not be (u.s.e.d) because of (5.2).

Remark. From the above example, by taking the sup norm in (5.3) over $\|v\| = 1$, we get that (C, P) is also (u.w.e.d). Hence (C, P) is (u.w.e.d) but not (u.s.e.d).

Remark. The connection between the three concepts studied in this paper is summarized in the below diagram

$$(u.s.e.d) \not\Rightarrow (u.e.d) \Rightarrow (u.w.e.d) \Leftarrow (u.s.e.d)$$

$$(u.s.e.d) \not\Leftarrow (u.e.d) \not\Leftarrow (u.w.e.d) \not\Rightarrow (u.s.e.d).$$

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Rényi Entropy in Measuring Information Levels in Voronoï Tessellation Cells with Application in Digital Image Analysis

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Abstract

This work introduces informative and interesting Voronoï regions through measures utilizing probability density functions and qualities of Voronoï cells of digital image point patterns. Global mesh cell quality exhibits a fairly horizontal behaviour in its range of convergence across several categories of digital images. Simulation results unambiguously show that Shannon entropy does not expose the most information in Voronoï meshes although it's in the range $1 < \beta \leq 2.5$ for which information is maximized. Mesh information is seen to be generally a non-linear, non-decreasing function of image point patterns. Some important mathematical theorems on quantities and optimality conditions are proved.

Keywords: Generator, quality, Voronoï mesh, pattern, entropy, information.
2010 MSC No: Primary 54E05, Secondary 20L05, 35B36.

1. Introduction

This article introduces an approach to measuring the information levels Voronoï tessellation (mesh) cells via Rényi entropy. The focus is on the Rényi entropy of Voronoï meshes with varying quality. Let $p(x_1), \ldots, p(x_i), \ldots, p(x_n)$ be the probabilities of a sequence of events $x_1, \ldots, x_i, \ldots, x_n$ and let $\beta \geq 1$. Then the Rényi entropy (Rényi, 2011) $H_\beta(X)$ of a set of event X is defined by

$$H_\beta(X) = \frac{1}{1-\beta} \ln \sum_{i=1}^{n} p^{\beta}(x_i) \text{ (Rényi entropy).}$$

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Rényi's entropy is based on the work by R.V.L. Hartley (Hartley, 1928) and H. Nyquist (Nyquist, 1924) on the transmission of information. A proof that $H_\beta(X)$ approaches Shannon entropy as $\beta \longrightarrow 1$ is given in (Bromiley *et al.*, 2010), *i.e.*,

$$\lim_{\beta \longrightarrow 1} \frac{1}{1-\beta} \ln \sum_{i=1}^{n} p^{\beta}(x_i) = -\sum i = 1^n p_i \ln p_i.$$

The information of order β contained in the observation of the event x_i with respect to the random variable X is defined by $H(X)$. In our case, it is information level of the observation of the quality of a Voronoï mesh cell viewed as random event that is considered in this study. The principle application of the proposed approach to measuring the information levels of mesh cells is the tessellation of digital images.

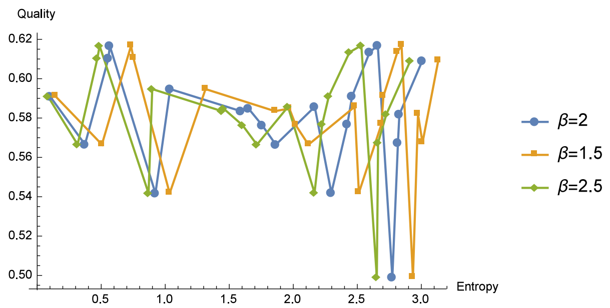


Figure 1. Rényi entropy

A main result reported in this study is the correspondence between image quality and Rényi entropy for different types of tessellated digital images. In other words, the correspondence between the Rényi entropy of mesh cells relative to the quality of the cells varies for different classes of images. For example, with Voronoï tessellations of images of humans, Rényi entropy tends to be higher for higher quality mesh cells (see, *e.g.*, the plot in Fig. 1 for different Rényi entropy levels, ranging from $\beta = 1.5$ to 2.5 in 0.5 increments).

2. Literature Review on Voronoï Diagrams

It is known that generating meshes is a fundamental and necessary step in several domains such as engineering, computing, geometric and scientific applications (Leibon & Letscher, 2000; Owen, 1998; Liu & Liu, 2004). No matter what their domain application and the specific terminology used, the resultant meshes have structures or volumes that result from the geometry of surfaces, dimension of the space and placement or organization of generators(Ebeida & Mitchell, 2012; Mitchell, 1993; Persson, 2004). Meshes may be generated for purposes of image processing and segmentation (Arbeláez & Cohen, 2006), clustering (Ramella *et al.*, 1998), data compression, quantization, analysis of territorial behavior of animals (Persson, 2004; Persson & Strang, 2004; Du *et al.*, 1999) to name a few. Applications of meshes are growing but works in the direction of exploiting pattern nature and information are lacking. We are therefore of the view that understanding the pattern and the underlying process could greatly benefit applications.

Voronoï diagrams were introduced by the Ukrainian mathematician G. Voronoï (Voronoï, 1903, 1907, 1908) (elaborated in the context of proximity and quality spaces spaces in (Peters, 2015*b*,*c*,*a*; A-iyeh & Peters, 2015; Peters, 2016)) provide a means of covering a space with regular polygons. The process allows us to understand fundamental properties of elements of the space by exploiting properties of the meshes. The properties of the space may otherwise have remained inaccessible.

In telecommunications, Voronoï diagrams have furnished a tool for analysis of binary linear block codes (Agrell, 1996) governing regions of block code and performance of Gaussian channels.

In musicology, Voronoï diagrams have demonstrated their utility (McLean, 2007). For example, they have been successfully applied in automatic grouping of polyphony (Hamanaka & Hirata, 2002). Other works bordering on applications of Voronoï meshes are in reservoir modeling (Møller & Skare, 2001) and cancer diagnosis (Demir & Yener, 2005).

The fact that the partitioning algorithm divides the plane into Euclidean neighborhoods permits exploitation of proximity relations while offering the flexibility of modeling the space as a continuous image-like point pattern representing the space. Given the substantial utility of Voronoï tessellations their applicability in additional areas including point pattern detection and image analysis is currently being investigated vigorously.

In this work meshes are generated for the purposes of characterizing a point pattern information using multiple measures for the individual mesh cells. The major focus here goes beyond tessellating a space with meshes. Additionally we search for important cues that may be fundamental for basic pattern understanding which in turn may lead to identifying and understanding the underlying pattern.

3. Preliminaries

In this section, the grounding theory entropy, quality of cells and Voronoï diagrams based on point pattern distributions is set. Some useful definitions are given prior to facilitate the process.

3.1. Notation and Definitions

A subset of points in $\mathbb{R}^n$ is denoted by S. A partition of the space of $S \subseteq \mathbb{R}^n$ according to the Voronoï criterion into contiguous non-overlapping polygons is denoted by the set $\{\mathbb{V} = \mathcal{F}, \mathcal{E}, S = \mathbb{N}\}$ where $\mathcal{F}, \mathcal{E}$ are the faces and edges of graph regions respectively. Also, properties of cells such as length of edges of polygons are represented by l_i, area by A, quality of cells by q_i and entropy by H_R.

Definition 3.1. Given a point pattern set $S \subseteq \mathbb{R}^n$ of three or more non-collinear points and a distance function d_n, the set $\{\mathbb{V}, S = \{\mathbb{N}\}$ is called a Voronoï tessellation of S if $\mathbb{V}_i \cap \mathbb{V}_j \neq \varnothing$ for $i \neq j \subset S$. A Voronoï tessellation is a set of polygons with their edges and vertices that partition a given space of points.

Definition 3.2. The Voronoï region of an image point is a polygon about that site. The set of all regions partition a plane of image points based on a distance function $\| \cdot \|$. This results in a covering of the plane with polygons about the points.

Definition 3.3. Consider the set $S = \{s_1, ..., s_k\}$, a plane (v_i, v_j) is a Voronoï edge of the Voronoï region $\mathbb{V}_i$ if and only if there exists a point x such that the circle centered at x and circumscribing v_i and v_j does not contain in its interior any other point of $\mathbb{V}_i$. A Voronoï edge is a half plane equidistant from two sites and bounds some part of the Voronoï diagram. Every edge is incident upon exactly two vertices and every vertex upon at least three edges.

Definition 3.4. A Voronoï neighborhood of a point p in the vicinity of point q is the locus of bisectors or half planes equidistant from p and q. The union of half planes H_q^p (H_p^q) is the locus of points nearer to p than to q. The intersection of half planes $\bigcap_{q \in S, q \neq p} H_p^q$ defines a region generated at p.

Definition 3.5. A Voronoï vertex is the center of a circumcircle through three sites.

Definition 3.6. A set of points S is a convex set if there is a line connecting each pair of points within S.

Definition 3.7. The convex hull of Voronoï regions about S is the smallest set which contains the Voronoï regions as well as the union of the regions.

Definition 3.8. A point pattern is a set of points of the signal representing locations of signal features. For example sets of corners, keypoints etc. are referred to as point or dot patterns.

Definition 3.9. The quality of a Voronoï cell is a dimensionless real number assigned to the cell based on the extent to which the sides of the cell match.

Definition 3.10. An open pattern point is a point such that a disk centered on it contains the point as an interior point.

Definition 3.11. A closed pattern point is a point such that a disk centered on it contains the point as well a boundary.

Definition 3.12. Let $\mathbb{V}$ be a Voronoï diagram in $\mathbb{R}^2$. The skeleton of $\mathbb{V}_i \in \mathbb{V}$, is the open set $S(\Omega)$ from which the Voronoï diagram is generated.

Definition 3.13. The Voronoï quality of visual information given by a point generator is defined as the aggregate of measure of cells comprising the tessellation. In other words it shows the organization of a point pattern.

Definition 3.14. A point pattern is feasible when there exists a constant $t > 0$ such that at least one quality measure of the Voronoï cells is at least t.

3.2. Voronoï Diagrams

The spatial distribution of point sets informs the nature and organizations of the pattern. This in turn influences the graph geometry of the Voronoï diagram the point set. Assume we have a finite set S of point locations called sites s_i in a space $\mathbb{R}^n$. Computing the Voronoï diagram with respect to S entails partitioning the space of S into Voronoï regions $\mathbb{V}(s_i)$ in such a way that the region $\mathbb{V}(s_i)$ contains all points of S that are closer to s_i than to any other object s_j, $i \neq j$ in S.

More elaborately, given the generator set

$$S = \{s_1, \ldots, s_k : i \in \mathbb{N}\},$$

the Voronoï region $\mathbb{V}(s_i)$ is defined by

$$\mathbb{V}(s_i) = \{x \in \mathbb{R}^n : \|x - s_i\| \leq \|x - s_k\|, s_k \in S, i \neq k\},$$

where $\|.,.\|$ is the Euclidean norm (distance between vectors). The set

$$\mathbb{V}(S) = \bigcup_{s_i \in S} \mathbb{V}(s_i)$$

is called the n-dimensional Voronoï diagram generated by the point set S. In $\mathbb{R}^2$, this effectively covers the plane with convex and non overlapping graphs, one for each generating point in S. By the definition of a Voronoï region above, the region about a site x satisfies

$$d(x, s_i) \leq d(x, s_k) \Leftrightarrow ||x - s_i||^2 \leq ||x - s_k||^2 \forall s_i \in S.$$

Manipulating the expression of a Voronoï region gives

$$\mathbb{V}(s_i) := \{(s_k - s_i)x \leq \frac{||s_k||^2 - ||s_i||^2}{2}, s_k \in S\}.$$

The immediate expression is recognized as an ordinary linear system of equations when S is finite (Goberna *et al.*, 2012). For a partitioned space in which all the individual regions are triangles, the optimal tessellation of the point set which maximizes the minimum angle in each triangular graph is the Delaunay triangulation. The Delaunay triangulation of S is the triangulation DT(S) where the circum-circles of all cells contain only the three points forming the triangle. Since a Delaunay image triangulation can be obtained from the corresponding Voronoï image graph our focus shall be on the latter. Point patterns in Delaunay image triangulations are informative and can be used to study the nature of the underlying tessellated process.

The advantage of Voronoï diagrams in studying patterns is that it associates the local neighborhood of a point with the information in the region inclosed by the point as opposed to point estimates only. Consequently measures may be aggregated for global pattern information gathering.

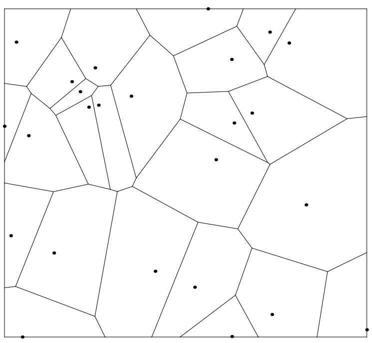


Figure 2. Voronoï mesh pattern

Fig. 3.2 displays a Voronoï diagram generated by a point set (not shown) in $\mathbb{R}^2$. The diagram shows how a space partitioned into regions of influences about the generators in the form convex non-intersecting polygons. The nature of the pattern influences the distribution of the point set as well as the structure of the partitioned space. For example polygons in regions of higher point densities are of smaller sizes or areas compared to polygons of regions with with lower point densities.

4. Patterns and Information Theory

Information in signals and patterns is commonly characterized using information theoretic approaches such as entropy and characteristics of a transformed space of the pattern such as quality measures of Voronoï cells. In the following subsections we present those tools.

4.1. Entropy

Entropy has long been an indicator of information and information content whose utility has since extended to other fields besides thermodynamics where it emerged. In thermodynamics, it was first used for understanding molecular structure. Entropy now finds applications in several other fields including portfolio selection and financial decision making (Zhou *et al.*, 2013), distribution analysis (Chapman, 1970) where it's founded on probability density functions derived from random variables.

Some general observations on entropic information are in order before proceeding. If all the realizations of a random variable have equal chance of being observed, then the variables have equal probabilities. Relating this to Voronoí cells this means we have a simple pattern formed by repetition of a unit. Consequently the same information is contained in all cells of the pattern. This scenario corresponds to maximization of entropy.

When a measure of information in a pattern is maximized the variations of the pattern primitives must be minimal and one variable or cell and its attribute is representative of the pattern. This situation also means there is no other information in the pattern other than the fact that the random variables of the pattern are uniformly distributed. On the contrary variations in a random variable indicates interestingness, disorder, complexity or randomness in the pattern and most importantly a distribution of variables that is anything but uniform.

4.1.1. Renyi Entropy

Renyi entropy is a general information criterion of which Shannon entropy and others are special cases (Xu & Erdogmuns, 2010). This generality is useful in diversity and dissimilarity characterization (Rao, 1982) of pattern structure. Recall that the area of a Voronoï cell satisfies $0 < A_i \leq \infty$ and so the probability $Pr(.)$ of the area random variable assuming a value in the range of areas is defined in $0 \leq Pr(A_i) \leq 1$. Let A_T be the total planar surface area of a Voronoï tessellation $\mathbb{V}$. It follows that the probability of the random variable A_i is defined by

$$Pr(A_i) = \frac{A_i}{A_T},$$

and

$$\sum_i Pr(A_i) = 1.$$

A general entropy criterion utilizing the probability densities of the random variables is defined by:

$$H = \frac{1}{1-\beta} ln \sum_{i=1}^{n} Pr_i^{\beta},$$

where $Pr(A_i) = Pr_i$. A noteworthy property of Renyi entropy is majorization. Assume two finite probability vectors P and Q of length $1 < k \leq n$. P is said to majorize Q if

$$P_1 + P_2 + \cdots + P_k \geq Q_1 + Q_2 + \cdots + Q_k.$$

This means that P exhibits a stronger tendency towards uniformity than Q and thus has more entropy. This is an important indicator for understanding the nature of the distribution of a random variable.

4.2. *Cell Quality*

Mesh quality in the literature is sufficiently developed with guarantees for triangular and tetrahedral elements (Bern & Eppstein, 1995). However this is not so for mesh elements of four or more sides as well as hexahedra. As a result this research is necessitated in the direction of mesh elements from planar Voronoï diagrams which mostly have four or more sides towards their quality guarantees. This is where the potential utility and impact of mesh qualities in this work is directed. The quality of a mesh depicts a way of investigating pattern organization with a measure of geometric structure. The quality q of a cell is defined by the lengths of the sides of the polygon l_i and its area A. To illustrate consider a quadrilateral Voronoï cell. Its quality is defined by

$$q = 4\frac{A}{l_1^2 + l_2^2 + l_3^2 + l_4^2}.$$

Quality factors of different kinds of polygons are adopted to the criteria of (Shewchuk, 2002; Bhatia & Lawrence, 1990; Knupp, 2001). Quality measures are defined to assume values in $0 \leq q \leq 1$. A quality value of zero corresponding to a degenerate mesh region whilst a value of one corresponds to a region with equal polygonal side lengths.

5. Theorems and Observations on Voronoï Diagrams

Let $\{q_i\}, i = 1, 2, .., n < \infty$ be the set of qualities of cells resulting from a Voronoï tessellation.

Theorem 5.1. *Qualities of cells satisfy the inequality*

$$(q_1 + q_2 + q_3 + ... + q_n)^2 \leq n^2.$$

Proof. Without loss of generality assume $n = 4$. Notice that $q_i \in [0, 1]$

$$(q_1 + q_2 + q_3 + q_4)^2 = q_1^2 + 2q_1q_2 + q_2^2 + q_1q_3 + q_1q_4 + q_2q_3 + q_2q_4 + q_3^2 + 2q_3q_4 + q_4^2q_i^2, q_iq_j \leq 1.$$

Each of the individual terms is potentially less than its maximum value since all the qualities may not have $q_i = 1$. So the squared sum of the qualities is equal to n^2 if and only if all cells have a quality of 1. The quality inequality must be as it is to take care of qualities other than the extremes of zero and unity. Thus we must have

$$(q_1 + q_2 + q_3 + ... + q_n)^2 \leq n^2,$$

for $n < \infty$. □

Theorem 5.2. *For a Voronoï cell of quality $q_i = 1$ there exists a point inside the cell to which all vertices are equidistant.*

Proof. See (A-iyeh & Peters, 2015). □

Theorem 5.3. *For every Voronoï cell with $q = 1$ there exists a polygon whose edge lengths are not unequal.*

Proof. See (A-iyeh & Peters, 2015). □

Lemma 5.1. *Let $A(\mathcal{V}_s)$ be the area of the smallest polygon in a Voronoï mesh and let $A(\mathcal{V}_l)$ be the area of the polygon with the largest area in the same mesh with intermediate polygonal areas $A(\mathcal{V}_1)\ldots,A(\mathcal{V}_n)$. Then*

$$A(\mathcal{V}_s) \subseteq A(\mathcal{V}_l)$$

and

$$A(\mathcal{V}_s) \subseteq A(\mathcal{V}_1) \subseteq A(\mathcal{V}_2)\cdots \subseteq A(\mathcal{V}_n) \subseteq A(\mathcal{V}_l)$$

for a mesh with $n+2$ polygons.

Lemma 5.2. *The sequence of all ordered elements of the projections of sets A_1 and B_1, i.e., $\{a_n\}$ and $\{b_n\}$, $n = 1,2,3,...$ form a metric space.*

Consider polygonal elements of R^n with elements $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$. Let $\rho(A_1, B_1) = Inf\{\|x-y\| : x \in A_1, y \in B_1\}$ be the distance between functions of bounded elements A_1 and B_1 of the space. Again let $pr_n(A_1) = Inf\{x \in A_1 | \exists x_1, x_2 ..., x_{n-1} \in R : x = (x_1, x_2 ..., x_{n-1}) \in A_1\}$ be the projection of set A_1 onto the nth-coordinate space of R^n and $\Delta_{l_1 \ldots l_{n-1}}$ represents polygons(half-open meshes) of the form $(l_1 h, l_1 h + h] \times \cdots \times (l_{n-1} h, l_{n-1} h + h]$. h is the edge length and $l_1, l_2, ..., l_{n-1}$ are integers.

Theorem 5.4. *If A_1 and B_1 be bounded polygons in a Voronoï with with a function of the polygons $\rho(A_1, B_1) = \delta_0 > 0$, then a family of polygons $\{\triangle\}_{k=1}^{N}$, $\triangle_k \subseteq \mathbb{R}^{n-1}$ exists such that*

$$pr_{\mathbb{R}^{n-1}}(A \cup B) \subseteq \prod_{i=1}^{N},$$

for any $\triangle$ if $x \in A$, $y \in B$, $pr_{\mathbb{R}^{x-1}x}, pr_{(R)^{n-1}y} \in \triangle_k$, then $|x_n - y_n| = |pr_n x - pr_n y| \geq \delta = \frac{\delta_0}{2}$.

Proof. Assume $h \in (0, \delta_0(2n)^{-1/2})$. Let $D_{k_1 \ldots k_{n-1}} = \triangle_{k_1 \ldots k_{n-1} \mathbb{R}}$. Then $D_{k_1 \ldots k_{n-1}}$ possesses the following properties

1. $\bigcup_{k_1,\ldots k_{n-1} \in \mathbb{Z}} D_{k_1,\ldots k_{n-1}} = \mathbb{R}^n$
2. $D_i \cap D_j = \varnothing$
3. $\forall D = D_{k_1,\ldots k_{n-1}}$ and $\forall x, y \in D$ if $\rho(x,y) \geq \delta_0$, then $\rho(pr_n x, pr_y) \geq \delta_0/2$

Consider $x, y \in D$ and assume $|x_n - y_n| = |pr_n x - pr_n y| < \delta_0/2$. Then we have $\rho(x, y) = [(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2]^{-1/2} \leq (h^2 + \cdots + h^2 + \delta_0^2/4)$. For $h \in (0, \delta_0(2n)^{-1/2})$, $\rho(x, y) = (\delta_0^2(n-1)/(2n) + \delta_0^2/4)^{-1/2} < \delta_0$. This is untrue. Hence, property 3 is proved.

For property 2, $A \cup B \neq \varnothing$ and the union of the bounded sets is bounded, so $\bigcup_{i=1}^{N} \supseteq A \cup B$. Thus the union of all the polygons covers the space $\mathbb{R}^n$ and that proves property 1. $pr_{\mathbb{R}^{n-1}}(\bigcup_{i=1}^{N} D_i) = pr_{\mathbb{R}^{n-1}}(\bigcup_{k=1}^{N} \triangle_k \mathbb{R}) \supseteq pr_{\mathbb{R}^{n-1}}(A \cup B)$ and $\bigcup_{k=1}^{N} \supseteq pr_{\mathbb{R}^{n-1}}(A \cup B)$. These statements imply that for $x \in A$, $y \in B$ we can find $\triangle_k$ such that $\rho(x, y) \geq \delta_0$ by assumption, so that $\rho(pr_n x, pr_n y) = |x_n - y_n| \geq \delta_0/2 = \delta$. □

Theorem 5.5. *Symmetry is a condition for optimality of Voronoï meshes.*

Proof. Note that $\mathbb{V}$ for a site s can be expressed as $\mathbb{V}(s_i) := \{(s_k - s_i)x \leq \frac{||s_k||^2 - ||s_i||^2}{2}, s_k \in S\}$. To show optimality we need

$$\frac{\partial \mathbb{V}(s_i)}{\partial s_i}.$$

This gives

$$\frac{\partial V(s)}{\partial s} = -x = -\frac{2||s_i||}{2}.$$

The immediate expression is equivalent to

$$x = \begin{cases} s_i, & \text{if } x \geq 0, \\ s_i, & \text{if } x < 0, \end{cases}$$

which is a mathematical expression for symmetry. □

Property 1. Given a measure function $q(.)$ for a Voronoï diagram of an $n \geq 3$ point set the Voronoï tessellation consists of quality functions equal in number to the number of Voronoï cells.

Property 2. The Voronoï diagram of a set S consisting of $n \geq 3$ non-collinear objects with a measure q for the polygons has at most $2n - 5$ vertices and $3n - 6$ edges, respectively.

Theorem 5.6. *The quality of a scaled Voronoï cell is scale invariant.*

Proof. Consider a triangular cell with quality $q = 1$ before scaling. Now assume the edges of the cell have been scaled with a multiplier $m > 0$. The quality before scaling is given by

$$q = 4\sqrt{3}\frac{0.5l^2}{l^2 + l^2 + l^2} = 1.$$

The quality, after scaling, is expressed by

$$q = 4\sqrt{3}\frac{0.5(ml)^2\sqrt{\frac{3}{4}}}{(ml)^2 + (ml)^2 + (ml)^2} = 1.$$

□

6. Applications

The utility of Voronoï tessellations has often been limited to space partitioning and not understanding the pattern as evidenced by numerous articles. Owing to this an abysmal number of works explore the potential of Voronoï diagrams beyond space partitions. Even fewer works examine properties of Voronoï cells with the viewpoint of understanding underlying nature of patterns. We attempt a way of representing part of a signal space from a point set sample distribution that summarizes the pattern by its equivalent Voronoï signature. These points in the pattern form generators for Voronoï diagrams. Keypoint image patterns of buildings, animals, humans and mountains as previously utilized in (A-iyeh & Peters, 2015) were sampled from images of dimensions M by N to summarize the signals. These point patterns consist of 50 units corresponding to the most prominent in the images. To establish a fair basis for cross analysis the same number of point sets is sampled for all images. In addition all the image signals are gray scale of their respective categories from the dataset of (Wang *et al.*, 2001) (Fig. 4).
With the preamble in place we tessellate and cover the pattern spaces with Voronoï polygons. It is expected that since point patterns are distinct their Voronoï diagrams would exhibit discriminatory properties. This could be key in pattern discrimination using the computed quantities.

Upon identifying the subset representing an image space, we apply the Voronoï partition algorithm to the generators in the signal space. The result is a tessellated space of Voronoï polygons. Open polygons are typical of Voronoï partitions as such in the mathematical formulation of some derived features of the tessellated spaces we adopt techniques that allow the infinite polygons as well as the finite ones to be well behaved.
To help examine the nature and bahaviour of patterns, plots of various quantities are given. There are as many qualities as cells so we define a global quality index or fidelity to capture the geometry of the pattern. Using all cell qualities in a tessellation it is defined by

$$q_{all} = \frac{1}{n}\sum_{i=1}^{n} q_i,$$

where n is the total number of cells and q_i is the quality of cell i. This enables a one-to-one correspondence between quantities.

Due to the finite nature of digital image, we limit the geometrical extent of the point patterns to their convex sets. The information content of images are assessed using a general entropy criterion. A special case of the the general entropy criterion H occurs when $\beta = 2$. This is the so called Renyi entropy denoted here H_R. Simulation results are included for $\beta = 2, 1.5, 2.5$. This range of β captures a range of entropies including the Shannon entropy at $\beta = 2$.

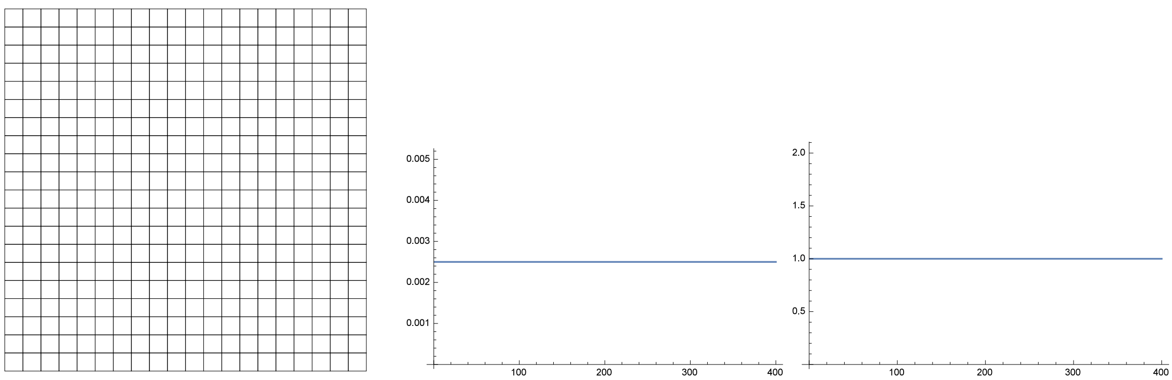
The choice of β in the neighborhood of 2 is not arbitrary. The reasons are two fold; on the one hand we are close to Shannon entropy which enables us to obtain information on the distribution of elements. On the other hand it gives us information on how units of a point pattern influence their distribution. Just as l_0 and l_∞ norms represent extremes of the smallest and largest elements of a set H_0 and H_∞ are the extremes of information measures of which $H_{0<p<\infty}$ gives a tradeoff.

The simulation process is summarized in the following algorithm.

Mesh Quality(q)
for each Voronoï region $\mathbb{V}_i \in \mathbb{V}$ of S **do**
  Access the number of sides and coordinates of the vertices of the polygon.
  Using the coordinates, compute the lengths l_i and Area A of the polygon.
  Use l_i and A_i in the appropriate expression to compute its quality q_i.
end for
$Q = \{q_i\}$
Mesh Entropy(H)
for each Voronoï region $\mathbb{V}_i \in \mathbb{V}$ **do**
  Compute Pr_i
  Use Pr_i to compute H_i
  $H = \{H_i\}$
end for

Remark 6.1. *The assumption made here is that the lengths of the sides of every Voronoï region polygon are measurable. Unfortunately, this is not always the case in, for example, Voronoï tessellations of 2D digital images, since some of the sides of Voronoï region polygons along the borders of an image have infinite length and border polygons have unbounded areas. To cope with this problem, the lengths of all border polygons are a measured relative to one or more image borders.*

Example 1. Consider a completely regular pattern tessellated as shown in Fig. 3.

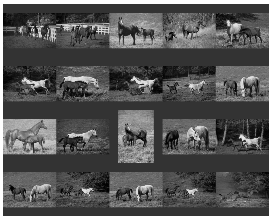


3.1: Mesh 3.2: Probability 3.3: Quality

Figure 3. Perfectly Regular Image Graph Space and Quantities

In Fig. 3 all Voronoï cells have the same area resulting in a uniform distribution of their probabilities. Also all cells have the same quality. Now there are 400 cells in the tessellation and so H attains its maximum value of 5.99146 and the global quality index also attains its maximum value of unity. From the distribution of the probability of cells and their qualities it's straight forward to see that a plot of general entropy against global quality indices would be a straight horizontal line.


4.1: Building set


4.2: Horse set


4.3: Human set

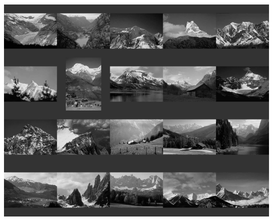
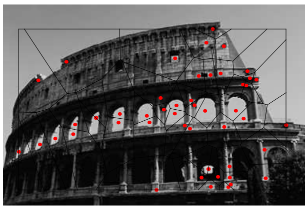
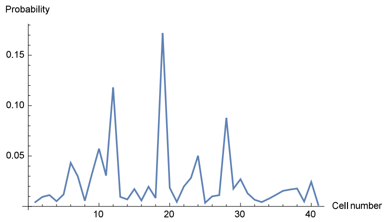
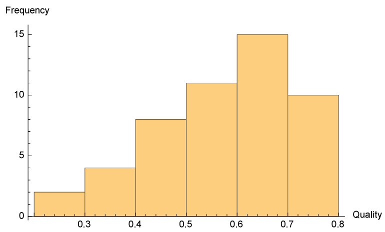

4.4: Mountain set

Figure 4. Data sets


5.1: Mesh



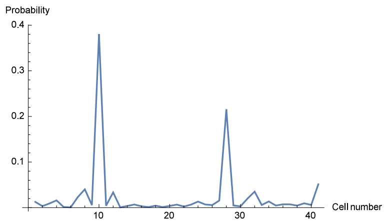
5.2: Probability



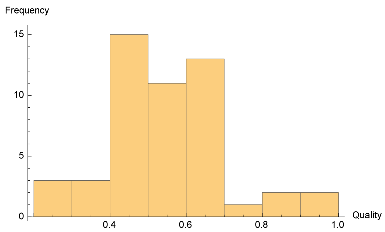
5.3: Quality

Figure 5. Image Graph Spaces


6.1: Mesh



6.2: Probability



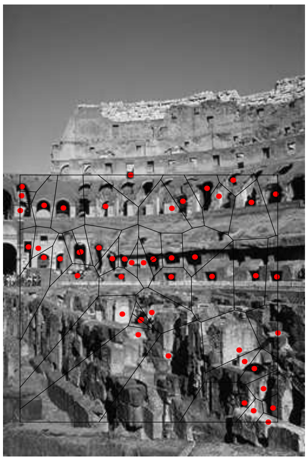
6.3: Quality

Figure 6. Image Graph Spaces

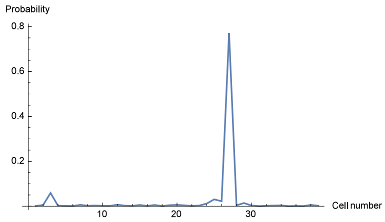
7. Results and Discussion

Most polygons typically have non-zero area so Pr_i is defined for all regions in the plane. In the following image Voronoï graphs, probability functions of cells, image cell qualities and plots of quantities are shown. Also quality of cells and information are studied by examining the nature of the plots. The results of our simulations are shown for only three images per category of the data set given in Fig. 4 for space reasons although the results are presented for the entire data set of 20 images per category amounting to 80 images in total. Corresponding cell area probabilities and distribution of cell qualities are shown next to tessellated spaces in Fig. 5-Fig. 16 in that order.

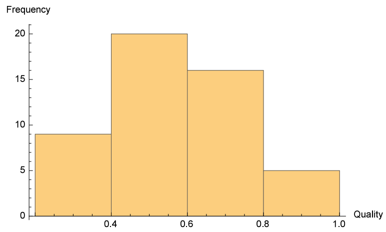
Point patterns consist of a maximum of 50 keypoints and so the resulting cells are usually 50 in number. Notice the nature of the distributions of probabilities and qualities. Probability distri-



7.1: Mesh



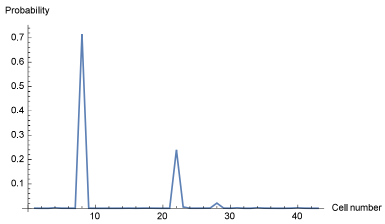
7.2: Probability



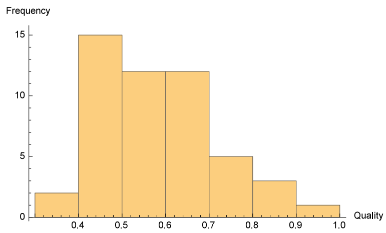
7.3: Quality

Figure 7. Image Graph Spaces

8.1: Mesh



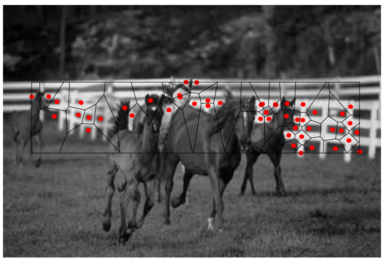
8.2: Probability



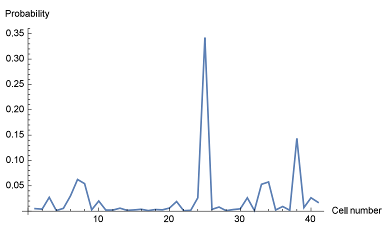
8.3: Quality

Figure 8. Image Graph Spaces

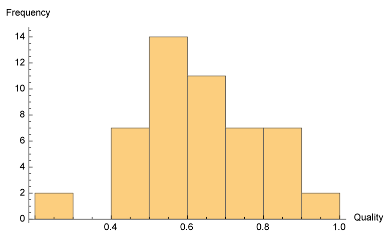
butions range from the extreme of only a few influential cells to cells exhibiting higher tendencies of equal influences. This corresponds to a few large peaks on the probability distributions and a spread out distribution respectively. The qualities of the cells portray the exhibited behaviour.



9.1: Mesh

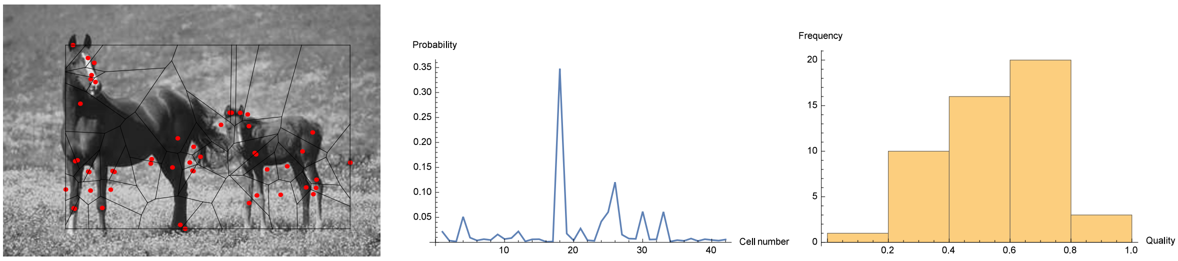


9.2: Probability



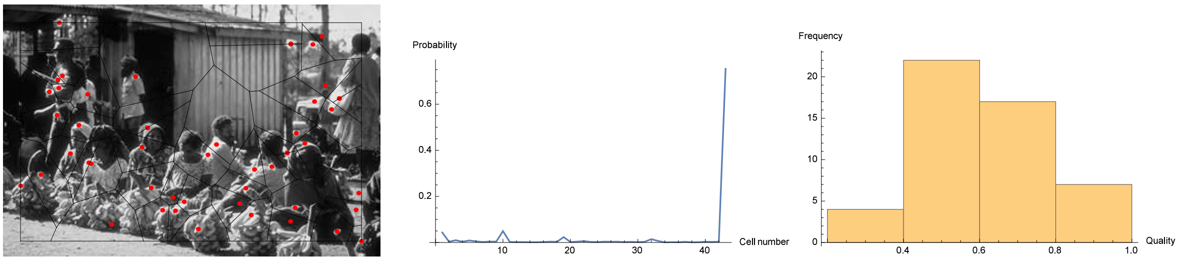
9.3: Quality

Figure 9. Image Graph Spaces



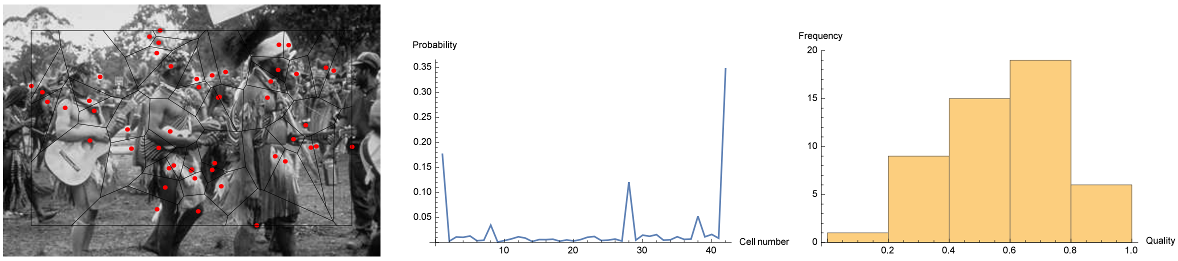
10.1: Mesh 10.2: Probability 10.3: Quality

Figure 10. Image Graph Spaces



11.1: Mesh 11.2: Probability 11.3: Quality

Figure 11. Image Graph Spaces



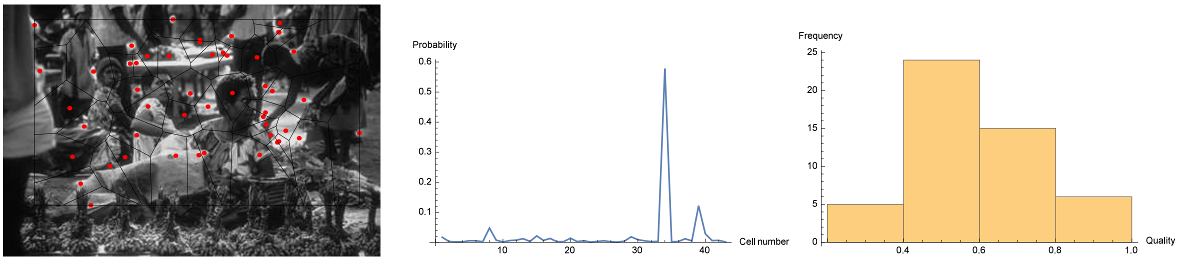
12.1: Mesh 12.2: Probability 12.3: Quality

Figure 12. Image Graph Spaces

Entropies of tessellations and global quality indices are condensed into the following plots. For 50 Voronoï cells exihibiting a uniform probability distribution the maximum value possible for Renyi entropy is 3.912. All entropy values fall short of this value. Plots of entropies and global qualities are shown for the buildings, horses, humans and mountain scenery categories in Fig. 17. Notice the flat nature of the global qualities for the images. Renyi entropies as a function of the images is non-decreasing.

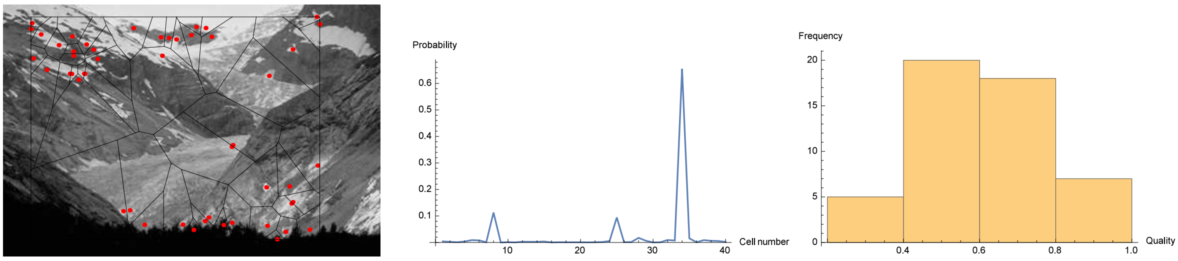
In the following, plots of global qualities, Renyi entropies and plots of entropies against qualities are shown.

Notice the monotonically increasing entropies and global qualities in Fig. 17. Also observe that



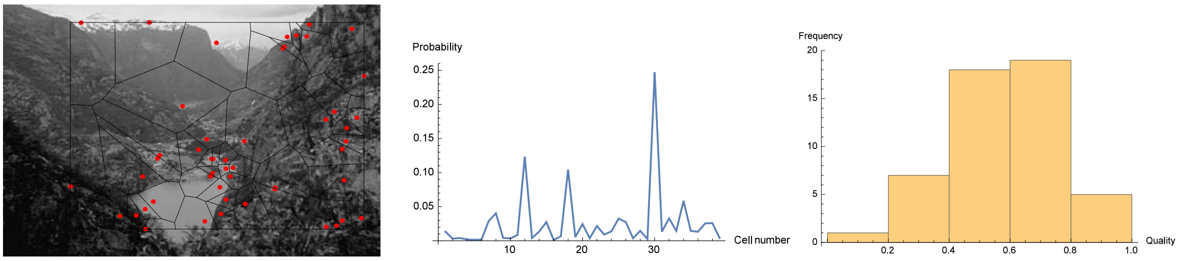
13.1: Mesh 13.2: Probability 13.3: Quality

Figure 13. Image Graph Spaces



14.1: Mesh 14.2: Probability 14.3: Quality

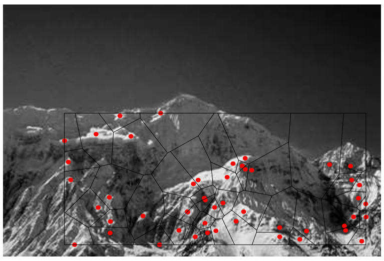
Figure 14. Image Graph Spaces



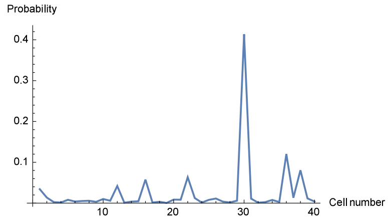
15.1: Mesh 15.2: Probability 15.3: Quality

Figure 15. Image Graph Spaces

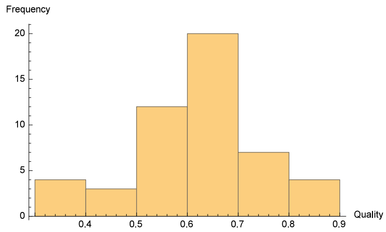
the quantities are distinct across categories. Most importantly entropic information is decreases for $\beta = 1.5, 2.0, 2.5$ in that order. Recall that $\beta = 2$ yields Shannon entropy from the general entropy criterion H. It is interesting to note the oscillating (Fig. 18) as opposed to uniform relationship between entropy and global quality. This confirms the departure of the images from the less interesting case of completely regular patterns.



16.1: Mesh

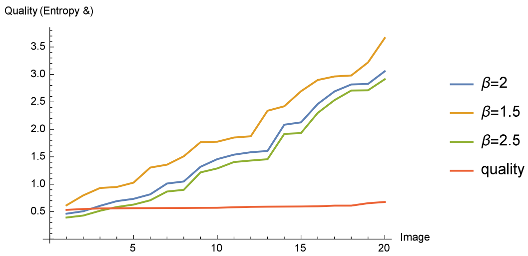


16.2: Probability

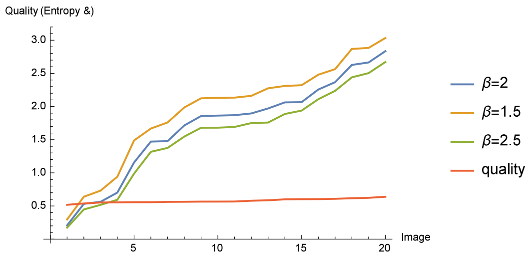


16.3: Quality

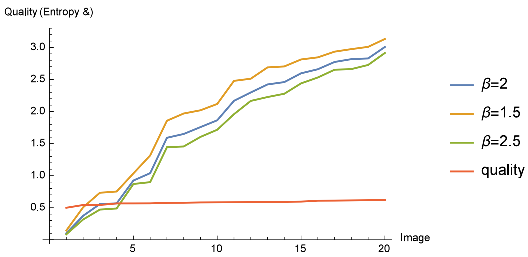
Figure 16. Image Graph Spaces



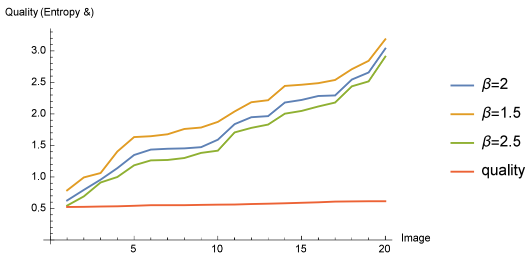
17.1: Buildings



17.2: Horses



17.3: Humans



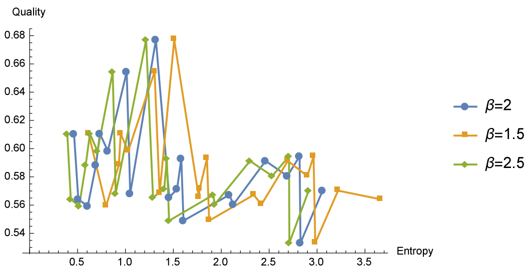
17.4: Mountains

Figure 17. Quantity Relations

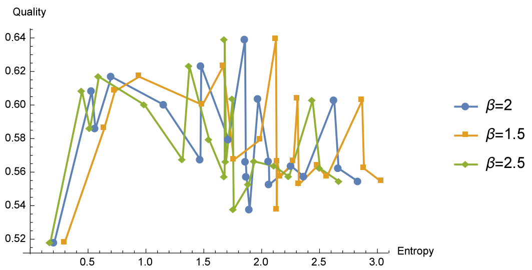
8. Conclusion and Future Work

Non-linear probability distribution distribution functions as opposed to uniform ones are observed. However, recall that a uniform distribution maximizes the entropy so that implies that the point patterns are more informative and interesting compared to completely regular patterns. Although the patterns are not uniform the information parameter range of $1 < \beta \leq 2.5$ maximizes the information content of Voronoï cells. This shows that the Renyi entropy is more informative than Shannon entropy. This is due to the variations in pattern structure. Owing to the non-linear relationship between entropy and cell qualities, we see that the patterns are not simple patterns because of the variations.

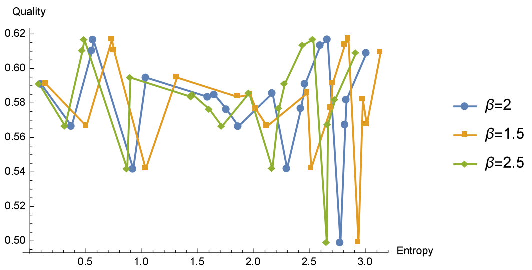
Notice that the global qualities q_{all} for all image categories practically follow a linear distribution with a gradient close to zero. So given a global quality of a tessellation converging in the



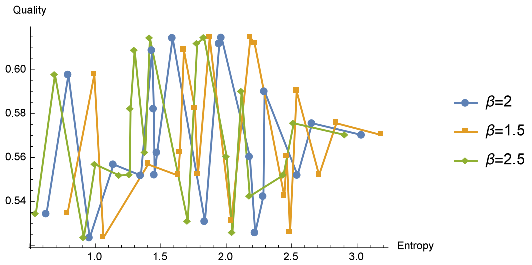
18.1: Buildings



18.2: Horses



18.3: Humans



18.4: Mountains

Figure 18. Quality Signatures

neighborhood of $0.5 \leq q_{all} < 1.0$, the point pattern is not completely regular and could be from a digital image. This range of global qualities observed shows that point pattern primitives of digital images may not be simple and completely regular features.

Image point patterns with global quality coefficients in the range $0.5 \leq q_{all} < 1.0$ are stable. This indicates that the image physical system is sufficiently modeled. This is the so called fidelity of solution of the physical system of differential equations represented by the mesh. A completely regular pattern with a global index or fidelity of unity is the most stable (Fig. 3) so that an unstable system has an index of zero or close to zero.

Since the point patterns are not completely regular they contain more information than regular ones because their global indices are less than unity and their entropies are less than the maximum value.

Notwithstanding this quality guarantees for meshes of four or more sides which is hardly studied and much less developed is seen to be stable and guaranteed in the reported range.

Finally it has been shown that the distribution of digital image point patterns is anything but uniform. Therefore future work should reveal the applicable distribution(s).

It goes without saying that although the method is simple and effective in characterizing pattern information and structure the assignment of zero probabilities to infinite Voronoï cells is a disadvantage. This however is a natural consequence of Voronoï partitioning for which the choice has to be made whether the information is attributed to a few infinite cells or otherwise.

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Further on Fuzzy Pseudo Near Compactness and *ps-ro* Fuzzy Continuous Functions

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Abstract

Main objective of this paper is to study further properties of fuzzy pseudo near compactness via *ps-ro* closed fuzzy sets, fuzzy nets and fuzzy filterbases. It is shown by an example that *ps-ro* fuzzy continuous and fuzzy continuous functions do not imply each other. Several characterizations of *ps-ro* fuzzy continuous function are obtained in terms of a newly introduced concept of *ps-ro* interior operator, *ps-ro q-nbd* and its graph.

Keywords: Fuzzy pseudo near compactness, fuzzy net, fuzzy filterbase, *ps-ro* interior, *ps-ro q-nbd*.
2010 MSC: 03E72, 54A40, 54D30, 54C08.

1. Introduction

In (Ray & Chettri, 2010), while finding interplay between a fuzzy topological space (fts, for short) (X, τ) and its corresponding strong α-level topology(general) on X, the concept of pseudo regular open(closed) fuzzy sets and *ps-ro* fuzzy topology on X was introduced, members of which are called *ps-ro* open fuzzy sets and their complements are *ps-ro* closed fuzzy sets on (X, τ). In (Ray & Chettri, 2011), in terms of above fuzzy sets, a fuzzy continuous type function called *ps-ro* fuzzy continuous function and a compact type notion called fuzzy pseudo near compactness were introduced and different properties were studied.

In this paper, fuzzy pseudo near compactness has been studied via *ps-ro* closed fuzzy sets, fuzzy nets and fuzzy filterbases. Further, it is shown by an example that *ps-ro* fuzzy continuous and fuzzy continuous functions are independent of each other. An interior-type operator called *ps-ro* interior is introduced and several properties of such functions are studied interms of this operator, *ps-ro q-nbd* and its graph.

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We state a few known definitions and results here that we require subsequently. A fuzzy point x_α is said to q-coincident with a fuzzy set A, denoted by $x_\alpha qA$ if $\alpha + A(x) > 1$. If A and B are not q-coincident, we write $A \not{q} B$. A fuzzy set A is said to be a q-neighbourhood (in short, *q-nbd.*) of a fuzzy point x_α if there is a fuzzy open set B such that $x_\alpha qB \leqslant A$ (Pao-Ming & Ying-Ming, 1980). Let f be a function from a set X into a set Y. Then the following holds:
(i) $f^{-1}(1 - B) = 1 - f^{-1}(B)$, for any fuzzy set B on Y.
(ii) $A_1 \leqslant A_2 \Rightarrow f(A_1) \leqslant f(A_2)$, for any fuzzy sets A_1 and A_2 on X. Also, $B_1 \leqslant B_2 \Rightarrow f^{-1}(B_1) \leqslant f^{-1}(B_2)$, for any fuzzy sets B_1 and B_2 on Y.
(iii) $ff^{-1}(B) \leqslant B$, for any fuzzy set B on Y and the equality holds if f is onto. Also, $f^{-1}f(A) \geqslant A$, for any fuzzy set A on X, equality holds if f is one-to-one (Chang, 1968). For a function $f : X \to Y$, the graph $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$, for each $x \in X$, where X and Y are any sets. Let X, Y be fts and $g : X \to X \times Y$ be the graph of the function $f : X \to Y$. Then if A, B are fuzzy sets on X and Y respectively, $g^{-1}(A \times B) = A \wedge f^{-1}(B)$(Azad, 1981). Let Z, X, Y be fts and $f_1 : Z \to X$ and $f_2 : Z \to Y$ be two functions. Let $f : Z \to X \times Y$ be defined by $f(z) = (f_1(z), f_2(z))$ for $z \in Z$, where $X \times Y$ is provided with the product fuzzy topology. Then if B, U_1 U_2 are fuzzy sets on Z, X, Y respectively such that $f(B) \leqslant U_1 \times U_2$, then $f_1(B) \leqslant U_1$ and $f_2(B) \leqslant U_2$ (Bhattacharyya & Mukherjee, 2000). A function f from a fts (X, τ) to fts (Y, σ) is said to be fuzzy continuous, if $f^{-1}(\mu)$ is fuzzy open on X, for all fuzzy open set μ on Y (Chang, 1968). For a fuzzy set μ in X, the set $\mu^\alpha = \{x \in X : \mu(x) > \alpha\}$ is called the strong α-level set of X. In a fts (X, τ), the family $i_\alpha(\tau) = \{\mu^\alpha : \mu \in \tau\}$ for all $\alpha \in I_1 = [0, 1)$ forms a topology on X called strong α-level topology on X (Lowen, 1976), (Kohli & Prasannan, 2001). A fuzzy open(closed) set μ on a fts (X, τ) is said to be pseudo regular open(closed) fuzzy set if the strong α-level set μ^α is regular open(closed) in $(X, i_\alpha(\tau))$, $\forall \alpha \in I_1$. The family of all pseudo regular open fuzzy sets form a fuzzy topology on X called *ps-ro* fuzzy topology on X which is coarser than τ. Members of *ps-ro* fuzzy topology are called *ps-ro* open fuzzy sets and their complements are known as *ps-ro* closed fuzzy sets on (X, τ) (Ray & Chettri, 2010). A function f from a fts (X, τ_1) to another fts (Y, τ_2) is pseudo fuzzy *ro* continuous (in short, *ps-ro* fuzzy continuous) if $f^{-1}(U)$ is *ps-ro* open fuzzy set on X for each pseudo regular open fuzzy set U on Y. For a fuzzy set A, $\wedge\{B : A \leqslant B,\ B$ is *ps-ro* closed fuzzy set on $X\}$ is called fuzzy *ps*-closure of A. In a fts (X, τ), a fuzzy set A is said to be a *ps-ro nbd.* of a fuzzy point x_α, if there is a *ps-ro* open fuzzy set B such that $x_\alpha \in B \leqslant A$. In addition, if A is *ps-ro* open fuzzy set, the *ps-ro nbd.* is called *ps-ro* open *nbd.* A fuzzy set A is called *ps-ro* quasi neighborhood or simply *ps-ro q-nbd.* of a fuzzy point x_α, if there is a *ps-ro* open fuzzy set B such that $x_\alpha qB \leqslant A$. In addition, if A is *ps-ro* open, the *ps-ro q-nbd.* is called *ps-ro* open *q-nbd.* Let $\{S_n : n \in D\}$ be a fuzzy net on a fts X. i.e., for each member n of a directed set $(D, \leqslant)$, S_n be a fuzzy set on X. A fuzzy point x_α on X is said to be a fuzzy *ps*-cluster point of the fuzzy net if for every $n \in D$ and every *ps-ro* open *q-nbd.* V of x_α, there exists $m \in D$, with $n \leqslant m$ such that $S_m qV$. A collection $\mathcal{B}$ of fuzzy sets on a fts (X, τ) is said to form a fuzzy filter base in X if for every finite subcollection $\{B_1, B_2, ..., B_n\}$ of $\mathcal{B}$, $\bigwedge_{i=1}^{n} B_i \neq 0$ (Ray & Chettri, 2011).

2. Fuzzy Pseudo Near Compactness

It is easy to observe, as pseudo regular open fuzzy sets form a base for *ps-ro* fuzzy topology, replacing *ps-ro* open cover by pseudo regular open cover, we may obtain pseudo near compactness.

Definition 2.1. Let x_α be a fuzzy point on a fts X. A fuzzy net $\{S_n : n \in (D, \geqslant)\}$ on X is said to ps-converge to x_α, written as $S_n \stackrel{ps}{\to} x_\alpha$ if for each ps-ro open q-nbd. W of x_α, there exists $m \in D$ such that $S_n q W$ for all $n \geqslant m$,$(n \in D)$.

Definition 2.2. Let x_α be a fuzzy point on a fts X. A fuzzy filterbase $\mathcal{B}$ is said to
(i) ps-adhere at x_α written as $x_\alpha \leqslant ps\text{-}ad.\mathcal{B}$ if for each ps-ro open q-nbd. U of x_α and each $B \in \mathcal{B}$, BqU.
(ii) ps-converge to x_α, written as $\mathcal{B} \stackrel{ps}{\to} x_\alpha$ if for each ps-ro open q-nbd. U of x_α, there coresponds some $B \in \mathcal{B}$ such that $B \leqslant U$.

Theorem 2.1. A fts (X, τ) is fuzzy pseudo nearly compact iff every $\{B_\alpha : \alpha \in \Lambda\}$ of ps-ro closed fuzzy sets on X with $\wedge_{\alpha\in\Lambda} B_\alpha = 0$, there exist a finite subset Λ_0 of Λ such that $\wedge_{\alpha\in\Lambda_0} B_\alpha = 0$.
Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a ps-ro open cover of X. Now, $\wedge_{\alpha\in\Lambda}(1 - U_\alpha) = (1 - \vee_{\alpha\in\Lambda} U_\alpha) = 0$. As $\{1 - U_\alpha : \alpha \in \Lambda\}$ is a collection of ps-ro closed fuzzy sets on X, by given condition, there exist a finite subset Λ_0 of Λ such that $\wedge_{\alpha\in\Lambda_0}(1 - U_\alpha) = 0 \Rightarrow 1 - \vee_{\alpha\in\Lambda_0} U_\alpha = 0$. i.e., $1 = \vee_{\alpha\in\Lambda_0} U_\alpha$. So, X is fuzzy pseudo nearly compact.
Conversely, let $\{B_\alpha : \alpha \in \Lambda\}$ be a family of ps-ro closed fuzzy sets on X with $\wedge_{\alpha\in\Lambda} B_\alpha = 0$. Then $1 = 1 - \wedge_{\alpha\in\Lambda} B_\alpha \Rightarrow 1 = \vee_{\alpha\in\Lambda}(1 - B_\alpha)$. By given condition there exist a finite subset Λ_0 of Λ such that $1 = \vee_{\alpha\in\Lambda_0}(1 - B_\alpha) \Rightarrow 1 = (1 - \wedge_{\alpha\in\Lambda_0} B_\alpha)$. Hence, $\wedge_{\alpha\in\Lambda_0} B_\alpha \leqslant (\wedge_{\alpha\in\Lambda_0} B_\alpha) \wedge (1 - \wedge_{\alpha\in\Lambda_0} B_\alpha) = 0$. Consequently, $\wedge_{\alpha\in\Lambda_0} B_\alpha = 0$.

Theorem 2.2. For a fuzzy set A on a fts, the following are equivalent:
(a) Every fuzzy net in A has fuzzy ps-cluster point in A.
(b) Every fuzzy net in A has a ps-convergent fuzzy subnet.
(c) Every fuzzy filterbase in A ps-adheres at some fuzzy point in A.
Proof. $(a) \Rightarrow (b)$**:** Let $\{S_n : n \in (D, \geqslant)\}$ be a fuzzy net in A having fuzzy ps-cluster point at $x_\alpha \leqslant A$. Let $Q_{x_\alpha} = \{A : A$ is ps-ro open q-nbd. of $x_\alpha\}$. For any $B \in Q_{x_\alpha}$, some $n \in D$ can be chosen such that $S_n q B$. Let E denote the set of all ordered pairs (n, B) with the property that $n \in D$, $B \in Q_{x_\alpha}$ and $S_n q B$. Then $(E, \succ)$ is a directed set where $(m, C) \succ (n, B)$ iff $m \geqslant n$ in D and $C \leqslant B$. Then $T : (E \succ) \to (X, \tau)$ given by $T(n, B) = S_n$, is a fuzzy subnet of $\{S_n : n \in (D, \geqslant)\}$. Let V be any ps-ro open q-nbd. of x_α. Then there exists $n \in D$ such that $(n, V) \in E$ and hence $S_n q V$. Now, for any $(m, U) \succ (n, V)$, $T(m, U) = S_m q U \leqslant V \Rightarrow T(m, U) q V$. Hence, $T \stackrel{ps}{\to} x_\alpha$.
$(b) \Rightarrow (a)$ If a fuzzy net $\{S_n : n \in (D, \geqslant)\}$ in A does not have any fuzzy ps-cluster point, then there is a ps-ro open q-nbd. U of X_α and $n \in D$ such that $S_n \not{q} U, \forall\, m \geqslant n$. Then clearly no fuzzy subnet of the fuzzy net can ps-converge to x_α.
$(c) \Rightarrow (a)$ Let $\{S_n : n \in (D, \geqslant)\}$ be a fuzzy net in A. Consider the fuzzy filter base $\mathcal{F} = \{T_n : n \in D\}$ in A, generated by the fuzzy net, where $T_n = \{S_m : m \in (D, \geqslant)$ and $m \geqslant n\}$. By (c), there exist a fuzzy point $a_\alpha \leqslant A \wedge (ps\text{-}ad\mathcal{F})$. Then for each ps-ro open q-nbd. U of a_α and each $F \in \mathcal{F}$, UqF, i.e., $UqT_n, \forall\, n \in D$. Hence, the given fuzzy net has fuzzy ps-cluster point a_α.
$(a) \Rightarrow (c)$ Let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a fuzzy filterbase in A. For each $\alpha \in \Lambda$, choose a fuzzy point $x_{F_\alpha} \leqslant F_\alpha$, and construct the fuzzy net $S = \{x_{F_\alpha} : F_\alpha \in \mathcal{F}\}$ in A with $(\mathcal{F}, \succ\succ)$ as domain, where for two members $F_\alpha, F_\beta \in \mathcal{F}$, $F_\alpha \succ\succ F_\beta$ iff $F_\alpha \leqslant F_\beta$. By (a), the fuzzy net has a fuzzy ps-cluster

point say $x_t \leqslant A$, where $0 < t \leqslant 1$. Then for any *ps-ro* open *q-nbd*. U of x_t and any $F_\alpha \in \mathcal{F}$, there exists $F_\beta \in \mathcal{F}$ such that $F_\beta >> F_\alpha$ and $x_{F_\beta} qU$. Then $F_\beta qU$ and hence $F_\alpha qU$. Thus $\mathcal{F}$ adheres at x_t.

Theorem 2.3. If a fts is fuzzy pseudo nearly compact, then every fuzzy filterbase on X with at most one ps-adherent point is ps-convergent.
Proof. Let $\mathcal{F}$ be a fuzzy filterbase with at most one ps-adherent point in a fuzzy pseudo nearly compact fts X. Then by Theorem (2.2), $\mathcal{F}$ has at least one ps-adherent point. Let x_α be the unique ps-adherent point of $\mathcal{F}$. If $\mathcal{F}$ does not ps-converge to x_α, then there is some *ps-ro* open *q-nbd*. U of x_α such that for each $F \in \mathcal{F}$ with $F \leqslant U$, $F \wedge (1-U) \neq 0$. Then $\mathcal{G} = \{F \wedge (1-U) : F \in \mathcal{F}\}$ is a fuzzy filterbase on X and hence has a ps-adherent point y_t(say) in X. Now, $U \not{q} G$, for all $G \in \mathcal{G}$, so that $x_\alpha \neq y_t$. Again, for each *ps-ro* open *q-nbd*. V of y_t and each $F \in \mathcal{F}$, $Vq(F \wedge (1-U)) \Rightarrow VqF \Rightarrow y_t$ is a ps-adherent point of $\mathcal{F}$, where $x_\alpha \neq y_t$. This shows that y_t is another ps-adherent point of $\mathcal{F}$, which is not the case.

3. *ps-ro* Fuzzy Continuous Functions

We begin this section by introducing an interior-type operator, called ps-interior operator and observe a few useful properties of that operator.

Definition 3.1. The union of all *ps-ro* open fuzzy sets, each contained in a fuzzy set A on a fts X is called fuzzy ps-interior of A and is denoted by *ps-int*(A). So, *ps-int*$(A) = \vee\{B : B \leqslant A,\ B$ is *ps-ro* open fuzzy set on $X\}$

Some properties of *ps-int* operator are furnished below. The proofs are straightforward and hence omitted.

Theorem 3.1. For any fuzzy set A on a fts (X, τ), the following hold:
(a) *ps-int*(A) is the largest *ps-ro* open fuzzy set contained in A.
(b) *ps-int*$(0) = 0$, *ps-int*$(1) = 1$.
(c) *ps-int*$(A) \leqslant A$.
(d) A is *ps-ro* open fuzzy set iff $A =$ *ps-int*(A).
(e) *ps-int*$($*ps-int*$(A)) =$ *ps-int*(A).
(f) *ps-int*$(A) \leqslant$ *ps-int*(B), $if A \leqslant B$.
(g) *ps-int*$(A \wedge B) =$ *ps-int*$(A) \wedge$ *ps-int*(B).
(h) *ps-int*$(A \vee B) \geqslant$ *ps-int*$(A) \vee$ *ps-int*(B).
(i) *ps-int*$($*ps-int*$(A)) =$ *ps-int*(A).
(j) $1 -$ *ps-int*$(A) =$ *ps-cl*$(1 - A)$.
(k) $1 -$ *ps-cl*$(A) =$ *ps-int*$(1 - A)$.

Now, we recapitulate the definition of *ps-ro* fuzzy continuous functions.

Definition 3.2. A function f from fts (X, τ_1) to fts (Y, τ_2) is pseudo fuzzy *ro* continuous (in short, *ps-ro* fuzzy continuous) if $f^{-1}(U)$ is *ps-ro* open fuzzy set on X for each pseudo regular open fuzzy set U on Y.

The following Example shows that *ps-ro* fuzzy continuity and fuzzy continuity do not imply each other.

Example 3.1. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Let A, B and C be fuzzy sets on X defined by $A(a) = 0.2, A(b) = 0.4, A(c) = 0.4$, $B(t) = 0.4, \forall t \in X$ and $C(t) = 0.2, \forall t \in X$. Let D and E be fuzzy sets on Y defined by $D(t) = 0.2, \forall t \in Y$ and $E(x) = 0.6, E(y) = 0.7, E(z) = 0.7$. Clearly, $\tau_1 = \{0, 1, A, B, C\}$ and $\tau_2 = \{0, 1, D, E\}$ are fuzzy topologies on X and Y respectively. In the corresponding topological space $(X, i_\alpha(\tau_1)), \forall \alpha \in I_1 = [0, 1)$, the open sets are $\phi, X, A^\alpha, B^\alpha$ and C^α, where $A^\alpha = \begin{cases} X, & \text{for } \alpha < 0.2 \\ \{b, c\}, & \text{for } 0.2 \leqslant \alpha < 0.4 \\ \phi, & \text{for } \alpha \geqslant 0.4 \end{cases}$, $B^\alpha = \begin{cases} X, & \text{for } \alpha < 0.4 \\ \phi, & \text{for } \alpha \geqslant 0.4 \end{cases}$ and $C^\alpha = \begin{cases} X, & \text{for } \alpha < 0.2 \\ \phi, & \text{for } \alpha \geqslant 0.2 \end{cases}$
For $0.2 \leqslant \alpha < 0.4$, the closed sets are on $(X, i_\alpha(\tau_1))$ are ϕ, X and $\{a\}$. Therefore, $int(cl(A^\alpha)) = X$. So, A^α is not regular open on $(X, i_\alpha(\tau_1))$ and hence, A is not pseudo regular open fuzzy sets on (X, τ_1) for $0.2 \leqslant \alpha < 0.4$. Similarly, it can be seen that $0, 1, B$ and C are pseudo regular open fuzzy set on (X, τ_1). Therefore, *ps-ro* fuzzy topology on X is $\{0, 1, B, C\}$. Again, E is not pseudo regular open fuzzy set for $0.6 \leqslant \alpha < 0.7$ on Y. Therefore, *ps-ro* fuzzy topology on Y is $\{0, 1, D\}$. Now, $ps\text{-}cl(B) = 1 - B$ and $ps\text{-}cl(C) = 1 - B$ where, $(1 - B)(t) = 0.6, \forall t \in X$. Define a function $f : X \to Y$ by $f(a) = x, f(b) = y$ and $f(c) = z$. Then, $f^{-1}(D)(t) = 0.2 = C(t), \forall t \in X$. Hence, $f^{-1}(U)$ is *ps-ro* open fuzzy set on X, for every *ps-ro* open fuzzy set U on Y. Therefore, f is *ps-ro* fuzzy continuous function. But, f is not fuzzy continuous as $f^{-1}(E)$ is not fuzzy open on X. Clearly, every *ps-ro* open fuzzy set is fuzzy open but not conversely, as for an example here A is fuzzy open but not *ps-ro* open fuzzy on X. This implies that a fuzzy continuous function need not be *ps-ro* fuzzy continuous. Hence, *ps-ro* fuzzy continuous and fuzzy continuous functions are independent of each other.

The following couple of results give characterizations of *ps-ro* fuzzy continuous functions.

Theorem 3.2. Let (X, τ) and (Y, σ) be two fts. For a function $f : X \to Y$, the following are equivalent:
(a) f is *ps-ro* fuzzy continuous.
(b) Inverse image of each *ps-ro* open fuzzy set on Y under f is *ps-ro* open on X.
(c) For each fuzzy point x_α on X and each *ps-ro* open *nbd*. V of $f(x_\alpha)$, there exists a *ps-ro* open fuzzy set U on X, such that $x_\alpha \leqslant U$ and $f(U) \leqslant V$.
(d) For each *ps-ro* closed fuzzy set F on Y, $f^{-1}(F)$ is *ps-ro* closed on X.
(e) For each fuzzy point x_α on X, the inverse image under f of every *ps-ro nbd*. of $f(x_\alpha)$ on Y is a *ps-ro nbd*. of x_α on X.
(f) For all fuzzy set A on X, $f(ps\text{-}cl(A)) \leqslant ps\text{-}cl(f(A))$.
(g) For all fuzzy set B on Y, $ps\text{-}cl(f^{-1}(B)) \leqslant f^{-1}(ps\text{-}cl(B))$.
(h) For all fuzzy set B on Y, $f^{-1}(ps\text{-}int(B)) \leqslant ps\text{-}int(f^{-1}(B))$.
Proof. $(a) \Rightarrow (b)$ Let f be *ps-ro* fuzzy continuous and μ be any *ps-ro* open fuzzy set on Y. Then $\mu = \vee \mu_i$, where μ_i is pseudo regular open fuzzy set on Y, for each i. Now, $f^{-1}(\mu) = f^{-1}(\vee_i \mu_i) = \vee_i f^{-1}(\mu_i)$. f being *ps-ro* fuzzy continuous, $f^{-1}(\mu_i)$ is *ps-ro* open fuzzy set and consequently,

$f^{-1}(\mu)$ is *ps-ro* open fuzzy set on X.
$(b) \Rightarrow (a)$ Let the inverse image of each *ps-ro* open fuzzy set on Y under f be *ps-ro* open fuzzy set on X. Let U be a pseudo regular open fuzzy set on Y. Every pseudo regular open fuzzy set being *ps-ro* open fuzzy set, the result follows.
$(b) \Rightarrow (c)$ Let V be any *ps-ro* open *nbd*. of $f(x_\alpha)$ on Y. Then there is a *ps-ro* open fuzzy set V_1 on Y such that $f(x_\alpha) \leqslant V_1 \leqslant V$. By hypothesis, $f^{-1}(V_1)$ is *ps-ro* open fuzzy set on X. Again, $x_\alpha \leqslant f^{-1}(V_1) \leqslant f^{-1}(V)$. So, $f^{-1}(V)$ is a *ps-ro nbd*. of x_α, such that $f(f^{-1}(V)) \leqslant V$, as desired.
$(c) \Rightarrow (b)$ Let V be any *ps-ro* open fuzzy set on Y and $x_\alpha \leqslant f^{-1}(V)$. Then $f(x_\alpha) \leqslant V$ and so by given condition, there exists *ps-ro* open fuzzy set U on X such that $x_\alpha \leqslant U$ and $f(U) \leqslant V$. Hence, $x_\alpha \leqslant U \leqslant f^{-1}(V)$. i.e., $f^{-1}(V)$ is a *ps-ro nbd*. of each of the fuzzy points contained in it. Thus $f^{-1}(V)$ is *ps-ro* open fuzzy set on X.
$(b) \Leftrightarrow (d)$ Obvious.
$(b) \Rightarrow (e)$ Suppose, W is a *ps-ro* open *nbd*. of $f(x_\alpha)$. Then there exists a *ps-ro* open fuzzy set U on Y such that $f(x_\alpha) \leqslant U \leqslant W$. Then $x_\alpha \leqslant f^{-1}(U) \leqslant f^{-1}(W)$. By hypothesis, $f^{-1}(U)$ is *ps-ro* open fuzzy set on X and hence the result is obtained.
$(e) \Rightarrow (b)$ Let V be any *ps-ro* open fuzzy set on Y. If $x_\alpha \leqslant f^{-1}(V)$ then $f(x_\alpha) \leqslant V$ and so $f^{-1}(V)$ is a *ps-ro nbd*. of x_α.
$(d) \Rightarrow (f)$ $ps\text{-}cl(f(A))$ being a *ps-ro* closed fuzzy set on Y, $f^{-1}(ps\text{-}cl(f(A)))$ is *ps-ro* closed fuzzy set on X. Again, $f(A) \leqslant ps\text{-}cl(f(A))$. So, $A \leqslant f^{-1}(ps\text{-}cl(f(A)))$. As $ps\text{-}cl(A)$ is the smallest *ps-ro* closed fuzzy set on X containing A, $ps\text{-}cl(A) \leqslant f^{-1}(ps\text{-}cl(f(A)))$. Hence, $f(ps\text{-}cl(A)) \leqslant ff^{-1}(ps\text{-}cl(f(A))) \leqslant ps\text{-}cl(f(A))$.
$(f) \Rightarrow (d)$ For any *ps-ro* closed fuzzy set B on Y, $f(ps\text{-}cl(f^{-1}(B))) \leqslant ps\text{-}cl(f(f^{-1}(B))) \leqslant ps\text{-}cl(B) = B$. Hence, $ps\text{-}cl(f^{-1}(B)) \leqslant f^{-1}(B) \leqslant ps\text{-}cl(f^{-1}(B))$. Thus, $f^{-1}(B)$ is *ps-ro* closed fuzzy set on X.
$(f) \Rightarrow (g)$ For any fuzzy set B on Y, $f(ps\text{-}cl(f^{-1}(B))) \leqslant ps\text{-}cl(f(f^{-1}(B))) \leqslant ps\text{-}cl(B)$. Hence, $ps\text{-}cl(f^{-1}(B)) \leqslant f^{-1}(ps\text{-}cl(B))$.
$(g) \Rightarrow (f)$ Let $B = f(A)$ for some fuzzy set A on X. Then $ps\text{-}cl(f^{-1}(B)) \leqslant f^{-1}(ps\text{-}cl(B)) \Rightarrow ps\text{-}cl(A) \leqslant ps\text{-}cl(f^{-1}(B)) \leqslant f^{-1}(ps\text{-}cl(f(A)))$. So, $f(ps\text{-}cl(A)) \leqslant ps\text{-}cl(f(A))$.
$(b) \Rightarrow (h)$ For any fuzzy set B on Y, $f^{-1}(ps\text{-}int(B))$ is *ps-ro* open fuzzy set on X. Also, $f^{-1}(ps\text{-}int(B)) \leqslant f^{-1}(B)$. So, $f^{-1}(ps\text{-}int(B)) \leqslant ps\text{-}int(f^{-1}(B))$.
$(h) \Rightarrow (b)$ Let B be any *ps-ro* open fuzzy set on Y. So, $ps\text{-}int(B) = B$. Now, $f^{-1}(ps\text{-}int(B)) \leqslant ps\text{-}int(f^{-1}(B)) \Rightarrow f^{-1}(B) \leqslant ps\text{-}int(f^{-1}(B)) \leqslant f^{-1}(B)$. Hence, $f^{-1}(B)$ is *ps-ro* open fuzzy set on X.

Theorem 3.3. Let (X, τ) and (Y, σ) be two fts. A function $f : X \to Y$ is f is *ps-ro* fuzzy continuous iff for every fuzzy point x_α on X and every *ps-ro* open fuzzy set V on Y with $f(x_\alpha)qV$ there exists a *ps-ro* open fuzzy set U on X with $x_\alpha qU$ and $f(U) \leqslant V$.
Proof. Let f be *ps-ro* fuzzy continuous and x_α a fuzzy point on X, V a *ps-ro* open fuzzy set V on Y with $f(x_\alpha)qV$. So, $V(f(x)) + \alpha > 1 \Rightarrow f^{-1}(V)(x) + \alpha > 1$. So, $x_\alpha q(f^{-1}(V))$. Now, $ff^{-1}(V) \leqslant V$ is always true. Choosing $U = f^{-1}(V)$ we have, $f(U) \leqslant V$ with $x_\alpha qU$.
Conversely, let the condition hold. Let V be any *ps-ro* open fuzzy set on Y. To prove $f^{-1}(V)$ is *ps-ro* open fuzzy set on X, we shall prove $1 - f^{-1}(V)$ is *ps-ro* closed fuzzy set on X. Let x_α be any fuzzy point on X such that $x_\alpha > 1_X - f^{-1}(V)$. So, $(1 - f^{-1}(V))(x) < \alpha \Rightarrow V(f(x)) + \alpha > 1$. So,

$f(x_\alpha)qV$. By given condition, there exists a *ps-ro* open fuzzy set on U such that $x_\alpha qU$ and $f(U) \leqslant V$. Now, $U(t) + (1 - f^{-1}(V))(t) \leqslant V(f(t)) + 1 - V(f(t)) = 1, \forall\, t$. Hence, $U \not{q}(1 - f^{-1}(V))$. Consequently, x_α is not a fuzzy *ps*-cluster point of $1 - f^{-1}(V)$. This proves $1 - f^{-1}(V)$ is a *ps-ro* closed fuzzy set on X

Theorem 3.4. Let X, Y, Z be *fts*. For any functions $f_1 : Z \to X$ and $f_2 : Z \to Y$, a function $f : Z \to X \times Y$ is defined as $f(x) = (f_1(x), f_2(x))$ for $x \in Z$, where $X \times Y$ is endowded with the product fuzzy topology. If f is *ps-ro* fuzzy continuous then f_1 and f_2 are both *ps-ro* fuzzy continuous.

Proof. Let U_1 be a *ps-ro q-nbd*. of $f_1(x_\alpha)$ on X, for any fuzzy point x_α on Z. Then $U_1 \times 1_Y$ is a *ps-ro q-nbd*. of $f(x_\alpha) = (f_1(x_\alpha), f_2(x_\alpha))$ on $X \times Y$. By *ps-ro* continuity of f, there exists *ps-ro q-nbd*. V of x_α on Z such that $f(V) \leqslant U_1 \times 1_Y$. Then $f(V)(t) \leqslant (U_1 \times 1_Y)(t) = U_1(t) \wedge 1_Y(t) = U_1(t), \forall\, t \in Z$. So, $f_1(V) \leqslant U_1$. Hence, f_1 is *ps-ro* fuzzy continuous. Similarly, it can be shown that f_2 is also *ps-ro* fuzzy continuous.

Theorem 3.5. Let $f : X \to Y$ be a function from a *fts* X to another *fts* Y and $g : X \to X \times Y$ be the graph of the function f. Then f is *ps-ro* fuzzy continuous if g is so.

Proof. Let g be *ps-ro* fuzzy continuous and B be *ps-ro* open fuzzy set on Y. By Lemma 2.4 of (Azad, 1981), $f^{-1}(B) = 1_X \wedge f^{-1}(B) = g^{-1}(1_X \times B)$. Now, as $1_X \times B$ is *ps-ro* open fuzzy set on $X \times Y$, $f^{-1}(B)$ becomes *ps-ro* open fuzzy set on X. Hence, f is *ps-ro* fuzzy continuous.

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Coefficient Estimates for New Subclasses of m-Fold Symmetric Bi-univalent Functions

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Abstract

In this paper, we introduce and investigate two subclasses $\mathcal{A}_{\Sigma_m}(\lambda;\alpha)$ and $\mathcal{A}_{\Sigma_m}(\lambda;\beta)$ of Σ_m consisting of analytic and m-fold symmetric bi-univalent functions in the open unit disc $\mathbb{U}$. For functions in each of the subclasses introduced in this paper, we obtain the coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$.

Keywords: Univalent functions, Bi-univalent functions, Coefficient estimates, m-fold symmetric bi-univalent functions.
2010 MSC: 30C45, 30C50.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the open unit disk

$$\mathbb{U} = \{z : \ z \in \mathbb{C} \text{ and } |z| < 1\}$$

and normalized by the conditions $f(0) = 0$, $f'(0) = 1$ and having the following form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Also let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see Duren (1983)).

The Koebe One Quarter Theorem (e.g.,see (Duren, 1983)) ensures that the image of $\mathbb{U}$ under every univalent function $f(z) \in \mathcal{A}$ contains the disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

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and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f),\ r_0(f) \geqslant \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$ given by the Taylor-Maclaurin series expansion (1.1).

For a brief history and examples of functions in the class Σ, see (Srivastava *et al.*, 2010) (see also (Brannan & Taha, 1988), (Lewin, 1967), (Taha, 1981)).

In fact, the aforecited work of Srivastava et al. (Srivastava *et al.*, 2010) essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Ali et al. (Ali *et al.*, 2012), Srivastava et al. (Srivastava *et al.*, 2015*b*)(see also (Akın & Sümer-Eker, 2014), (Deniz, 2013), (Frasin & Aouf, 2011), (Srivastava, 2012), Xu et al (Xu *et al.*, 2012*a*), (Xu *et al.*, 2012*b*) and the references cited in each of them).

Let $m \in \mathbb{N} = \{1, 2, ...\}$. A domain E is said to be *m-fold symmetric* if a rotation of E about the origin through an angle $2\pi/m$ carries E on itself (e.g.,see (Goodman, 1983)). It follows that, a function $f(z)$ analytic in $\mathbb{U}$ is said to be m-fold symmetric in $\mathbb{U}$ if for every z in $\mathbb{U}$

$$f(e^{2\pi i/m}z) = e^{2\pi i/m}f(z).$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$.

A simple argument shows that $f \in \mathcal{S}_m$ is characterized by having a power series of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1} \qquad (z \in \mathbb{U},\ m \in \mathbb{N}). \tag{1.2}$$

Each bi-univalent function generates an m-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of f is given as in (1.2) and the series expansion for f^{-1}, which has been recently proven by Srivastava et al.(Srivastava *et al.*, 2014), is given as follows

$$g(w) = w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right]w^{2m+1} \tag{1.3}$$

$$- \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \cdots$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$.

Recently, certain subclasses of m-fold bi-univalent functions class Σ_m similar to subclasses of Σ introduced and investigated by Sümer Eker (Sümer-Eker, 2016), Altınkaya and Yalçın (Altınkaya & Yalçın, 2015), Srivastava et al (Srivastava *et al.*, 2015*a*).

The aim of this paper is to introduce new subclasses of the function class bi-univalent functions in which both f and f^{-1} are m-fold symmetric analytic functions and derive estimates on initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

2. Coefficient Estimates for the function class $\mathcal{A}_{\Sigma_m}(\lambda;\alpha)$

Definition 2.1. A function $f(z) \in \Sigma_m$ given by (1.2) is said to be in the class $\mathcal{A}_{\Sigma_m}(\lambda;\alpha)$ $(0 < \alpha \leqslant 1,\ 0 \leqslant \lambda \leqslant 1)$ if the following conditions are satisfied:

$$\left| arg\left(\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}) \tag{2.1}$$

and

$$\left| arg\left(\frac{wg'(w)}{g(w)} + \frac{\lambda w^2 g''(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}) \tag{2.2}$$

where the function g is given by (1.3).

Theorem 2.1. *Let $f \in \mathcal{A}_{\Sigma_m}(\lambda;\alpha)$ $(0 < \alpha \leqslant 1,\ 0 \leqslant \lambda \leqslant 1)$ be given by (1.2). Then*

$$|a_{m+1}| \leqslant \frac{2\alpha}{m\sqrt{2\alpha[1+2\lambda(m+1)] + (1-\alpha)[1+\lambda(m+1)]^2}} \tag{2.3}$$

and

$$|a_{2m+1}| \leqslant \frac{\alpha(m+1)\,[1+|\alpha-1|]}{m^2[1+2\lambda(m+1)]}. \tag{2.4}$$

Proof. From (2.1) and (2.2) we have

$$\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)} = [p(z)]^\alpha \tag{2.5}$$

and for its inverse map, $g = f^{-1}$, we have

$$\frac{wg'(w)}{g(w)} + \frac{\lambda w^2 g''(w)}{g(w)} = [q(w)]^\alpha \tag{2.6}$$

where $p(z)$ and $q(w)$ are in familiar Caratheodory Class $\mathcal{P}$ (see for details (Duren, 1983)) and have the following series representations:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{2.7}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots . \tag{2.8}$$

Comparing the corresponding coefficients of (2.5) and (2.6) yields

$$m[1+\lambda(m+1)]a_{m+1} = \alpha p_m, \tag{2.9}$$

$$2m[1+\lambda(2m+1)]a_{2m+1} - m[1+\lambda(m+1)]a_{m+1}^2 = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2}p_m^2, \tag{2.10}$$

$$-m[1+\lambda(m+1)]a_{m+1} = \alpha q_m \tag{2.11}$$

and

$$m[(2m+1)+\lambda(m+1)(4m+1)]a_{m+1}^2 - 2m[1+\lambda(2m+1)]a_{2m+1} = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2}q_m^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$p_m = -q_m \tag{2.13}$$

and

$$2m^2[1+\lambda(m+1)]^2 a_{m+1}^2 = \alpha^2(p_m^2+q_m^2). \tag{2.14}$$

Also from (2.10), (2.12) and (2.14), we get

$$2m^2[1+2\lambda(m+1)]a_{m+1}^2 = \alpha(p_{2m}+q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2).$$

Therefore, we have

$$a_{m+1}^2 = \frac{\alpha^2(p_{2m}+q_{2m})}{m^2\left[2\alpha[1+2\lambda(m+1)] + (1-\alpha)[1+\lambda(m+1)]^2\right]}. \tag{2.15}$$

Note that, according to the Caratheodory Lemma (see (Duren, 1983)), $|p_m| \leqslant 2$ and $|q_m| \leqslant 2$ for $m \in \mathbb{N}$. Now taking the absolute value of (2.15) and applying the Caratheodory Lemma for coefficients p_{2m} and q_{2m} we obtain

$$|a_{m+1}| \leqslant \frac{2\alpha}{m\sqrt{2\alpha[1+2\lambda(m+1)] + (1-\alpha)[1+\lambda(m+1)]^2}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted (2.3).

To find bounds on $|a_{2m+1}|$, we multiply $(2m+1)+\lambda(m+1)(4m+1)$ and $1+\lambda(m+1)$ to the relations (2.10) and (2.12) respectively and on adding them we obtain:

$$\begin{aligned}4m^2[1+\lambda(2m+1)][1+2\lambda(m+1)]a_{2m+1} \\ = \alpha\left\{[(2m+1)+\lambda(m+1)(4m+1)]\,p_{2m} + [1+\lambda(m+1)]\,q_{2m}\right\} \\ +\frac{\alpha(\alpha-1)}{2}\left\{[(2m+1)+\lambda(m+1)(4m+1)]\,p_m^2 + [1+\lambda(m+1)]q_m^2\right\}.\end{aligned}$$

Now using $p_m^2 = q_m^2$ and the Caratheodory Lemma again for coefficients p_m, p_{2m} and q_{2m} we obtain

$$|a_{2m+1}| \leqslant \frac{\alpha(m+1)[1+|\alpha-1|]}{m^2[1+2\lambda(m+1)]}.$$

This completes the proof of the Theorem 2.1.

3. Coefficient Estimates for the function class $\mathcal{A}_{\Sigma_m}(\lambda;\beta)$

Definition 3.1. A function $f(z) \in \Sigma_m$ given by (1.2) is said to be in the class $\mathcal{A}_{\Sigma_m}(\lambda;\beta)$ $(0 \leqslant \lambda \leqslant 1, 0 \leqslant \beta < 1)$ if the following conditions are satisfied:

$$Re\left\{\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)}\right\} > \beta \qquad (z \in \mathbb{U}) \tag{3.1}$$

and

$$Re\left\{\frac{wg'(w)}{g(w)} + \frac{\lambda w^2 g''(w)}{g(w)}\right\} > \beta \qquad (w \in \mathbb{U}) \tag{3.2}$$

where the function $g(w)$ is given by (1.3).

Theorem 3.1. *Let $f \in \mathcal{A}_{\Sigma_m}(\lambda;\beta)$ $(0 \leqslant \lambda \leqslant 1,\ 0 \leqslant \beta < 1)$ be given by (1.2). Then*

$$|a_{m+1}| \leqslant \sqrt{\frac{2(1-\beta)}{m^2[1+2\lambda(m+1)]}} \tag{3.3}$$

and

$$|a_{2m+1}| \leqslant \frac{(1-\beta)(m+1)}{m^2[1+2\lambda(m+1)]}. \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that

$$\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)} = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$\frac{wg'(w)}{g(w)} + \frac{\lambda w^2 g''(w)}{g(w)} = \beta + (1-\beta)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.7) and (2.8), respectively. Equating coefficients (3.5) and (3.6) yields

$$m[1+\lambda(m+1)]a_{m+1} = (1-\beta)p_m, \tag{3.7}$$

$$2m[1+\lambda(2m+1)]a_{2m+1} - m[1+\lambda(m+1)]a_{m+1}^2 = (1-\beta)p_{2m}, \tag{3.8}$$

$$-m[1+\lambda(m+1)]a_{m+1} = (1-\beta)q_m \tag{3.9}$$

and

$$m[(2m+1)+\lambda(m+1)(4m+1)]a_{m+1}^2 - 2m[1+\lambda(2m+1)]a_{2m+1} = (1-\beta)q_{2m}. \tag{3.10}$$

From (3.7) and (3.9) we get

$$p_m = -q_m \tag{3.11}$$

and

$$2m^2[1+\lambda(m+1)]^2 a_{m+1}^2 = (1-\beta)^2(p_m^2+q_m^2). \tag{3.12}$$

Also from (3.8) and (3.10), we obtain

$$2m^2[1+2\lambda(m+1)]a_{m+1}^2 = (1-\beta)(p_{2m}+q_{2m}). \tag{3.13}$$

Thus we have

$$\begin{aligned} |a_{m+1}^2| &\leqslant \frac{(1-\beta)}{2m^2[1+2\lambda(m+1)]}(|p_{2m}|+|q_{2m}|) \\ &\leqslant \frac{2(1-\beta)}{m^2[1+2\lambda(m+1)]}, \end{aligned}$$

which is the bound on $|a_{m+1}|$ as given in the Theorem 3.1.

In order to find the bound on $|a_{2m+1}|$, we multiply $(2m+1)+\lambda(m+1)(4m+1)$ and $1+\lambda(m+1)$ to the relations (3.8) and (3.10) respectively and on adding them we obtain:

$$\begin{aligned} &4m^2[1+\lambda(2m+1)][1+2\lambda(m+1)]a_{2m+1} \\ &= (1-\beta)\{[(2m+1)+\lambda(m+1)(4m+1)]p_{2m} + [1+\lambda(m+1)]q_{2m}\} \end{aligned}$$

or equivalently

$$a_{2m+1} = \frac{(1-\beta)[(2m+1)+\lambda(m+1)(4m+1)]p_{2m} + [1+\lambda(m+1)]q_{2m}}{4m^2[1+\lambda(2m+1)][1+2\lambda(m+1)]}$$

Applying the Caratheodory Lemma for the coefficients p_{2m} and q_{2m}, we find

$$|a_{2m+1}| \leqslant \frac{(1-\beta)(m+1)}{m^2[1+2\lambda(m+1)]},$$

which is the bound on $|a_{2m+1}|$ as asserted in Theorem 3.1.

Remark. For 1-fold symmetric bi-univalent functions, if we put $\lambda = 0$ in our Theorems, we obtain the Theorem 2.1 and the Theorem 3.1 which were given by Brannan and Taha (Brannan & Taha, 1988).

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Subordination Properties of Certain Subclasses of Multivalent Functions Defined By Srivastava-Wright Operator

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Abstract

Some subordination properties are investigated for functions belonging to each of the subclasses $\mathcal{V}(\lambda, A, B)$ and $\mathcal{W}(\lambda, A, B)$ of analytic p- valent functions involving the Srivastava-Wright operator in the open unit disk, $\mathbb{U}$ with suitable restrictions on the parameters λ, A and B. The authors also derive certain subordination results involving the Hadamard product (or convolution) of the associated functions. Relevant connections of the main results to various known results are established.

Keywords: Multivalent function, Srivastava-Wright Operator, Convex function, Differential subordination, Argument estimates.
2010 MSC: 30C45, 30C50, 30C55.

1. Introduction

Let $\mathcal{A}_k(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n \quad (p < k; p, k \in \mathbb{N} := \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic and $p-$valent in the unit disc, $\mathbb{U} := \mathbb{U}(1)$, where $\mathbb{U}(r) = \{z \in \mathbb{C} : |z| < r\}$. Also, let $\mathcal{A}(p) = \mathcal{A}_{p+1}(p)$ and $\mathcal{A} = \mathcal{A}(1)$. For the functions $f \in \mathcal{A}_k(p)$ of the form (1.1) and $g \in \mathcal{A}_k(p)$ given by $g(z) = z^p + \sum_{n=k}^{\infty} b_n z^n$, the *Hadamard product (or convolution) of f and g* is defined by

$$(f * g)(z) := z^p + \sum_{n=k}^{\infty} a_n b_n z^n, \ z \in \mathbb{U}.$$

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If f and g are two analytic functions in $\mathbb{U}$, we say that *f is subordinate to g*, written symbolically as $f(z) \prec g(z)$, if there exists a *Schwarz function w,* which (by definition) is analytic in $\mathbb{U}$, with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$.

If the function g is univalent in $\mathbb{U}$, then we have the following equivalence, (c.f (Miller & Mocanu, 1981, 2000)):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\alpha_1, A_1, \ldots, \alpha_q, A_q$ and $\beta_1, B_1, \ldots, \beta_s, B_s (q, s \in \mathbb{N})$ be positive and real parameters such that

$$1 + \sum_{i=1}^{s} B_i - \sum_{i=1}^{q} A_i > 0.$$

The *Wright generalized hypergeometric function*

$${}_q\Psi_s[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{q} \Gamma\left(\alpha_i + nA_i\right)}{\prod_{i=1}^{s} \Gamma\left(\beta_i + nB_i\right)} \frac{z^n}{n!} \ (z \in \mathbb{U}).$$

If $A_i = 1 (i = 1, ..., q)$ and $B_i = 1 (i = 1, ..., s)$, we have the following relationship:

$$\Omega_q\Psi_s[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] =_q F_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),$$

where ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ is the *generalized hypergeometric function* and

$$\Omega = \frac{\prod_{i=1}^{q} \Gamma\left(\beta_i\right)}{\prod_{i=1}^{s} \Gamma\left(\alpha_i\right)} \tag{1.2}$$

Now we define a function $\mathcal{WH}_p[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z]$ by

$$\mathcal{WH}_p[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] = \Omega\, z^p \ {}_q\Psi_s[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z]$$

and also consider the following linear operator

$$\theta_p^{q,s}[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] : \mathcal{A}_k(p) \to \mathcal{A}_k(p)$$

defined using the convolution

$$\theta_p^{q,s}[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}]f(z) = \mathcal{WH}_p[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] * f(z).$$

We note that, for a function f of the form (1.1), we have

$$\theta_p^{q,s}[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}]f(z) = z^p + \sum_{n=k}^{\infty} \Omega\sigma_{n,p}(\alpha_1) a_n z^n, \tag{1.3}$$

where Ω is given by (1.2) and $\sigma_{n,p}(\alpha_1)$ is defined by

$$\sigma_{n,p}(\alpha_1) = \frac{\Gamma(\alpha_1 + A_1(n-p))\ldots\Gamma(\alpha_q + A_q(n-p))}{\Gamma(\beta_1 + B_1(n-p))\ldots\Gamma(\beta_s + B_s(n-p))(n-p)!}. \tag{1.4}$$

If for convenience, we write

$$\theta_p^{q,s}(\alpha_1)f(z) = \theta_p^{q,s}[(\alpha_1, A_1)\ldots(\alpha_q, A_q);(\beta_1, B_1)\ldots(\beta_s, B_s)]f(z)$$

then we can easily verify from (1.3) that

$$zA_1\left(\theta_p^{q,s}(\alpha_1)f(z)\right)' = \alpha_1\theta_p^{q,s}(\alpha_1+1)f(z) - (\alpha_1 - pA_1)\theta_p^{q,s}(\alpha_1)f(z)\ (A_1 > 0). \tag{1.5}$$

For $A_i = 1 (i = 1, ..., q)$ and $B_i = 1 (i = 1, ..., s)$, we obtain $\theta_p^{q,s}[\alpha_1]f(z) = H_{p,q,s}f(z)$, which is known as the Dziok-Srivastava operator; it was introduced and studied by Dziok and Srivastava (Dziok & Srivastava, 1999, 2003). Also, for $f(z) \in \mathcal{A}$, the linear operator $\theta_1^{q,s}[\alpha_1]f(z) = \theta[\alpha_1]$ is popularly known in the current literature as the Srivastava-Wright operator; it was systematically and firmly investigated by Srivastava (Srivastava, 2007).(see also (Kiryakova, 2011; Dziok & Raina, 2004) and (Aouf *et al.*, 2010)).

Remark. For $f \in \mathcal{A}(p), A_i = 1(1 = 1, 2, \ldots, q), B_i = 1(i = 1, 2, \ldots, s), q = 2$ *and* $s = 1$ by specializing the parameters α_1, α_2 *and* β_1 the operator $\theta_p^{q,s}(\alpha_1)$ gets reduced to the following familiar operators:

(i) $\theta_p^{2,1}[a, 1; c]f(z) = L_p(a, c)f(z)$[see Saitoh (Saitoh, 1996)];

(ii) $\theta_p^{2,1}[\mu + p, 1; 1]f(z) = D^{\mu+p-1}f(z)\ (\mu > -p)$, where $D^{\mu+p-1}$ is the $\mu + p - 1$- the order *Ruscheweyh derivative* of a function $f \in \mathcal{A}(p)$. [see Kumar and Shukla (Kumar & Shukla, 1984*a*,*b*)]

(iii) $\theta_p^{2,1}[1 + p, 1; 1 + p - \mu]f(z)$, where the operator $\Omega_z^{\mu,p}$ is defined by [see Srivastava and Aouf (Srivastava & Aouf, 1992)];

$$\Omega_z^{\mu,p}f(z) = \frac{\Gamma(1+p-\mu)}{\Gamma(1+p)}z^{\mu}D_z^{\mu}f(z)\ (0 \le \mu < 1; p \in \mathbb{N}),$$

where D_z^{μ} is the *fractional derivative operator*.

(iv) $\theta_p^{2,1}[\nu + p, 1; \nu + p + 1]f(z) = J_{\nu,p}(f)(z)$, where $J_{\nu,p}$ is the generalized *Bernadi-Libera-Livingston-integral operator* (see (Bernardi, 1996; Libera, 1969; Livingston, 1966));

(v) $\theta_p^{2,1}[\lambda + p, a; c]f(z) = I_p^{\lambda}(a, c)f(z)\ (a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \lambda > -p)$, where $I_p^{\lambda}(a, c)$ is the *Cho-Kwon-Srivastava* operator (Cho *et al.*, 2004);

Definition 1.1. For the fixed parameters A and B, with $0 \le B < 1, -1 \le A < B$ and $0 \le \lambda < p, p \in \mathbb{N}$ and for a analytic $p-$ valent function of the form (1.1) we define the following subclasses:

$$\mathcal{V}(\lambda, A, B) = \left\{f \in \mathcal{A}_k(p) : \frac{1}{p-\lambda}\left(\frac{z\left[\theta_p^{q,s}(\alpha_1)f(z)\right]'}{\theta_p^{q,s}(\alpha_1)f(z)} - \lambda\right) \prec \frac{1+Az}{1+Bz}\right\} \tag{1.6}$$

and

$$\mathcal{W}(\lambda, A, B) = \left\{ f \in \mathcal{A}_k(p) : \frac{1}{p-\lambda}\left(1 + \frac{z[\theta_p^{q,s}(\alpha_1)f(z)]''}{[\theta_p^{q,s}(\alpha_1)f(z)]'} - \lambda\right) \prec \frac{1+Az}{1+Bz} \right\}. \tag{1.7}$$

The subclass $\mathcal{V}(\lambda, A, B)$ was discussed by Aouf et al., (Aouf *et al.*, 2010) for multivalent analytic functions with negative coefficients, also coefficients estimates, distortion theorem, the radii of $p-$valent starlikeness and $p-$valent convexity and modified Hadamard products were investigated. In (Murugusundaramoorthy & Aouf, 2013) Murugusundaramoorthy and Aouf obtained similar results for the meromorphic equivalent of the class $\mathcal{W}(\lambda, A, B)$. Sarkar et al., (Sarkar *et al.*, 2013) presented certain inclusion and convolution results involving the operator $\theta_p^{q,s}(\alpha_1)$ for functions belonging to certain favoured classes of analytic p-valent functions.
Motivated by the aforementioned works, in the present study we obtain certain strict subordination relationship involving the subclasses $\mathcal{V}(\lambda, A, B)$ and $\mathcal{W}(\lambda, A, B)$. Some subordination properties involving the linear operator defined in (1.3) are also considered. An argument estimate result is also obtained.

2. Preliminaries

Let $\mathcal{P}_m$ denote the class of function of the form

$$f(z) = 1 + a_m z^m + a_{m+1} z^{m+1} + \dots \tag{2.1}$$

that are analytic in the unit disc,$\mathbb{U}$. In proving our main results, we need each of the following definitions and lemmas.

Definition 2.1. (Wilf, 1961)
A sequence $\{b_n\}_{n\in\mathbb{N}}$ of complex numbers is said to be a *subordination factor sequence* if for each function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $z \in \mathbb{U}$, from the class of convex (univalent) functions in $\mathbb{U}$, denoted by S^c, we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z) \quad (\text{where} \quad a_1 = 1).$$

Lemma 2.1. *(Wilf, 1961) A sequence $\{b_n\}$ is a subordinating factor sequence if and only if*

$$Re\left(1 + 2\sum_{n=1}^{\infty} b_n z^n\right) > 0, \; z \in \mathbb{U}. \tag{2.2}$$

Lemma 2.2. *(Miller & Mocanu, 1981, 2000) Let the function h be analytic and convex (univalent) in $\mathbb{U}$ with $h(0) = 1$. Suppose also that the function ϕ given by (2.1). If*

$$\phi(z) + \frac{z\phi'(z)}{\gamma} \prec h(z) \quad (Re\gamma \geq 0, \; \gamma \in \mathbb{C}^*), \tag{2.3}$$

then

$$\phi(z) \prec \psi(z) = \frac{\gamma}{m} z^{-\frac{\gamma}{m}} \int_0^z t^{\frac{\gamma}{m}-1} h(t)\, dt \prec h(z)$$

and ψ *is the best dominant.*

Lemma 2.3. *(Nunokawa, 1993)*

Let the function p *be analytic in* $\mathbb{U}$*, such that* $p(0) = 1$ *and* $p(z) \neq 0$ *for all* $z \in \mathbb{U}$*. If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$|\arg p(z)| < \frac{\pi\delta}{2}, \quad \textit{for} \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\delta}{2} \quad (\delta > 0),$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\delta,$$

where

$$k \geq \frac{1}{2}\left(c + \frac{1}{c}\right), \quad \textit{when} \quad \arg p(z_0) = \frac{\pi\delta}{2}$$

and

$$k \leq -\frac{1}{2}\left(c + \frac{1}{c}\right), \quad \textit{when} \quad \arg p(z_0) = -\frac{\pi\delta}{2},$$

where

$$p(z_0)^{1/\delta} = \pm ic, \quad \textit{and} \quad c > 0.$$

Lemma 2.4. *(Whittaker & Watson, 1927)*

For the complex numbers a, b *and* c*, with* $c \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$*, the following identities hold:*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a, b; c; z), \ z \in \mathbb{U}, \tag{2.4}$$

$$\textit{for } Re\, c > Re\, b > 0, \tag{2.5}$$

$${}_2F_1(a, b; c; z) = (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \ z \in \mathbb{U}, \tag{2.6}$$

and

$$(b+1)\, {}_2F_1(1, b; b+1; z) = (b+1) + bz\, {}_2F_1(1, b+1; b+2; z), \ z \in \mathbb{U}. \tag{2.7}$$

3. Coefficient estimates and subordination results for the function classes $\mathcal{W}(\lambda, A, B)$ and $\mathcal{V}(\lambda, A, B)$

Unless otherwise mentioned, we shall assume throughout the sequel that $0 \le \lambda < p, p \in \mathbb{N}$ *and* $0 \le B < 1$. First, we will give sufficient conditions for a function to be in the classes $\mathcal{W}(\lambda, A, B)$.

Lemma 3.1. *A sufficient condition for an analytic p-valent function f of the form (1.1), to be in the class $\mathcal{W}(\lambda, A, B)$ is*

$$\sum_{n=k}^{\infty} \gamma_{n,p}|a_n| \le p(B-A)(p-\lambda) \tag{3.1}$$

where

$$\gamma_{n,p} = \Omega\sigma_{n,p}(\alpha_1)n[(n-p)(1+B)-(A-B)(p-\lambda)],\ (n \ge k). \tag{3.2}$$

Proof. An analytic $p-$valent function f of the form (1.1) belongs to the class $\mathcal{W}(\lambda, A, B)$, if and only if there exists a *Schwarz function* w, such that

$$\frac{1}{p-\lambda}\left(1+\frac{z[\theta_p^{q,s}(\alpha_1)f(z)]''}{[\theta_p^{q,s}(\alpha_1)f(z)]'}-\lambda\right) = \frac{1+Aw(z)}{1+Bw(z)},\ z \in \mathbb{U}.$$

Since $|w(z)| \le |z|$ for all $z \in \mathbb{U}$, the above relation is equivalent to

$$\left|\frac{[\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]'}{([\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]')B - (p-\lambda)(A-B)[\theta_p^{q,s}(\alpha_1)f(z)]'}\right| < 1.$$

Thus it is sufficient to show that

$$\begin{aligned}&\left|[\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]'\right| \\ &- \left|([\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]')B - (p-\lambda)(A-B)[\theta_p^{q,s}(\alpha_1)f(z)]'\right| < 0,\ z \in \mathbb{U}.\end{aligned}$$

Indeed, letting $|z| = r$ $(0 < r < 1)$ and using (3.1), we have

$$\begin{aligned}&\left|[\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]'\right| - \\ &\left|[\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]' - (p-\lambda)(A-B)[\theta_p^{q,s}(\alpha_1)f(z)]'\right| \\ &\le \sum_{n=k}^{\infty} n(n-p)\Omega\sigma_{n,p}(\alpha_1)|a_n|r^n - (B-A)p(p-\lambda)\, r^{p-1} \\ &+ \sum_{n=k}^{\infty} n[(n-p)B-(A-B)(p-\lambda)]\Omega\sigma_{n,p}(\alpha_1)|a_n|r^n = r^{p-1}\Big(\sum_{n=k}^{\infty}\gamma_{n,p}|a_n|r^{n-p+1} - (B-A)p(p-\lambda)\Big) < 0.\end{aligned}$$

Hence $f \in \mathcal{W}(\lambda, A, B)$. □

Similarly, we have the following Lemma which gives sufficient condition for a function to be in the class $\mathcal{V}(\lambda, A, B)$.

Lemma 3.2. *A sufficient condition for an analytic p-valent function f of the form (1.1), to be in the class $\mathcal{V}(\lambda, A, B)$ is*

$$\sum_{n=k}^{\infty} \delta^*_{n,p}|a_n| \leq (B-A)(p-\lambda) \tag{3.3}$$

where

$$\delta^*_{n,p} = \Omega\sigma_{n,p}(\alpha_1)[(n-p)(1+B)-(A-B)(p-\lambda)],\ (n \geq k). \tag{3.4}$$

Our next result provides a sharp subordination result involving the functions of the class $\mathcal{W}(\lambda, A, B)$.

Theorem 3.1. *Let the sequence $\{\gamma_{n,p}\}_{n\in\mathbb{N}}$ defined in (3.2) be a nondecreasing sequence. If a function f of the form (1.1) belong to the class $\mathcal{W}(\lambda, A, B)$. and $g \in \mathcal{S}^c$, then*

$$\left(\epsilon\left(z^{1-p}\right) * g\right)(z) \prec g(z), \tag{3.5}$$

and

$$Re\left(z^{1-p}f(z)\right) > -\frac{1}{2\epsilon},\ z \in \mathbb{U}, \tag{3.6}$$

$$\textit{whenever} \quad \epsilon = \frac{\gamma_{k,p}}{2\left[(B-A)p(p-\lambda)\right]+\gamma_{k,p}}.$$

Moreover, if $(k-p)$ is even, then the number ϵ cannot be replaced by a larger number.

Proof. Supposing that the function $g \in \mathcal{S}^c$ is of the form

$$g(z) = \sum_{n=1}^{\infty} b_n z^n,\ z \in \mathbb{U} \quad (\text{where} \quad b_1 = 1),$$

then

$$\sum_{n=1}^{\infty} d_n b_n z^n = \left(\epsilon\left(z^{1-p}f\right) * g\right)(z) \prec g(z),$$

where

$$d_n = \begin{cases} \epsilon, & \text{if} \quad n = 1, \\ 0, & \text{if} \quad 2 \leq n \leq k-p, \\ \epsilon a_{n+p-1}, & \text{if} \quad n > k-p. \end{cases}$$

Now, using the Definition 2.1, the subordination result in (3.5) holds if $\{d_n\}$ is a subordinating factor sequence. Since $\{\gamma_{n,p}\}_{n\in\mathbb{N}}$ is a nondecreasing sequence we have,

$$\begin{aligned}\operatorname{Re}\left(1+2\sum_{n=1}^{\infty} d_n z^n\right) = \operatorname{Re}\Bigg(1 + \frac{\gamma_{k,p}}{p(p-\lambda)(B-A)+\gamma_{k,p}} z + \\ \sum_{n=k}^{\infty} \frac{\gamma_{k,p}}{p(p-\lambda)(B-A)+\gamma_{k,p}} a_n z^{n-p}\Bigg) \geq \\ 1 - \frac{\gamma_{k,p}}{p(p-\lambda)(B-A)+\gamma_{k,p}} r - \\ \frac{r}{p(p-\lambda)(B-A)+\gamma_{k,p}} \sum_{n=k}^{\infty} \delta_{n,p}|a_n|, \ |z| = r < 1.\end{aligned} \tag{3.7}$$

Thus, by using Lemma 3.1 in (3.7) we obtain

$$\begin{aligned}\operatorname{Re}\left(1+2\sum_{n=1}^{\infty} c_n z^n\right) \geq 1 - \frac{\gamma_{k,p}}{p(B-A)(p-\lambda)+\gamma_{k,p}} r - \\ \frac{r}{p(B-A)(p-\lambda)+\gamma_{k,p}} (B-A)p(p-\lambda) > 0, \ z \in \mathbb{U},\end{aligned}$$

which proves the inequality (2.2), hence also the subordination result asserted by (3.5). The inequality (3.6) asserted by Theorem 3.1 would follow from (3.5) upon setting

$$g(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n, \ z \in \mathbb{U}.$$

We also observe that whenever the functions of the form

$$f_{n,p}(z) = z^p + \frac{(B-A)p(p-\lambda)}{\gamma_{n,p}} z^n, \ z \in \mathbb{U} \ (n \geq k),$$

belongs the class $\mathcal{W}(\lambda, A, B)$ and if $(k-p)$ is a even number, then

$$z^{1-p} f_{k,p}(z)\Big|_{z=-1} = -\frac{1}{2\epsilon},$$

and the constant ϵ is the best estimate. □

Using the same techniques as in the proof of Theorem 3.1, we have the following result.

Theorem 3.2. *Let the sequence* $\{\delta^*_{n,p}\}_{n\in\mathbb{N}}$ *defined by* (3.4) *be a nondecreasing sequence. If the function g of the form* (1.1) *belongs to the class* $\mathcal{V}(\lambda, A, B)$ *and* $h \in \mathcal{S}^c$, *then*

$$\left(\mu\left(z^{1-p} f\right) * h\right)(z) \prec h(z), \tag{3.8}$$

and

$$Re\left(z^{1-p}f(z)\right) > -\frac{1}{2\mu},\ z \in \mathbb{U}, \tag{3.9}$$

where

$$\mu = \frac{\delta^*_{k,p}}{2\left[(B-A)(p-\lambda)\right] + \delta^*_{k,p}}.$$

Moreover, if $(k-p)$ is even, then the number μ cannot be replaced by a larger number.

4. Subordination Properties of the operator $\theta_p^{q,s}(\alpha_1)$

In this section we obtain certain subordination properties involving the operator $\theta_p^{q,s}(\alpha_1)$.

Theorem 4.1. *For $f \in \mathcal{A}_k(p)$ let the operator $\mathcal{Q}$ be defined by*

$$\mathcal{Q}f(z) := \left[1 - \tau - \tau\frac{(\alpha_1 - pA_1)}{A_1}\theta_p^{q,s}(\alpha_1)f(z)\right] + \frac{\tau\alpha_1}{A_1}\left[\theta_p^{q,s}(\alpha_1+1)f(z)\right], \tag{4.1}$$

for $A_1 \neq 0$ and $\tau > 0$.

(i) If

$$\frac{\mathcal{Q}^{(j)}f(z)(p-j)!}{z^{p-j}p!} \prec (1-\tau+\tau p)\frac{1+Az}{1+Bz} \quad (0 \leq j \leq p), \tag{4.2}$$

, then

$$\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)(p-j)!\right]^{(j)}}{z^{p-j}p!} \prec \widetilde{g}(z) \prec \frac{1+Az}{1+Bz}, \tag{4.3}$$

where for m positive, $\widetilde{g}$ is given by

$$\widetilde{g}(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\frac{1-\tau+\tau p}{\tau m}+1;\frac{Bz}{1+Bz}\right), & if \quad B \neq 0, \\ 1 + \frac{Az(1-\tau+\tau p)}{1-\tau+\tau(m+p)}, & if \quad B = 0, \end{cases}$$

and $\widetilde{g}$ is the best dominant of (4.3).

(ii)

$$Re\left(\frac{\mathcal{Q}^{(j)}f(z)}{z^{p-j}}\right) > \frac{p!}{(p-j)!}\sigma,\ z \in \mathbb{U} \tag{4.4}$$

where

$$\sigma = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1,1;\frac{1-\tau+\tau p}{\tau m}+1;\frac{B}{B-1}\right), & if \quad B \neq 0, \\ 1 - \frac{A(1-\tau+\tau p)}{1-\tau+\tau(p+m)}, & if \quad B = 0. \end{cases}$$

The inequality (4.4) is the best possible.

Proof. From (1.5) and (4.1) we easily obtain

$$Q^{(j)}f(z) = (1-\tau+\tau j)\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)} + \tau z\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j+1)},\ z \in \mathbb{U}. \tag{4.5}$$

Letting

$$g(z) := \frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}(p-j)!}{z^{p-j}p!}.$$

with $f \in \mathcal{A}_k(p)$, then g is analytic in $\mathbb{U}$ and has the form (2.1). Also, note that

$$(1-\tau+\tau p)\left[g(z) + \frac{\tau}{1-\tau+\tau p}zg'(z)\right] = \frac{Q^{(j)}f(z)(p-j)!}{z^{p-j}p!}. \tag{4.6}$$

Then, by (4.2) we have

$$g(z) + \frac{\tau}{1-\tau+\tau p}zg'(z) \prec \frac{1+Az}{1+Bz}.$$

Now, by using Lemma 2.2 for $\gamma = \dfrac{1-\tau+\tau p}{\tau}$ and whenever $\gamma > 0$, by a changing of variables followed by the use of the identities (2.5), (2.6) and (2.7), we deduce that

$$\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}(p-j)!}{z^{p-j}p!} \prec \widetilde{g}(z) = \frac{(1-\tau+\tau p)}{\tau m}z^{-\frac{(1-\tau+\tau p)}{\tau m}}\int_0^z t^{\frac{(1-\tau+\tau p)}{\tau m}-1}\frac{1+At}{1+Bt}\,dt$$

$$= \begin{cases} \dfrac{A}{B} + \left(1-\dfrac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\dfrac{1-\tau+\tau p}{\tau m}+1;\dfrac{Bz}{1+Bz}\right), & \text{if} \quad B \neq 0, \\ 1 + \dfrac{A(1-\tau+\tau p)}{1-\tau+\tau(p+m)}z, & \text{if} \quad B = 0, \end{cases}$$

which proves the assertion (4.3) of our Theorem.

Next, in order to prove the assertion (4.4), it sufficies to show that

$$\inf\{\operatorname{Re}\widetilde{g}(z) : z \in \mathbb{U}\} = \widetilde{g}(-1). \tag{4.7}$$

Indeed, for $|z| \le r < 1$ we have

$$\operatorname{Re}\frac{1+Az}{1+Bz} \ge \frac{1-Ar}{1-Br},$$

and setting

$$\chi(s,z) = \frac{1+Asz}{1+Bsz} \quad \text{and} \quad d\mu(s) = \frac{1-\tau+\tau p}{\tau m}s^{\frac{1-\tau+\tau p}{\tau m}-1}\,ds \quad (0 \le s \le 1)$$

which is a positive measure on the closed interval $[0,1]$ whenever $\tau > 0$, we get

$$\widetilde{g}(z) = \int_0^1 \chi(s,z)\,d\mu(s),$$

and

$$\operatorname{Re}\widetilde{g}(z) \geq \int_0^1 \frac{1 - Asr}{1 - Bsr} d\mu(s) = \widetilde{g}(-r), \ |z| \leq r < 1.$$

Letting $r \to 1^-$ in the above inequality we obtain the assertion (4.7) of our Theorem. The estimate in (4.4) is the best possible since the function $\widetilde{g}$ is the best dominant of (4.3). □

Taking $q = 2$ and $s = 1$,for $A_i = B_i = 1, \alpha_1 = 1, \alpha_2 = \beta_1$ and $A = 1 - \frac{2\alpha(p-j)!}{(1-\tau+\tau p)p!}$ and $B = -1$ in Theorem 4.1 we get the following result:

Corollary 4.1. *Let $Qf(z) = (1-\tau)f(z) + \tau z f'(z)$, where $f \in \mathcal{A}_k(p)$. For $\tau > 0$*

$$Re\frac{Q^{(j)}f(z)(p-j)!}{z^{p-j}p!} > \alpha, \ z \in \mathbb{U} \quad \left(0 \leq \alpha < \frac{(1-\tau+\tau p)p!}{(p-j)!}, \ 0 \leq j \leq p\right),$$

implies that

$$Re\frac{f^{(j)}(z)}{z^{p-j}} > \frac{\alpha}{1-\tau+\tau p} +$$
$$\left[\frac{p!}{(p-j)!} - \frac{\alpha}{1-\tau+\tau p}\right]\left[{}_2F_1\left(1, 1; \frac{1-\tau+\tau p}{\tau m} + 1; \frac{1}{2}\right) - 1\right], \ z \in \mathbb{U}.$$

The above inequality is the best possible.

Theorem 4.2. *For $f \in \mathcal{A}_k(p)$ let the operator Q be given by (4.1), and let $\tau > 0$.*

(i) If

$$Re\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}}{z^{p-j}} > \rho, \ z \in \mathbb{U} \quad \left(\rho < \frac{p!}{(p-j)!}\right),$$

then

$$Re\frac{Q^{(j)}f(z)}{z^{p-j}} > \rho(1-\tau+\tau p), \ |z| < R,$$

where

$$R = \left[\sqrt{1 + \left(\frac{\tau m}{1-\tau+\tau p}\right)^2} - \frac{\tau m}{1-\tau+\tau p}\right]^{\frac{1}{m}}. \tag{4.8}$$

(ii) If

$$Re\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}}{(-1)^j z^{-p-j}} < \rho, \ z \in \mathbb{U} \quad \left(\rho > \frac{p!}{(p-j)!}, \right),$$

then

$$Re\frac{Q^{(j)}f(z)}{z^{p-j}} < \rho(1-\tau+\tau p), \ |z| < R.$$

The bound R is the best possible.

Proof. (i) Defining the function Φ by

$$\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}}{z^{p-j}} =: \rho + \left[\frac{p!}{(p-j)!} - \rho\right]\Phi(z), \tag{4.9}$$

then Φ is an analytic function of the form (2.1) with positive real part in $\mathbb{U}$. Differentiating (4.9) with respect to z and using (4.5) we have

$$\frac{Q^{(j)}f(z)}{z^{p-j}} - \rho(1-\tau+\tau p) = \left[\frac{p!}{(p-j)!} - \rho\right]\left[(1-\tau+\tau p)\Phi(z) + \tau z\Phi'(z)\right]. \tag{4.10}$$

Now, by applying in (4.10) the following well-known estimate (MacGregor, 1963)

$$\frac{|z\Phi'(z)|}{\mathrm{Re}\,\Phi(z)} \leq \frac{2mr^m}{1-r^{2m}}, \quad |z| = r < 1, \tag{4.11}$$

we have

$$\mathrm{Re}\left[\frac{Q^{(j)}f(z)}{z^{p-j}} - \rho(1-\tau+\tau p)\right] \geq \tag{4.12}$$
$$\mathrm{Re}\,\Phi(z)\left[\frac{p!}{(p-j)!} - \rho\right]\left[(1-\tau+\tau p) - \frac{2\tau mr^m}{1-r^{2m}}\right], \quad |z| = r < 1.$$

Now, it is easy to see that the right hand side of (4.12) is positive whenever $r < R$, where R is given by (4.8). In order to show that the bound R is the best possible, we consider the function $f \in \mathcal{A}_k(p)$ defined by

$$\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}}{z^{p-j}} = \rho + \left[\frac{p!}{(p-j)!} - \rho\right]\frac{1+z^m}{1-z^m}.$$

Then,

$$\frac{Q^{(j)}f(z)}{z^{p-j}} - \rho(1-\tau+\tau p) =$$
$$\frac{\frac{p!}{(p-j)!} - \rho}{(1-z^m)^2}\left[(1-\tau+\tau p)\left(1-z^{2m}\right) + 2\tau mz^m\right] = 0,$$

for $z = R\exp^{\frac{i\pi}{m}}$, and the first part of the Theorem is proved.
Similarly, we can prove part (ii) of the Theorem.

□

5. An argument estimate

In this section we obtain an argument estimate involving the operator $\theta_p^{q,s}(\alpha_1)$ and connected with the linear operator Q.

Theorem 5.1. *For $f \in \mathcal{A}_k(p)$, let the operator Q be defined by (4.1), and let $0 \le \tau < \dfrac{1}{1-p}$. If*

$$\left|\arg \frac{Q^{(j)} f(z)}{z^{p-j}}\right| < \frac{\pi\delta}{2},\ z \in \mathbb{U} \quad \big(\delta > 0,\ 0 \le j \le p\big), \tag{5.1}$$

then

$$\left|\arg \frac{\left[\theta_p^{q,s}(\alpha_1) f(z)\right]^{(j)}}{z^{p-j}}\right| < \frac{\pi\delta}{2},\ z \in \mathbb{U}.$$

Proof. For $f \in \mathcal{A}_k(p)$, if we let

$$q(z) := \frac{\left[\theta_p^{q,s}(\alpha_1) f(z)\right]^{(j)}}{z^{p-j}} \frac{(p-j)!}{p!},$$

then q is of the form (2.1) and it is analytic in $\mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg q(z)| < \frac{\pi\delta}{2},\ |z| < |z_0| \quad \text{and} \quad |\arg q(z_0)| = \frac{\pi\delta}{2} \quad (\delta > 0)\,,$$

then, accorollaryding to Lemma 2.3 we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik\delta \quad \text{and} \quad q(z_0)^{1/\delta} = \pm ic \quad (c > 0).$$

Also, from the equality (4.5) we get

$$\frac{Q^{(j)} f(z_0)}{z_0^{p-j}} = \frac{p!}{(p-j)!}(1-\tau+\tau p)q(z_0)\left[1 + \frac{\tau}{1-\tau+\tau p}\frac{z_0 q'(z_0)}{q(z_0)}\right].$$

If $\arg q(z_0) = \dfrac{\pi\delta}{2}$, then

$$\arg \frac{Q^{(j)} f(z_0)}{z_0^{p-j}} = \frac{\pi\delta}{2} + \arg\left(1 + \frac{\tau}{1-\tau+\tau p} ik\delta\right) = \frac{\pi\delta}{2} + \tan^{-1}\left(\frac{\tau}{1-\tau+\tau p} k\delta\right) \ge \frac{\pi\delta}{2},$$

whenever $k \ge \dfrac{1}{2}\left(c + \dfrac{1}{c}\right)$ and $0 \le \tau < \dfrac{1}{1-p}$, and this last inequality contradicts the assumption (5.1).

Similarly, if $\arg q(z_0) = -\dfrac{\pi\delta}{2}$, then we obtain

$$\arg \frac{Q^{(j)} f(z_0)}{z_0^{p-j}} \le -\frac{\pi\delta}{2},$$

which also contradicts the assumption (5.1).

Consequently, the function q need to satisfy the inequality $|\arg q(z)| < \dfrac{\pi\delta}{2}$, $z \in \mathbb{U}$, i.e. the conclusion of our theorem. □

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On a Class of Harmonic Univalent Functions Defined by Using a New Differential Operator

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Abstract

In this paper, a new class of complex-valued harmonic univalent functions defined by using a new differential operator is introduced. We investigate coefficient bounds, distortion inequalities, extreme points and inclusion results for this class.

Keywords: Harmonic functions, univalent functions, starlike and convex functions, differential operator.
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1. Introduction

Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics (e.g. see Choquet (Choquet, 1945), Dorff (Dorff, 2003), Duren (Duren, 2004)). A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense- preserving in D is that $\left|h^{'}(z)\right| > \left|g^{'}(z)\right|$ in D; see (Clunie & Sheil-Small, 1984).

Denote by SH the class of functions $f = h+\overline{g}$ that are harmonic univalent and sense-preserving in the unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in SH$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{1.1}$$

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Therefore

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}, \quad |b_1| < 1.$$

Note that SH reduces to the class S of normalized analytic univalent functions in U if the co-analytic part of f is identically zero.

In 1984 Clunie and Sheil-Small (Clunie & Sheil-Small, 1984) investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on SH and its subclasses such as Avcı and Zlotkiewicz (Avcı& Zlotkiewicz, 1990), Silverman (Silverman, 1998), Silverman and Silvia (Silverman, 1999), Jahangiri (Jahangiri, 1999) studied the harmonic univalent functions.

The differential operator $D^n_{\alpha,\mu}(\lambda, w)$ $(n \in \mathbb{N}_0)$ was introduced by Bucur et al. (Bucur *et al.*, 2015). For $f = h + \overline{g}$ given by (1.1), we define the following differential operator:

$$D^n_{\alpha,\mu}(\lambda, w) f(z) = D^n_{\alpha,\mu}(\lambda, w) h(z) + (-1)^n \overline{D^n_{\alpha,\mu}(\lambda, w) g(z)},$$

where

$$D^n_{\alpha,\mu}(\lambda, w) h(z) = z + \sum_{k=2}^{\infty} \left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n a_k z^k$$

and

$$D^n_{\alpha,\mu}(\lambda, w) g(z) = \sum_{k=1}^{\infty} \left[(k+1)(\mu w^\lambda - \alpha) + k\right]^n b_k z^k,$$

where $\mu, \lambda, w \geq 0$, $0 \leq \alpha \leq \mu w^\lambda$, with $D^n_{\alpha,\mu}(\lambda, w) f(0) = 0$.

Motivated by the differential operator $D^n_{\alpha,\mu}(\lambda, w)$, we define generalization of the differential operator for a function $f = h + \overline{g}$ given by (1.1).

$$D^0_{\alpha,\mu}(\lambda, w) f(z) = D^0 f(z) = h(z) + \overline{g(z)},$$

$$D^1_{\alpha,\mu}(\lambda, w) f(z) = (\alpha - \mu w^\lambda)(h(z) + \overline{g(z)}) + (\mu w^\lambda - \alpha + 1)(z h'(z) - \overline{z g'(z)}),$$

$$\vdots$$

$$D^n_{\alpha,\mu}(\lambda, w) f(z) = D\left(D^{n-1}_{\alpha,\mu}(\lambda, w) f(z)\right). \tag{1.2}$$

If f is given by (1.1), then from (1.2), we see that

$$D^n_{\alpha,\mu}(\lambda, w) f(z) = z + \sum_{k=2}^{\infty} \left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} \left[(k+1)(\mu w^\lambda - \alpha) + k\right]^n \overline{b_k z^k}. \tag{1.3}$$

When , $w = \alpha = 0$, we get modified Salagean differential operator (Salagean, 1983).

Denote by $SH(\lambda, w, n, \alpha, \beta)$ the subclass of SH consisting of functions f of the form (1.1) that satisfy the condition

$$\Re\left(\frac{D^{n+1}_{\alpha,\mu}(\lambda, w) f(z)}{D^n_{\alpha,\mu}(\lambda, w) f(z)}\right) \geq \beta; \quad (0 \leq \beta < 1), \tag{1.4}$$

where $D^n_{\alpha,\mu}(\lambda, w)f(z)$ is defined by (1.3).

We let the subclass $\overline{SH}(\lambda, w, n, \alpha, \beta)$ consisting of harmonic functions $f_n = h + \overline{g}_n$ in SH so that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \ g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad a_k, \ b_k \geq 0. \tag{1.5}$$

By suitably specializing the parameters, the classes $SH(\lambda, w, n, \alpha, \beta)$ reduces to the various subclasses of harmonic univalent functions. Such as,

(i) $SH(0, 0, 0, 0, 0) = SH^*(0)$ (Avcı (Avcı& Zlotkiewicz, 1990), Silverman (Silverman, 1998), Silverman and Silvia (Silverman, 1999)),

(ii) $SH(0, 0, 0, 0, \beta) = SH^*(\beta)$ (Jahangiri (Jahangiri, 1999)),
$SH(0, 0, 0, 0, \beta) = \overline{S}_H(1, 0, \beta)$ (Yalçın (Yalçın, 2005)),

(iii) $SH(0, 0, 1, 0, 0) = KH(0)$ (Avcı (Avcı& Zlotkiewicz, 1990), Silverman (Silverman, 1998), Silverman and Silvia (Silverman, 1999)),

(iv) $SH(0, 0, 1, 0, \beta) = KH(\beta)$ (Jahangiri (Jahangiri, 1999)),
$SH(0, 0, 1, 0, \beta) = \overline{S}_H(2, 1, \beta)$ (Yalçın (Yalçın, 2005)),

(v) $SH(0, 0, n, 0, \beta) = H(n, \beta)$ (Jahangiri et al. (Jahangiri *et al.*, 2002)),
$SH(0, 0, n, 0, \beta) = \overline{S}_H(n + 1, n, \beta)$ (Yalçın (Yalçın, 2005)),

The object of the present paper is to give sufficient condition for functions $f = h+\overline{g}$ where h and g are given by (1.1) to be in the class $SH(\lambda, w, n, \alpha)$; and it is shown that this coefficient condition is also necessary for functions belonging to the subclass $\overline{SH}(\lambda, w, n, \alpha, \beta)$. Also, we obtain coefficient bounds, distortion inequalities, extreme points and inclusion results for this class.

2. Coefficient Bounds

Theorem 2.1. *Let $f = h + \overline{g}$ be so that h and g are given by (1.1). Furthermore, let*

$$\sum_{k=2}^{\infty}(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n |a_k| + \sum_{k=1}^{\infty}(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n |b_k| \leq 1-\beta, \tag{2.1}$$

where $\mu, \lambda, w \geq 0$, $0 \leq \alpha \leq \mu w^{\lambda}$, $n \in \mathbb{N}_0$, $0 \leq \beta < 1$. Then f is sense-preserving, harmonic univalent in U and $f \in SH(\lambda, w, n, \alpha, \beta)$.

Proof. If $z_1 \neq z_2$,

$$\left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right| \geq 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right| = 1 - \left|\frac{\sum_{k=1}^{\infty} b_k \left(z_1^k - z_2^k\right)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k \left(z_1^k - z_2^k\right)}\right|$$

$$> 1 - \frac{\sum_{k=1}^{\infty} k\,|b_k|}{1 - \sum_{k=2}^{\infty} k\,|a_k|}$$

$$\geq 1-\frac{\sum\limits_{k=1}^{\infty}\frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n}}{1-\beta}|b_k|}{1-\sum\limits_{k=2}^{\infty}\frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n}}{1-\beta}|a_k|}$$

$$\geq 0,$$

which proves univalence. Note that f is sense preserving in U. This is because

$$\begin{aligned}
|h'(z)| &\geq 1-\sum_{k=2}^{\infty}k|a_k||z|^{k-1} > 1-\sum_{k=2}^{\infty}\frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n}}{1-\beta}|a_k| \\
&\geq \sum_{k=1}^{\infty}\frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n}}{1-\beta}|b_k| > \sum_{k=1}^{\infty}k|b_k||z|^{k-1} \\
&\geq |g'(z)|.
\end{aligned}$$

Using the fact that $\Re(w)\geq\beta$ if and only if $|1-\beta+w|\geq|1+\beta-w|$, it suffices to show that

$$\left|(1-\beta)D^{n}_{\alpha,\mu}(\lambda,w)+D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)\right|-\left|(1+\beta)D^{n}_{\alpha,\mu}(\lambda,w)-D^{n+1}_{\alpha,\mu}(\lambda,w)\right|\geq 0. \tag{2.2}$$

Substituting for $D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)$ and $D^{n}_{\alpha,\mu}(\lambda,w)f(z)$ in (2.2), we obtain

$$\begin{aligned}
&\left|(1-\beta)D^{n}_{\alpha,\mu}(\lambda,w)+D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)\right|-\left|(1+\beta)D^{n}_{\alpha,\mu}(\lambda,w)f(z)-D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)\right| \\
\geq\ & 2(1-\beta)|z|-\sum_{k=2}^{\infty}\left[(k+1-\beta)+(k-1)(\mu w^{\lambda}-\alpha)\right]\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n}|a_k||z|^{k} \\
&-\sum_{k=1}^{\infty}\left[(k-1+\beta)+(k-1)(\mu w^{\lambda}-\alpha)\right]\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n}|b_k||z|^{k} \\
&-\sum_{k=2}^{\infty}\left[(k-1-\beta)+(k-1)(\mu w^{\lambda}-\alpha)\right]\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n}|a_k||z|^{k} \\
&-\sum_{k=1}^{\infty}\left[(k+1+\beta)+(k-1)(\mu w^{\lambda}-\alpha)\right]\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n}|b_k||z|^{k} \\
\geq\ & 2(1-\beta)|z|\left(1-\sum_{k=2}^{\infty}\frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n}}{1-\beta}|a_k|\right. \\
&\left.-\sum_{k=1}^{\infty}\frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n}}{1-\beta}|b_k|\right).
\end{aligned}$$

This last expression is non-negative by (2.1), and so the proof is completed. □

Theorem 2.2. *Let $f_n = h + \overline{g}_n$ be given by (1.5). Then $f_n \in \overline{SH}(\lambda, n, \alpha)$ if and only if*

$$\sum_{k=2}^{\infty}(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k + \sum_{k=1}^{\infty}(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k \leq 1-\beta, \quad (2.3)$$

where $\mu, \lambda, w \geq 0$, $0 \leq \alpha \leq \mu w^{\lambda}$, $n \in \mathbb{N}_0$, $0 \leq \beta < 1$.

Proof. The "if" part follows from Theorem 2.1 upon noting that $\overline{SH}(\lambda, w, n, \alpha, \beta) \subset SH(\lambda, w, n, \alpha, \beta)$. For the "only if" part, we show that $f \notin \overline{SH}(\lambda, w, n, \alpha, \beta)$ if the condition (2.3) does not hold. Note that a necessary and sufficient condition for $f_n = h + \overline{g}_n$ given by (1.5), to be in $\overline{SH}(\lambda, w, n, \alpha, \beta)$ is that the condition (1.4) to be satisfied. This is equivalent to

$$\Re\left\{\frac{(1-\beta)z - \sum_{k=2}^{\infty}(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k z^k}{z - \sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k z^k + \sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k \overline{z}^k}\right.$$
$$\left.\frac{-\sum_{k=1}^{\infty}(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k \overline{z}^k}{z - \sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k z^k + \sum_{k=1}^{\infty}(k+1)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k \overline{z}^k}\right\} \geq 0.$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$ we must have

$$\frac{(1-\beta) - \sum_{k=2}^{\infty}(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k r^{k-1}}{1 - \sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k r^{k-1} + \sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k r^{k-1}}$$
$$\frac{-\sum_{k=1}^{\infty}(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k r^{k-1}}{1 - \sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k r^{k-1} + \sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k r^{k-1}} \geq 0. \quad (2.4)$$

If the condition (2.3) does not hold, then the numerator in (2.4) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.4) is negative. This contradicts the required condition for $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$ and so the proof is complete. □

3. Distortion Inequalities and Extreme Points

Theorem 3.1. *Let $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. Then for $|z| = r < 1$ we have*

$$|f_n(z)| \leq (1+b_1)\, r + \left(\frac{(1-\beta)}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^{n}} - \frac{(1+\beta)\left[2\left(\mu w^{\lambda}-\alpha\right)+1\right]^{n}}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^{n}} b_1\right) r^2,$$

and

$$|f_n(z)| \geq (1-b_1)\, r - \left(\frac{(1-\beta)}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^{n}} - \frac{(1+\beta)\left[2\left(\mu w^{\lambda}-\alpha\right)+1\right]^{n}}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^{n}} b_1\right) r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. Taking the absolute value of f_n we have

$$\begin{aligned}
|f_n(z)| &\leq (1+b_1)\,r + \sum_{k=2}^{\infty}(a_k+b_k)\,r^k \\
&\leq (1+b_1)\,r + \sum_{k=2}^{\infty}(a_k+b_k)\,r^2 \\
&= (1+b_1)\,r + \frac{(1-\beta)\,r^2}{(2-\beta)\left[\mu w^\lambda-\alpha+2\right]^n}\sum_{k=2}^{\infty}\frac{(2-\beta)\left[\mu w^\lambda-\alpha+2\right]^n}{(1-\beta)}[a_k+b_k] \\
&\leq (1+b_1)\,r + \frac{(1-\beta)\,r^2}{(2-\beta)\left[\mu w^\lambda-\alpha+2\right]^n} \\
&\quad \times\sum_{k=2}^{\infty}\left(\frac{(k-\beta)\left[(k-1)(\mu w^\lambda-\alpha)+k\right]^n}{1-\beta}a_k\right. \\
&\quad \left.+\frac{(k+\beta)\left[(k-1)(\mu w^\lambda-\alpha)+k\right]^n}{1-\beta}b_k\right) \\
&\leq (1+b_1)\,r + \frac{(1-\beta)}{(2-\beta)\left[\mu w^\lambda-\alpha+2\right]^n}\left(1-\frac{(1+\beta)\left[2\left(\mu w^\lambda-\alpha\right)+1\right]^n}{1-\beta}b_1\right)r^2 \\
&\leq (1+b_1)\,r + \left(\frac{(1-\beta)}{(2-\beta)\left[\mu w^\lambda-\alpha+2\right]^n}-\frac{(1+\beta)\left[2\left(\mu w^\lambda-\alpha\right)+1\right]^n}{(2-\beta)\left[\mu w^\lambda-\alpha+2\right]^n}b_1\right)r^2.
\end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 3.1. □

Corollary 3.1. *Let f_n of the form (1.5) be so that $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. Then*

$$\left\{w : |w| < \frac{(2-\beta)\left[\mu w^\lambda-\alpha+2\right]^n-1+\beta}{(2-\beta)\left[\mu w^\lambda-\alpha+2\right]^n}\right.$$

$$\left.-\frac{(2-\beta)\left[\mu w^\lambda-\alpha+2\right]^n-(1+\beta)\left[2(\mu w^\lambda-\alpha)+1\right]^n}{(2-\beta)\left[\mu w^\lambda-\alpha+2\right]^n}\right\} \subset f_n(U).$$

Theorem 3.2. *Let f_n be given by (1.5). Then $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$ if and only if*

$$f_n(z) = \sum_{k=1}^{\infty}\left(X_k h_k(z) + Y_k g_{n_k}(z)\right),$$

where

$$h_1(z) = z, \quad h_k(z) = z - \frac{1-\beta}{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha)+k\right]^n} z^k; \quad (k \geq 2),$$

$$g_{n_k}(z) = z + (-1)^n \frac{1-\beta}{(k+\beta)\left[(k+1)(\mu w^\lambda - \alpha)+k\right]^n} \overline{z}^k; \quad (k \geq 2),$$

$$\sum_{k=1}^{\infty} (X_k + Y_k) = 1, X_k \geq 0, Y_k \geq 0.$$

In particular, the extreme points of $\overline{SH}(\lambda, w, n, \alpha, \beta)$ *are* $\{h_k\}$ *and* $\{g_{n_k}\}$.

Proof. For functions f_n of the form (1.5) we may write

$$\begin{aligned} f_n(z) &= \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_{n_k}(z)) \\ &= \sum_{k=1}^{\infty} (X_k + Y_k)\, z - \sum_{k=2}^{\infty} \frac{1-\beta}{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha)+k\right]^n} X_k z^k \\ &\quad + (-1)^n \sum_{k=1}^{\infty} \frac{1-\beta}{(k+\beta)\left[(k+1)(\mu w^\lambda - \alpha)+k\right]^n} Y_k \overline{z}^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha)+k\right]^n}{1-\beta} \left(\frac{1-\beta}{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha)+k\right]^n} X_k \right) \\ &+ \sum_{k=1}^{\infty} \frac{(k+\beta)\left[(k+1)(\mu w^\lambda - \alpha)+k\right]^n}{1-\beta} \left(\frac{1-\beta}{(k+\beta)\left[(k+1)(\mu w^\lambda - \alpha)+k\right]^n} Y_k \right) \\ = &\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1, \text{ and so } f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta). \end{aligned}$$

Conversely, if $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$, then

$$a_k \leq \frac{1-\beta}{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha)+k\right]^n}$$

and

$$b_k \leq \frac{1-\beta}{(k+\beta)\left[(k+1)(\mu w^\lambda - \alpha)+k\right]^n}.$$

Setting

$$X_k = \frac{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha)+k\right]^n}{1-\beta} a_k; \ (k \geq 2),$$

$$Y_k = \frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} b_k; \quad (k \geq 1),$$

and

$$X_1 = 1 - \left(\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k\right)$$

where $X_1 \geq 0$. Then

$$f_n(z) = X_1 z + \sum_{k=2}^{\infty} X_k h_k(z) + \sum_{k=1}^{\infty} Y_k g_{n_k}(z)$$

as required. □

4. Inclusion Results

Theorem 4.1. *The class $\overline{SH}(\lambda, w, n, \alpha, \beta)$ is closed under convex combinations.*

Proof. Let $f_{n_i} \in \overline{SH}(\lambda, w, n, \alpha, \beta)$ for $i = 1, 2, ...,$ where f_{n_i} is given by

$$f_{n_i}(z) = z - \sum_{k=2}^{\infty} a_{k_i} z^k + (-1)^n \sum_{k=1}^{\infty} b_{k_i} \overline{z}^k.$$

Then by (2.3),

$$\sum_{k=2}^{\infty} \frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} a_{k_i} + \sum_{k=1}^{\infty} \frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} b_{k_i} \leq 1. \tag{4.1}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{k_i}\right) z^k + (-1)^n \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i b_{k_i}\right)\overline{z}^k.$$

Then by (4.1),

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta}\left(\sum_{i=1}^{\infty} t_i a_{k_i}\right) \\ &+\sum_{k=1}^{\infty} \frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-v}\left(\sum_{i=1}^{\infty} t_i b_{k_i}\right) \\ = &\sum_{i=1}^{\infty} t_i \left(\sum_{k=2}^{\infty} \frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} a_{k_i}\right. \\ &\left.+\sum_{k=1}^{\infty} \frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} b_{k_i}\right) \leq \sum_{i=1}^{\infty} t_i = 1. \end{aligned}$$

This is the condition required by (2.3) and so $\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. □

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Results on Approximation Properties in Intuitionistic Fuzzy Normed Linear Spaces

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Abstract

In this paper we introduce the notions approximation properties (APs) and bounded approximation properties (BAPs) in the setting of intuitionistic fuzzy normed linear spaces (IFNLSs). Further, we define strong intuitionistic fuzzy continuous and strong intuitionistic fuzzy bounded operators and using them we prove the existence of an IFNLS which does not have the approximation property. In addition, we give example of an IFNLS with the AP which fails to have the BAP.

Keywords: Intuitionistic fuzzy normed linear space, approximation property, bounded approximation property.
2010 MSC: 55M20, 54H25, 47H09.

1. Introduction

In analysis many problems we study are concerned with large classes of objects most of which turn out to be vector spaces or linear spaces. Since limit process is indispensable in such problems, a metric or topology may be induced in those classes. If the induced metric satisfies the translation invariance property, a norm can be defined in that linear space and we get a structure of the space which is compatible with that metric or topology. The resulting structure is a normed linear space. There are situations where crisp norm can not measure the length of a vector accurately and in such cases the notion of fuzzy norm happens to be useful. There has been a systematic development of fuzzy normed linear spaces (FNLSs) and one of the important development over FNLS is the notion of intuitionistic fuzzy normed linear space (IFNLS). The study of analytic propertis of IFNLSs, their topological structure and generalizations, therefore, remain well motivated areas of research.

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The idea of a fuzzy norm on a linear space was introduced by Katsaras (Katsaras, 1984). Felbin (Felbin, 1992) introduced the idea of a fuzzy norm whose associated metric is of Kaleva and Seikkala (Kaleva & Seikkala, 1984) type. Cheng and Mordeson (Cheng & Mordeson, 1994) introduced another notion of fuzzy norm on a linear space whose associated metric is Kramosil and Michalek (Kramosil & Michalek, 1975) type. Again, following Cheng and Mordeson, one more notion of fuzzy normed linear space was given by Bag and Samanta (Bag & Samanta, 2003*a*).

The notion of intuitionistic fuzzy set (IFS) introduced by Atanassov (Atanassov, 1986) has triggered some debate (for details, see (Cattaneo & Ciucci, 2006; Dubois *et al.*, 2005; Grzegorzewski & Mrowka, 2005)) regarding the use of the terminology "intuitionistic" and the term is considered to be a misnomer on the following account:

- The algebraic structure of IFSs in not intuitionistic, since negation is involutive in IFS theory.
- Intuitionistic logic obeys the law of contradiction, IFSs do not.

Also IFSs are considered to be equivalent to interval-valued fuzzy sets and they are particular cases of L-fuzzy sets. In response to this debate, Atanassov justified the terminology in (Atanassov, 2005). Apart from the terminological issues, research in intuitionistic fuzzy setting remains well motivated as IFSs give us a very natural tool for modeling imprecision in real life situations which can not be handled with fuzzy set theory alone and also IFS found its application in various areas of science and engineering.

With the help of arbitrary continuous t-norm and continuous t-conorm, Saadati and Park (Saadati & Park, 2006) introduced the concept of IFNLS. There has been further development over IFNLS, e.g., the topological structure of an intuitionistic fuzzy 2-normed space has been studied by Mursaleen and Lohani in (Mursaleen & Lohani, 2009). Recently, a number of interesting properties of IFNLS have been studied by Mursaleen and Mohiuddine (Mursaleen & Mohiuddine, 2009*a*,*b*,*c*,*d*). Further, generalizing the idea of Saadati and Park, an intuitionistic fuzzy n-normed linear space (IFnNLS) has been defined by Vijayabalaji et al. (Vijayabalaji *et al.*, 2007*b*). More properties of IFnNLS have been studied by N. Thillaigovindan, S. Anita Shanti and Y. B. Jun in (Vijayabalaji *et al.*, 2007*a*). Some more recent work in similar context can be found in (Debnath, 2015; Debnath & Sen, 2014*a*,*b*; Esi & Hazarika, 2012; Mursaleen *et al.*, 2010*a*; Sen & Debnath, 2011).

In classical Banach space theory, some most important properties are "Approximation properties" which were investigated by Grothendick (Grothendiek, 1955). We say that a Banach space X has the approximation property (AP) if, for every compact K and $\epsilon > 0$, there is a bounded finite rank operator $T : X \longrightarrow X$ such that $||T(x) - x|| < \epsilon$, for all $x \in K$, i.e. $I(x)$-the identity operator on X- can be approximated by finite rank operators uniformly on compact sets. Also X has the bounded approximation property (BAP) if for every compact K and $\epsilon > 0$, there is a bounded finite rank operator $T : X \longrightarrow X$ with $||T|| \leqslant \lambda$ such that $||T(x) - x|| < \epsilon$ for all $x \in K$ for some $\lambda > 0$. The APs play very crucial role in the study of infinite dimensional Banach space theory and also in the investigation of Schauder bases. Some of the important references from related works being (Choi *et al.*, 2009; Enflo, 1973; Kim, 2008; Mursaleen *et al.*, 2010*b*; Szarek, 1987).

Yilmaz (Yilmaz, 2010*a*) introduced the notion of the AP in fuzzy normed spaces and established some interesting results on it. Very recently Keun Young Lee (Lee, 2015) identified some limitations in Yilmaz's definitions regarding the continuity of fuzzy operators. He modified Yilmaz's definitions and studied approximation property (AP) and bounded approximation property (BAP) on fuzzy normed spaces.

In this article we address the questions raised by Keun Young Lee (Lee, 2015) and also generalize the work of Figel and Johnson (Figel & Johnson, 1973) in the context of AP and BAP in the new setting of IFNLS.

First we recall some basic definitions and results which will be used subsequently.

Definition 1.1. (Saadati & Park, 2006) The 5-tuple $(X, \mu, \nu, *, \circ)$ is said to be an IFNLS if X is a linear space, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm, and μ, ν fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

(a) $\mu(x, t) + \nu(x, t) \leqslant 1$,
(b) $\mu(x, t) > 0$,
(c) $\mu(x, t) = 1$ if and only if $x = 0$,
(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(e) $\mu(x, t) * \mu(y, s) \leqslant \mu(x + y, t + s)$,
(f) $\mu(x, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(g) $\lim_{t\to\infty} \mu(x, t) = 1$ and $\lim_{t\to 0} \mu(x, t) = 0$,
(h) $\nu(x, t) < 1$,
(i) $\nu(x, t) = 0$ if and only if $x = 0$,
(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(k) $\nu(x, t) \circ \nu(y, s) \geqslant \nu(x + y, t + s)$,
(l) $\nu(x, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(m) $\lim_{t\to\infty} \nu(x, t) = 0$ and $\lim_{t\to 0} \nu(x, t) = 1$.

In this case (μ, ν) is called an intuitionistic fuzzy norm. When no confusion arises, an IFNLS will be denoted simply by X.

Definition 1.2. (Debnath, 2012) Let X be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - \xi, t) > 1 - \varepsilon$ and $\nu(x_k - \xi, t) < \varepsilon$ for all $k \geqslant k_0$. It is denoted by $(\mu, \nu) - \lim x_k = \xi$.

Definition 1.3. (Saadati & Park, 2006) Let X be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\alpha \in (0, 1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_m, t) > 1 - \alpha$ and $\nu(x_k - x_m, t) < \alpha$ for all $k, m \geqslant k_0$.

Definition 1.4. (Debnath & Sen, 2014*a*) Let X be an IFNLS. Then X is said to be complete if and only if every Cauchy sequence of X is convergent.

Definition 1.5. (Lael & Nourouzi, 2007) Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. A subset S in X is said to be compact if each sequence of elements of S has a convergent subsequence.

Definition 1.6. (Debnath, 2012) Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. For $t > 0$, we define an open ball $B(x, r, t)$ with center at $x \in X$ and radius $0 < r < 1$, as

$$B(x, r, t) = \{y \in X : \mu(x - y, t) > 1 - r, \nu(x - y, t) < r\}.$$

Proof of the following lemma is similar to its analogue in case of fuzzy normed spaces (Bag & Samanta, 2003*b*).

Lemma 1.1. *Let* $(X, \mu, \nu, *, \circ)$ *be an IFNLS with the condition*

$$\mu(x, t) > 0 \text{ and } \nu(x, t) < 1 \text{ implies } x = 0, \text{ for all } t \in \mathbb{R}^+. \quad (1.1)$$

Let $\|x\|_\alpha = inf\{t \in \mathbb{R}^+ : \mu(x, t) > \alpha \text{ and } \nu(x, t) < 1 - \alpha\}$ *for each* $\alpha \in (0, 1)$*. Then* $\{\| \cdot \|_\alpha : \alpha \in (0, 1)\}$ *is an ascending class of norms on X. These norms are called* α*- norms on the intuitionistic fuzzy norm* (μ, ν)*.*

Definition 1.7. (Mursaleen *et al.*, 2010*a*) Let (x_n) be a sequence in an IFNLS $(X, \mu, \nu, *, \circ)$. It is said to be basis of X if for every $x \in X$ there exists a unique sequence (a_n) of scalars such that

$$(\mu, \nu) - \lim \textstyle\sum_{k=1}^{n} a_k x_k = x.$$

that is, for each $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists $n_0 = n_0(\alpha, \epsilon) \in N$ such that $n \geqslant n_0$ implies,

$$\mu(x - \textstyle\sum_{k=1}^{n} a_k x_k, \epsilon) > 1 - \alpha \text{ and } \nu(x - \textstyle\sum_{k=1}^{n} a_k x_k, \epsilon) < \alpha, \text{ where } x = \textstyle\sum_{k=1}^{\infty} a_k x_k.$$

2. Main Results

Now we are ready to discuss our main results. First we define some important notions in connection with approximation property in IFNLS.

Definition 2.1. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. A complete IFNLS is said to have the approximation property, briefly AP, if for every compact set K in X and for each $\alpha > (0, 1)$ and $\epsilon > 0$, there exists an operator T of finite rank such that

$$\mu\left(T_\alpha(x) - x, \epsilon\right) > 1 - \alpha \text{ and } \nu\left(T_\alpha(x) - x, \epsilon\right) < \alpha$$

for every $x \in K$.

Definition 2.2. Let λ be a real number. An IFNLS $(X, \mu, \nu, *, \circ)$ is said to have the λ-bounded approximation property, briefly λ-BAP, if for every compact set K in X and for each $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists an operator $T \in F(X, X, \lambda)$ such that

$$\mu\left(T(x) - x, \epsilon\right) > 1 - \alpha \text{ and } \nu\left(T(x) - x, \epsilon\right) < \alpha$$

for every $x \in K$.

Definition 2.3. Suppose that an IFNLS $(X, \mu, \nu, *, \circ)$ has a basis (x_n). For each positive integer m, the m^{th} natural projection P_m for x_m is the map

$$\sum_{n=1}^{\infty} a_n x_n \longrightarrow \sum_{n=1}^{m} a_n x_n \text{ from } (X,\mu,\nu,*,\circ) \text{ to } (X,\mu,\nu,*,\circ).$$

Definition 2.4. Let $(X,\mu,\nu,*,\circ)$ and $(Y,\mu^{'},\nu^{'},*,\circ)$ be two IFNLS and $T : X \longrightarrow Y$ be a linear operator where (μ,ν) and $(\mu^{'},\nu^{'})$ are intuitionistic fuzzy normed. Then

1. The operator T is called strongly intuitionistic fuzzy (shortly sif) continuous at $a \in X$ if, for given $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in X$,
$$\mu^{'}(T(x) - T(a),\epsilon) \geqslant \mu(x-a,\delta) \text{ and } \nu^{'}(T(x) - T(a),\epsilon) \leqslant \nu(x-a,\delta).$$
If T is sif-continuous at each point of X, then T is said to be sif-continuous on X.
2. The operator T is called strongly intuitionistic fuzzy bounded on X if there exists a positive real number M such that $\mu^{'}(T(x),t) \geqslant \mu(x,\frac{t}{M})$ and $\nu^{'}(T(x),t) \leqslant \nu(x,\frac{t}{M})$ for all $x \in X$ and $t \in \mathbb{R}$. We will denote the set of all strongly intuitionistic fuzzy (shortly sif) bounded operators form X to Y by $F(X,Y)$. Then $F(X,Y)$ is a vector space. For all $M > 0$, $F(X,Y,M)$ is denoted by
$$\{T \in F(X,Y) : \mu^{'}(T(x),t) \geqslant \mu(x,\tfrac{t}{M}), \nu^{'}(T(x),t) \leqslant \nu(x,\tfrac{t}{M}), \forall x \in X, \forall t \in \mathbb{R}\},$$
where M is a positive real number.
For some $M > 0$ if $\mathbb{S} = F(X,Y,M)$ then $\mathbb{S}$ is called a bounded subset of $F(X,Y)$. Again the set of all finite rank sif-bounded operators from X to Y is denoted by $\bar{F}(X,Y)$. Then $\bar{F}(X,Y)$ is subspace of $F(X,Y)$. Similarly, we can say that $\bar{F}(X,Y,M)$ is also a subspace of $F(X,Y,M)$ for some $M > 0$.

Proof of the following is similar to its fuzzy analogue in (Bag & Samanta, 2005).

Lemma 2.1. *Let $(X,\mu,\nu,*,\circ)$ and $(Y,\mu^{'},\nu^{'},*,\circ)$ be two IFNLSs satisfying condition 1.1 and $T : X \longrightarrow Y$ be a linear operator. Then T is sif-bounded if and only if it is uniformly bounded with respect to α- norms of (μ,ν) and $(\mu^{'},\nu^{'})$. That is, there exists some $M > 0$, independent of α, such that $||T(x)||_\alpha \leqslant M||x||_\alpha$, for all $\alpha \in (0,1)$.*

Remark. If $(X,\mu,\nu,*,\circ)$ and $(Y,\mu^{'},\nu^{'},*,\circ)$ be two IFNLSs satisfying the conditions:
$\mu(x,t) > 0$ and $\nu(x,t) < 1$ implies $x = 0$ for all $t \in \mathbb{R}^+$ and
for $x \neq 0$, $\mu(x,t)$ is continuous and strictly increasing on $\{t : 0 < \mu(x,t) < 1\}$, while $\nu(x,t)$ is continuous and strictly decreasing on $\{t : 0 < \mu(x,t) < 1\}$ and $M > 0$. Then we obtain

$$F(X,Y,M) = \{T \in F(X,Y) : ||T(x)||_\alpha \leqslant M||x||_\alpha, \forall x \in X, \forall \alpha \in (0,1)\}.$$

Hence $F(X,Y,M)$ and $\bar{F}(X,Y,M)$ are bounded convex subsets of $F(X,Y)$.

Theorem 2.1. *Let X be a Banach space and (x_n) be a Schauder basis in X. Then (x_n) is a basis for an IFNLS $(X,\mu,\nu,*,\circ)$ where*

$$\mu(x,t) = \begin{cases} \frac{t-||x||}{t+||x||}, & if\, t > ||x|| \\ 0, & if\, t \leqslant ||x||, \end{cases}$$

$$\nu(x,t) = \begin{cases} 1 - \frac{t-||x||}{t+||x||}, & if\, t > ||x|| \\ 1, & if\, t \leqslant ||x||, \end{cases}$$

and every natural projection is sif-continuous.

Proof. Given that (x_n) is a basis for an IFNLS $(X, \mu, \nu, *, \circ)$.

It is enough to show that -

natural projection $P_n : (X, \mu, \nu, *, \circ) \to (X, \mu, \nu, *, \circ)$ is sif-bounded for each $x \in N$.

Let $n \in N$, $t \in R$, $x \in X$.

Consider $M = ||P_n||$.

If $t \leqslant 0$, the result is trivial.

Assume that $t > 0$. Then it is enough to show that

$$\mu\left(P_n\left(x\right), t\right) \geqslant \mu\left(x, \tfrac{t}{M}\right) \text{ and } \nu\left(P_n\left(x\right), t\right) \leqslant \nu\left(x, \tfrac{t}{M}\right).$$

The proof of $\mu\left(P_n\left(x\right), t\right) \geqslant \mu\left(x, \tfrac{t}{M}\right)$ can be established in a similar manner as in Proposition 3.4 of (Lee, 2015).

Now considering for ν, we have

$$t > M||x||,$$

then

$$\nu\left(x, \tfrac{t}{M}\right) = 1 - \frac{\frac{t}{M} - ||x||}{\frac{t}{M} + ||x||}.$$

By the assumption,

$$t > M||x|| = ||P_n||||x|| \geqslant ||P_n\left(x\right)||$$

and

$$\frac{t - ||P_n(x)||}{t + ||P_n(x)||} \geqslant \frac{\frac{t}{M} - ||x||}{\frac{t}{M} + ||x||}.$$

Therefore, we have

$$\nu\left(P_n\left(x\right), t\right) = 1 - \frac{t - ||P_n(x)||}{t + ||P_n(x)||} \leqslant 1 - \frac{\frac{t}{M} - ||x||}{\frac{t}{M} + ||x||} = \nu\left(x, \tfrac{t}{M}\right).$$

Hence

$$\nu\left(P_n\left(x\right), t\right) \leqslant \nu\left(x, \tfrac{t}{M}\right).$$

Secondly,

$$t \leqslant ||Mx||,$$

then

$$\nu\left(Mx, t\right) = 1.$$

Thus,

$$\nu\left(P_n\left(x\right), t\right) \leqslant \nu\left(x, \tfrac{t}{M}\right).$$

□

So, we have the existence of an IFNLS having a basis such that every natural projection is sif-continuous. Now provide modified definitions of APs and BAPs in IFNLSs by incorporating the continuity of approximating operators.

Definition 2.5. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. Then X is said to be have the approximation property, briefly AP, if for every compact set K in X and for each $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists an operator $T \in \bar{F}(X, X)$ such that

$$\mu(T(x) - x, \epsilon) > 1 - \alpha \text{ and } \nu(T(x) - x, \epsilon) < \alpha$$

for every $x \in K$.

Definition 2.6. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS and λ be a positive real number. Then X is said to be have the λ- bounded approximation property, briefly λ - BAP, if for every compact set K in X and for each $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists an operator $T \in \bar{F}(X, X, \lambda)$ such that

$$\mu(T(x) - x, \epsilon) > 1 - \alpha \text{ and } \nu(T(x) - x, \epsilon) < \alpha$$

for every $x \in K$. We can also say that X has the BAP if X has the λ-BAP for some $\lambda > 0$.

Theorem 2.2. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. Then the following are equivalent.*

1. $(X, \mu, \nu, *, \circ)$ *has the AP.*
2. *If $(Y, \mu', \nu', *, \circ)$ is an IFNLS, then for every $T \in F(X, Y)$, every compact set K in $(X, \mu, \nu, *, \circ)$ and for each $\alpha \in (0, 1)$ and $t > 0$, there exists an operator $S \in \bar{F}(X, Y)$ such that*

$$\mu'(S(x) - T(x), t) > 1 - \alpha \text{ and } \nu'(S(x) - T(x), t) < \alpha$$

   for each $x \in K$.
3. *If $(Y, \mu', \nu', *, \circ)$ is an IFNLS, then for every $T \in F(Y, X)$, every compact set K in $(Y, \mu', \nu', *, \circ)$ and for each $\alpha \in (0, 1)$ and $t > 0$, there exists an operator $S \in \bar{F}(Y, X)$ such that*

$$\mu(S(y) - T(y), t) > 1 - \alpha \text{ and } \nu(S(y) - T(y), t) < \alpha$$

   for each $y \in K$.

Proof. $(i) \Rightarrow (ii)$

Let $T \in F(X, Y)$ and K be a compact set in $(X, \mu, \nu, *, \circ)$ and $\alpha \in (0, 1)$ and $t > 0$ and $t \in \mathbb{R}$. Then there exists a positive real number M such that

$$\mu'(T(x), t) \geqslant \mu\left(x, \frac{t}{M}\right) \text{ and } \nu'(T(x), t) \leqslant \nu\left(x, \frac{t}{M}\right)$$

for all $x \in X$.

Since $(X, \mu, \nu, *, \circ)$ has the AP, there exists an operator $R \in F(X, X)$ such that

$$\mu\left(R(x) - x, \frac{t}{M}\right) > 1 - \alpha \text{ and } \nu\left(R(x) - x, \frac{t}{M}\right) < \alpha$$

for every $x \in K$.

Now we put $S = TR$. Since T and R both are sif-bounded operators, therefore S is also a sif-bounded operator.

$$\begin{aligned}\mu'\left(S\left(x\right)-T\left(x\right),t\right) &= \mu'\left(TR\left(x\right)-T\left(x\right),t\right)\\ &\geqslant \mu\left(R\left(x\right)-x,\frac{t}{M}\right)\\ &> 1-\alpha.\end{aligned}$$

and

$$\begin{aligned}\nu'\left(S\left(x\right)-T\left(x\right),t\right) &= \nu'\left(TR\left(x\right)-T\left(x\right),t\right)\\ &\leqslant \nu\left(R\left(x\right)-x,\frac{t}{M}\right)\\ &< \alpha.\end{aligned}$$

for every $x \in K$.

$(i) \Rightarrow (iii)$

Let $T \in F(Y,X)$ and K be a compact set in $\left(Y,\mu',\nu',*,\circ\right)$ and $\alpha \in (0,1)$ and $t > 0$ and $t \in R$.

Since $(X,\mu,\nu,*,\circ)$ has the AP and $T(K)$ is compact set in $(X,\mu,\nu,*,\circ)$, there exists an operator $R \in \bar{F}(X,X)$ such that

$$\mu\left(R\left(x\right)-x,t\right) > 1-\alpha \text{ and } \nu\left(R\left(x\right)-x,t\right) < \alpha$$

for every $x \in T(K)$.

Now we put, $S = RT \in \bar{F}(Y,X)$. Then we have,

$$\begin{aligned}\mu\left(S\left(y\right)-T\left(y\right),t\right) &= \mu\left(RT\left(y\right)-T\left(y\right),t\right)\\ &> 1-\alpha.\end{aligned}$$

and

$$\begin{aligned}\nu\left(S\left(y\right)-T\left(y\right),t\right) &= \nu\left(RT\left(y\right)-T\left(y\right),t\right)\\ &< \alpha,\end{aligned}$$

for each $y \in K$.

Since (i) implies both (ii) and (iii), hence (i), (ii) and (iii) are equivalent.

Hence proposition is proved.

□

Proof of the following Lemma is similar to Lemma 4.2 of (Lee, 2015).

Lemma 2.2. *Let $(X,\mu,\nu,*,\circ)$ be an IFNLS and K be a subset in X. If K is a compact set in $(X,\mu,\nu,*,\circ)$, then for every $\alpha \in (0,1)$ and $t > 0$, there exists a finite set $\{x_1,x_2,\ldots,x_n\}$ in K such that for every $x \in K$ we have $x \in B(x_i,\alpha,t)$ for some x_i.*

Theorem 2.3. *Let $(X,\mu,\nu,*,\circ)$ be an IFNLS with intuitionistic fuzzy norm (μ,ν) and $M > 0$. Suppose that there exists a sequence $(T_n) \in \bar{F}(X,X,M)$ such that $T_n(x) \longrightarrow x$ for every $x \in X$, then $(X,\mu,\nu,*,\circ)$ has the AP.*

Proof. Let (T_n) be a sequence in $\bar{F}(X, X, M)$ such that

$$T_n(x) \longrightarrow x \text{ for every } x \in X.$$

Let $\alpha \in (0,1)$ and $t > 0$, and K be a compact set in $(X, \mu, \nu, *, \circ)$.

By the above Lemma, there exists a finite set $\{x_1, x_2, ..., x_n\} \subset K$ such that for $x \in K$ we have $x \in B(x_i, \alpha, t)$ for some x_i.

Then there exists $N_1, N_2 \in \mathbb{N}$ such that if $n \geqslant N_1, N_2$ we have,

$$\mu(T_n(x_i) - x_i, t) > 1 - \alpha \text{ and } \nu(T_n(x_i) - x_i, t) < \alpha$$

for each i.

Let $x \in K$ and choose i such that $x \in B(x_i, \alpha, t)$, that is,

$$\mu(x_i - x, t) > 1 - \alpha \text{ and } \nu(x_i - x, t) < \alpha.$$

Then for $n \geqslant N_1, N_2$,

$$\begin{aligned}
\mu(T_n(x) - x, t) &= \mu(T_n(x) + (-T_n(x_i)) + (T_n(x_i)) + (-x_i) + x_i + (-x), t) \\
&\geqslant min\left\{\mu\left(T_n(x - x_i), \frac{t}{3}\right), \mu\left(T_n(x_i) - x_i, \frac{t}{3}\right), \mu\left(x_i - x, \frac{t}{3}\right)\right\} \\
&\geqslant min\left\{\mu\left(x - x_i, \frac{t}{3M}\right), \mu\left(T_n(x_i) - x_i, \frac{t}{3}\right), \mu\left(x_i - x, \frac{t}{3}\right)\right\} \\
&> 1 - \alpha.
\end{aligned}$$

And

$$\begin{aligned}
\nu(T_n(x) - x, t) &= \nu(T_n(x) + (-T_n(x_i)) + (T_n(x_i)) + (-x_i) + x_i + (-x), t) \\
&\leqslant max\left\{\mu\left(T_n(x - x_i), \frac{t}{3}\right), \mu\left(T_n(x_i) - x_i, \frac{t}{3}\right), \mu\left(x_i - x, \frac{t}{3}\right)\right\} \\
&\leqslant max\left\{\mu\left(x - x_i, \frac{t}{3M}\right), \mu\left(T_n(x_i) - x_i, \frac{t}{3}\right), \mu\left(x_i - x, \frac{t}{3}\right)\right\} \\
&< \alpha.
\end{aligned}$$

Therefore, $\mu(T_n(x) - x, t) > 1 - \alpha$ and $\nu(T_n(x) - x, t) < \alpha$.

Hence $(X, \mu, \nu, *, \circ)$ has the AP. □

By using the above result we derive the following.

Theorem 2.4. *Suppose $(X, \mu, \nu, *, \circ)$ has a basis $\{x_n\}$ and every natural projection*

$$P_n : (X, (\mu, \nu)) \longrightarrow (X, (\mu, \nu))$$

is sif-continuous. Then $(X, \mu, \nu, *, \circ)$ has the AP but the converse is not necessarily true.

Theorem 2.5. *An IFNLS $(X, \mu, \nu, *, \circ)$ satisfying condition 1.1 has the AP if and only if for every compact set K in $(X, \mu, \nu, *, \circ)$ and for each $\alpha \in (0,1)$ and $\epsilon > 0$, there exists an operator $T \in \bar{F}(X, X)$ such that*

$$||T\,(x) - x||_\alpha < \epsilon$$

for every $x \in K$.

Theorem 2.6. *Let* $(X, \mu, \nu, *, \circ)$ *be an IFNLS satisfying condition 1.1 and* $\lambda > 0$. *Then* $(X, \mu, \nu, *, \circ)$ *has* λ*-BAP if and only if for every compact set* K *in* $(X, \mu, \nu, *, \circ)$ *and for each* $\alpha \in (0, 1)$ *and* $\epsilon > 0$, *there exists an operator* $T \in \bar{F}\,(X, X, \lambda)$ *such that*

$$||T\,(x) - x||_\alpha < \epsilon$$

for every $x \in K$.

Proof of the above two results follow from (Yilmaz, 2010*b*).

3. Examples

In this section, we give answers to the following interesting questions with proper examples:

1. Does every IFNLS have the AP?
2. Does in an IFNLSs the AP imply the BAP?

Now we are going to solve (in negative sense) the problem (i) and (ii) with the help of following two examples.

Example 3.1. As we know that there exists a Banach space $(X, ||\cdot||)$ which fails to have the approximation property, similarly there exists an IFNLS $(X, \mu, \nu, *, \circ)$ which fails to have the AP.

Let us define a function,

$$\mu, \nu : X \times \mathbb{R} \longrightarrow [0, 1] \text{ by}$$

$$\mu(x, t) = \begin{cases} 1, & \text{if } t > ||x|| \\ 0, & \text{if } t \leqslant ||x||. \end{cases}$$

and

$$\nu(x, t) = \begin{cases} 0, & \text{if } t > ||x|| \\ 1, & \text{if } t \leqslant ||x||. \end{cases}$$

where (μ, ν) is the intuitionistic fuzzy norm and $||x||_\alpha = ||x||$, for every $\alpha \in (0, 1)$.

Now suppose that $(X, \mu, \nu, *, \circ)$ has the AP.

Let $\alpha \in (0, 1)$ and $\epsilon > 0$ and K be a compact set in X. Since $||x||_\alpha = ||x||$ for each $\alpha \in (0, 1)$, K is compact in $(X, \mu, \nu, *, \circ)$. Then by Theorem 2.5, there exists an operator $T_\alpha \in \bar{F}\,(X, X)$ such that

$$||T\,(x) - x||_\alpha < \epsilon$$

for every $x \in K$.

Hence we have,

$$||T(x) - x|| = ||T(x) - x||_\alpha < \epsilon$$

for every $x \in K$, which is a contradiction as $(X, ||\cdot||)$ fails to have the approximation property.

$(X, \mu, \nu, *, \circ)$ has fails to have the AP.

As in Example 4.9 of (Lee, 2015), we give below an example of the existence of an IFNLS which has the AP but fails to have BAP.

Example 3.2. Enflo and Lidenstrauss (Enflo, 1973; Lindenstrauss, 1971) has proved the existence of a Banach Space X_0 which has the metric approximation property but its dual space X_0^* fails to have the approximation property. There is a sequence $(||\cdot||_n)$ of equivalent norms on X_0 so that $(X_0, ||\cdot||_n)$ fails to have the n- BAP. Consider $X_n = (X_0, ||\cdot||_n)$. Thus $(\sum \oplus X_n)_{l_2}$ fails to have the BAP where $(\sum \oplus X_n)_{l_2}$ is a Banach space whose elements are sequence of the form $(x_1, x_2, ...)$, where $\sum_{n=1}^{\infty} ||x_n||_n^2 < \infty$ and $x_n \in X_n$.

Now we consider, $X = (\sum \oplus X_n)_{l_2}$, and define $||x|| = \left(\sum_{n=1}^{\infty} ||x_n||_n^2\right)^{\frac{1}{2}}$ and $||x||_1 = sup_n ||x_n||$ for all $x = (x_1, x_2, ...) \in X$.

Let us defined a function,

$$\mu, \nu : X \times \mathbb{R} \longrightarrow [0, 1] \text{ by}$$

$$\mu(x,t) = \begin{cases} 1, & \text{if } t > ||x|| \\ \frac{1}{2}, & \text{if } ||x||_1 < t \leqslant ||x|| \\ 0, & \text{if } t \leqslant ||x||_1, \end{cases}$$

and

$$\nu(x,t) = \begin{cases} 0, & \text{if } t > ||x|| \\ \frac{1}{2}, & \text{if } ||x||_1 < t \leqslant ||x|| \\ 1, & \text{if } t \leqslant ||x||_1, \end{cases}$$

where (μ, ν) is the intuitionistic fuzzy norm.

Consider the α-norms as-

$$||x||_\alpha = \begin{cases} ||x||, & \text{if } 1 > \alpha > \frac{1}{2} \\ ||x||_1, & \text{if } 0 < \alpha \leqslant \frac{1}{2}. \end{cases}$$

Suppose that $(X, \mu, \nu, *, \circ)$ has the BAP. Let us assume that K be a compact set in $(X, ||\cdot||)$. Then we have to show that K is a compact set in $(X, \mu, \nu, *, \circ)$.

Let $\epsilon > 0$ and (x_n) be a sequence in K. As K is compact subset in $(X, ||\cdot||)$, there exists subsequence (x_{n_k}) in $(X, ||\cdot||)$. Therefore there exists an $x \in X$ and integers $\mu, \nu > 0$ such that for $k \geqslant \mu, \nu$

$$||x_{n_k} - x|| < \epsilon.$$

Since $||x||_1 \leqslant ||x||$ for all $x \in X$, therefore for $k \geqslant \mu, \nu$

$$||x_{n_k} - x||_\alpha < \epsilon$$

for all $\alpha \in (0,1)$.

Hence K is a compact set in $(X,\mu,\nu,*,\circ)$.

Next consider $\alpha \in \left(\frac{1}{2},1\right)$ and $\epsilon > 0$. As K is a compact set in $(X,\mu,\nu,*,\circ)$ and using $\alpha \in \left(\frac{1}{2},1\right)$ and $\epsilon > 0$ we have $\lambda > 0$ and $T_{\alpha,\epsilon} \in \bar{F}(X,X,\lambda)$ such that

$$||T_{\alpha,\epsilon}(x) - x||_\alpha < \epsilon \text{ for every } x \in K.$$

Then we have $||T_{\alpha,\epsilon}(x) - x|| < \epsilon$ and $||T_{\alpha,\epsilon}(x)|| \leqslant \lambda ||x||$, which is a contradiction as $(X, ||\cdot||)$ fails to have the BAP.

Hence $(X,\mu,\nu,*,\circ)$ has fails to have the BAP.

Finally, we have to show that $(X,\mu,\nu,*,\circ)$ has the AP. Let $\epsilon > 0$ and K be a compact subset in $(X,\mu,\nu,*,\circ)$. Again let $P_j : X \longrightarrow \left(\sum_{n=1}^{j} \oplus X_n\right)_{l_2}$ be the projection given by

$$P((x)) = (x_1, x_2, ..., x_j).$$

Since K is a compact set in X, therefore by Theorem 2.4 of (Choi *et al.*, 2009) there exists a natural number $m \in N$ and a finite rank operator $T' : (\sum_{n=1}^{m} \oplus X_n)_{l_2} \longrightarrow (\sum_{n=1}^{m} \oplus X_n)_{l_2}$ such that

$$||kT' P_m(x) - x|| < \epsilon$$

for every $x \in K$, where k is the map defined as $k : (\sum_{n=1}^{m} \oplus X_n)_{l_2} \longrightarrow X$ such that

$$k(x_1, x_2, ..., x_m) = (x_1, x_2, ..., x_m, 0, ...).$$

Now we put $T = kT' P_m$. As T is a finite rank operator defined as $T : X \longrightarrow X$ and $||x||_1 \leqslant ||x||$ for all $x \in X$, we have

$$||T(x) - x||_1 < \epsilon$$

that is, for every $\alpha \in (0,1)$, we have

$$||T(x) - x||_\alpha < \epsilon.$$

Next we have to show that T is sif-bounded on X. Since $(\sum_{n=1}^{m} \oplus X_n)_{l_2}$ and $(\sum_{n=1}^{m} \oplus X_n)_{l_\infty}$ are equivalent, there exists $M' > 1$ such that

$$(\sum_{n=1}^{m} ||x_n||_n^2)^{\frac{1}{2}} \leqslant M' sup_{1 \leqslant n \leqslant m} ||x_n||_n.$$

Then,

$$\begin{aligned}
\|T(x)\|_1 \leqslant ||T(x)|| &= \|kT' P_m(x)\| \\
&\leqslant ||kT'||(\sum_{n=1}^{m} \|x_n\|_n^2)^{\frac{1}{2}} \\
&\leqslant ||kT'||M' \sup_{1 \leqslant n \leqslant m} ||x_n||_n \\
&\leqslant ||kT'||M'||x||_1.
\end{aligned}$$

Taking $M = max\left\{||T||, ||kT'||, M'\right\}$, we have to show that

$$\mu\left(T\left(x\right),t\right) \geqslant \mu\left(x, \tfrac{t}{M}\right) \text{ and } \nu\left(T\left(x\right),t\right) \leqslant \nu\left(x, \tfrac{t}{M}\right)$$

for all $x \in X$ and $t \in \mathbb{R}$.

If $t \leqslant 0$, the result is trivial.

Assume that $t > 0$. Then it is enough to show that

$$\mu\left(T\left(x\right),t\right) \geqslant \mu\left(Mx,t\right) \text{ and } \nu\left(T\left(x\right),t\right) \leqslant \nu\left(Mx,t\right), \text{ for all } x \in X \text{ and } t \in \mathbb{R}.$$

Now first consider for μ:

For the first condition:

$$t > M||x||$$

then

$$\mu\left(Mx,t\right) = 1$$

By the assumption,

$$t > M||x|| \geqslant ||T||||x|| \geqslant ||T\left(x\right)||$$

we have

$$\mu\left(T\left(x\right),t\right) = 1.$$

Hence

$$\mu\left(T\left(x\right),t\right) \geqslant \mu\left(Mx,t\right).$$

For the second condition:

$$||Mx||_1 < t \leqslant ||Mx||$$

then

$$\mu\left(Mx,t\right) = \tfrac{1}{2}.$$

By the assumption

$$t > M||x||_1 \geqslant ||kT^{'}||M^{'}||x||_1 \geqslant ||T\left(x\right)||_1$$

we have

$$\mu\left(T\left(x\right),t\right) \geqslant \tfrac{1}{2}.$$

Hence

$$\mu\left(T\left(x\right),t\right) \geqslant \mu\left(Mx,t\right).$$

For the third condition :

$$t \leqslant M||x||$$

we have

$$\mu\left(Mx,t\right) = 0.$$

Then by the assumption trivially we obtain,

$$\mu\left(T\left(x\right),t\right) \geqslant \mu\left(Mx,t\right).$$

Next considering for ν :

For the first condition :

$$t > M||x||$$

then

$$\nu\left(Mx,t\right) = 0.$$

By the assumption,

$$t > M||x|| \geqslant ||T||||x|| \geqslant ||T\left(x\right)||$$

we have

$$\nu\left(T\left(x\right),t\right) = 0.$$

Hence

$$\nu\left(T\left(x\right),t\right) \leqslant \nu\left(Mx,t\right).$$

For the second condition:

$$||Mx||_1 < t \leqslant ||Mx||$$

then

$$\nu\left(Mx,t\right) = \tfrac{1}{2}$$

By the assumption

$$t > M||x||_1 \geqslant ||kT^{'}||M^{'}||x||_1 \geqslant ||T\left(x\right)||_1,$$

thus

$$\nu\left(T\left(x\right),t\right) \leqslant \tfrac{1}{2}$$

Hence

$$\nu\left(T\left(x\right),t\right) \leqslant \nu\left(Mx,t\right)$$

For the third condition :

$$t \leqslant M||x||_1.$$

Then

$$\nu\left(Mx,t\right) = 1.$$

By the assumption trivially we have,

$$\nu\left(T\left(x\right),t\right) \leqslant \nu\left(Mx,t\right).$$

Hence $(X,\mu,\nu,*,\circ)$ has the AP.

4. Conclusion

In this paper we introduced and investigated the concepts of AP and BAP in the context of an IFNLS. We have shown that there are IFNLSs which fail to have the AP and also there are IFNLSs with AP but not the BAP. The current results give us a better understanding of the analytical structure of an IFNLS.

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Co-Universal Algebraic Extension with Hidden Parameters

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Abstract

In the research of underlying algebraic structures of real world phenomena, we can find some behavior anomalies that depend on external parameters that are not ruled by their axiom systems. These are not visible straightaway and we have to deduce their existence from the effects they cause. To add them in mathematical constructions, we introduce co-universal extensions of algebras and co-algebras based upon the dual construction of the Kleisli category associated to a monad.

To illustrate this topic we introduce two applications. The first one is an artificial example. In the second application we analyze language algebraic structures with a method that states a bridge between language and logic blindly, that is to say, handling statements through their expressions in those languages satisfying some adequate conditions, and disregarding their meanings.

Keywords: Algebraic extensions, hidden parameters, algebraic language structures, co-monad, Kleisli categories, blind logic.
2010 MSC: 18C99, 18C20, 68T50, 03B65.

1. Introduction

When we investigate the mathematical structures of real world phenomena, we can observe some anomalies that depend on parameters that are not ruled by those axioms that define their algebraic structures. For instance, the states of a Turing machine, contexts when we interpret sentences in any language, environments, positions, etc. Recall that only tape symbols are the visible part of Turing machines. By contrast, moves and states are not displayed in their tapes. They work in the background as hidden parameters, but we can deduce their existence from the behavior changes they cause.

In positional notations, the meaning of each word or symbol depends on their position. For instance, consider the following sentences: 1) “*Programmers know how to write code fast;*” and 2) “*Programmers know how to write fast code.*” Both consist of the same words, but their meanings

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are order-dependent. We can consider both orders as hidden parameters, and the meanings of the former sentences depend on them. Accordingly, to define a map μ sending each sentence in the English language E into its meaning in M, we have to add a parameter set $\mathcal{H}$ the members of which are associated to orders, contexts and styles. Thus, the domain of μ must be the Cartesian product $E \times \mathcal{H}$; where E denotes the set of all English sentences.

We can also find hidden parameters in psychology, physics, and random phenomena. For instance, the probability of remembering a name increases with the occurrence frequency, or when some noticeable fact is associated to it. Thus, frequency and remarkable facts can work as hidden parameters that can modify probabilities. In section 4, we analyze an artificial example of this kind.

To learn and interpret any language, we have to handle abstractions and inferences between the definitions of sentence meanings (Tudor-Răzvan & Manolescu, 2011). The topic goes as follows. If two words, say W_1 and W_2, have the same meaning, when we swap them in any sentence, we obtain an equivalent one. We introduce language structure conditions to build the inverse method. Thus, we can find logical relations and abstractions between the meanings of W_1 and W_2 when we observe that some set of proper sentences T_1 becomes T_2 when we swap W_1 and W_2 and each member of T_2 is a proper sentence too. To know that T_2 consists of right sentences, we need not know their meanings. It is sufficient to find them in any scholar paper. The method works as a blind logic and can give rise to many ambiguities, that we can avoid deducing the existence of hidden parameters. This topic is an enlargement of what Newell stated in (Newell & Simon, 1976). We do not dive in this topic deeply, because we only expose these ideas to illustrate applications of co-universal algebraic extensions that we introduce.

The main aim of this article consists of introducing an algebraic device to enrich categories with sets of external (hidden) parameters that are not ruled by the axioms defining them. We term these constructions co-universal because are based upon co-monads together with the associated dual constructions of Kleisli categories. Well-known universal extensions of **Set**, associated to monads, are categories of sets with fuzzy subsets (Mawanda, 1988). These extensions of **Set** arise from an endofunctor that sends each ordinary set X into $X \times M$, where M is a monoid of truth-values. We introduce co-universal extensions by a similar endofunctor $X \mapsto X \times M$ such that M is the set of hidden parameters.

2. Preliminaries

To simplify expressions, we state some auxiliary definitions and notations. We write in bold face font those symbols denoting categories. In particular, **Set** denotes the category of ordinary sets and maps. We use the symbol ◂ as an end-of-definition marker.

Notation. For each couple of sets X and Y, we denote by $X^Y_{\geq n}$ the subset of X^Y defined as follows.

$$X^Y_{\geq n} = \{f \in X^Y \mid \#(\mathrm{img}(f)) \geq n\}.$$

For instance, $X^Y_{\geq 2}$ consists of every non-constant map in X^Y.

For each subcategory $\mathbf{C}$ of **Set** and every non-empty set $\mathcal{H}$, we denote the members of the set $(\mathrm{Hom}_{\mathbf{C}}(X, Y))^{\mathcal{H}}$ by symbols with the accent ˇ to indicate that are maps from an arbitrary set $\mathcal{H}$ into a homset. For each member $\breve{f}$ of $(\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}$ we write the values of the independent variable $\mathcal{H}$ as subscripts. Thus, the expression $\breve{f}_\alpha \in \mathrm{Hom}_{\mathbf{C}}(X,Y)$ denotes the image of $\alpha \in \mathcal{H}$ under $\breve{f}$.

Definition 2.1. Let $\mathbf{C}$ be a subcategory of **Set**. For every set $\mathcal{H}$ with cardinality greater than 1, we term $\mathcal{H}$-extension of $\mathbf{C}$ the category $\mathbf{C}[\mathcal{H}]$, with the same object class as $\mathbf{C}$, such that, for every couple of sets X and Y,

$$\mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(X,Y) = \mathrm{Hom}_{\mathbf{C}}(X,Y) \bigcup \left\{ \coprod_{\alpha\in\mathcal{H}} \{\breve{h}_\alpha\} \middle| \breve{h} \in (\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}_{\geq 2} \right\}. \tag{2.1}$$

Since $\mathbf{C}$ is a subcategory of $\mathbf{C}[\mathcal{H}]$, we only have to define those compositions involving morphisms in $\left\{\coprod_{\alpha\in\mathcal{H}}\{\breve{h}_\alpha\}\middle|\breve{h} \in (\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}_{\geq 2}\right\}$. We denote this composition by the infix symbol $\diamond$. For every couple of morphisms $f : X \to Y \in \mathrm{Hom}_{\mathbf{C}}(X,Y)$ and $\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\} \in \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(Y,Z)$ we define their composition as follows.

$$\left(\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\}\right) \diamond f = \coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha \circ f\} \tag{2.2}$$

Likewise, the composition of f and $\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\} \in \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(T,X)$ is

$$f \diamond \left(\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\}\right) = \coprod_{\alpha\in\mathcal{H}}\{f \circ \breve{g}_\alpha\} \tag{2.3}$$

Finally, we define the composition of two morphisms $\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\} \in \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(X,Y)$ and $\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\} \in \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(Y,Z)$ by

$$\left(\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\}\right) \diamond \left(\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}\right) = \coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha \circ \breve{f}_\alpha\}. \tag{2.4}$$

Since $\mathbf{C}$ is a subcategory of $\mathbf{C}[\mathcal{H}]$ with the same object class, identities are the same in both categories. ◁

Theorem 2.1. *Let* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *be two subcategories of* **Set**. *For every set* $\mathcal{H}$ *with cardinality greater than* 1, *and each functor* $T : \mathbf{C}_1 \to \mathbf{C}_2$, *the following statements hold.*

1) *There is an extension* $T^* : \mathbf{C}_1[\mathcal{H}] \to \mathbf{C}_2[\mathcal{H}]$ *of* T *with the same object-map.*

2) *If* $X_1 \xrightarrow{\sigma} T(X_2)$ *is a* T*-universal arrow, then* $X_1 \xrightarrow{\sigma} T^*(X_2)$ *is a* T^**-universal one.*

3) *If for every* $\alpha \in \mathcal{H}$, $X_1 \xrightarrow{\breve{\sigma}_\alpha} T(X_2)$ *is a* T*-universal arrow, then*

$$X_1 \xrightarrow{\coprod_{\alpha\in\mathcal{H}}\{\breve{\sigma}_\alpha\}} T^*(X_2)$$

is a T^**-universal one.*

Proof.

1) We define the extension T^* of T in the following terms. The object-maps of both T and T^* are the same. Recall that, by definition, $\mathrm{Obj}(\mathbf{C}_1) = \mathrm{Obj}(\mathbf{C}_1[\mathcal{H}])$. The images $T(f)$ and $T^*(f)$ of every morphism $f \in \mathrm{Mor}(\mathbf{C}_1)$ are the same. The image of each morphism $\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\} \in \mathrm{Mor}(\mathbf{C}_1[\mathcal{H}]) \setminus \mathrm{Mor}(\mathbf{C}_1)$ is given by

$$T^*\Big(\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}\Big) = \coprod_{\alpha\in\mathcal{H}}\{T(\breve{f}_\alpha)\} \tag{2.5}$$

The former definition is possible because, by equation (2.1), $\breve{f}_\alpha$ belongs to $\mathrm{Mor}(\mathbf{C}_1)$, for every $\alpha \in \mathcal{H}$.

It remains to be shown that T^* preserves morphism composition and identities. Since the restriction of T^* to $\mathrm{Mor}(\mathbf{C}_1)$ coincides with T, the extension T^* preserves identities and compositions between members of $\mathbf{C}_1$. We only have to show that T^* preserves morphism compositions involving some members of $\mathrm{Mor}(\mathbf{C}_1[\mathcal{H}]) \setminus \mathrm{Mor}(\mathbf{C}_1)$. For compositions like (2.2), taking into account (2.5),

$$\begin{aligned} T^*\Big(\coprod_{\alpha\in\mathcal{H}}\{(\breve{g}_\alpha \circ f)\}\Big) &= \coprod_{\alpha\in\mathcal{H}}\{T(\breve{g}_\alpha \circ f)\} = \coprod_{\alpha\in\mathcal{H}}\{T(\breve{g}_\alpha) \circ T(f)\} = \\ &\Big(\coprod_{\alpha\in\mathcal{H}}\{T(\breve{g}_\alpha)\}\Big) \diamond T(f) = T^*\Big(\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\}\Big) \diamond T^*(f) \end{aligned} \tag{2.6}$$

The proofs for compositions of the form (2.3) and (2.4) go as in the preceding case.

2) We have to show that, for every object Y and every morphism $f : X_1 \to T^*(Y)$ there is a unique $f^* : X_2 \to Y$ such that the following triangle commutes.

$$\begin{array}{ccc} X & \xrightarrow{\ \sigma\ } & T^*(X_2) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle T^*(f^*)} \\ & & T^*(Y) \end{array} \tag{2.7}$$

If $f \in \mathrm{Mor}(\mathbf{C}_1)$, by hypothesis, this condition must be satisfied. Now, suppose that $f = \coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}$. Since for every α, the morphism $\breve{f}_\alpha : X \to T^*(Y)$ belongs to $\mathrm{Mor}(\mathbf{C}_1)$, there is a unique $\breve{f}^*_\alpha : X_2 \to T^*(Y) = T(Y)$ such that the following diagram commutes.

$$\begin{array}{ccc} X_1 & \xrightarrow{\ \sigma\ } & T^*(X_2) = T(X_2) \\ & {\scriptstyle \breve{f}_\alpha}\searrow & \downarrow{\scriptstyle T^*(\breve{f}^*) = T(\breve{f}_\alpha)} \\ & & T^*(Y) = T(Y) \end{array} \tag{2.8}$$

By virtue of (2.2) the following triangle is also commutative

$$\begin{array}{ccc} X_1 & \xrightarrow{\ \sigma\ } & T^*(X_2) \\ & {\scriptstyle \coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}}\searrow & \downarrow{\scriptstyle T^*\left(\coprod_{\alpha\in\mathcal{H}}\{\breve{f}^*\}\right)} \\ & & T^*(Y)) \end{array} \tag{2.9}$$

3) Let $X_1 \xrightarrow{\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}} T^*(Y)$ be a $\mathbf{C}_1$-morphism. By assumption, for every $\alpha \in \mathcal{H}$, there is a $\mathbf{C}_1$-morphism $\breve{f}^*_\alpha$ such that the following diagram commutes.

$$\begin{array}{ccc} X_1 & \xrightarrow{\breve{\sigma}_\alpha} & T(X_2) \\ & \searrow^{\breve{f}_\alpha} & \downarrow{\scriptstyle T(\breve{f}_\alpha)} \\ & & T(Y) \end{array} \tag{2.10}$$

hence, the following triangle is also commutative.

$$\begin{array}{ccc} X_1 & \xrightarrow{\breve{\sigma}_\alpha} & T(X_2) \\ & \searrow^{\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}} & \downarrow{\scriptstyle T\left(\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}\right)} \\ & & T(Y) \end{array} \tag{2.11}$$

The uniqueness of $\coprod_{\alpha\in\mathcal{H}}\{\breve{f}^*_\alpha\}$ is a consequence of being unique each $\breve{f}^*_\alpha$ that satisfies the commutativity of (2.10), for every $\alpha \in \mathcal{H}$.

□

3. Co-universal algebraic extensions with hidden parameters

For every subcategory $\mathbf{C}$ of **Set**, being stable under Cartesian products, and each non-empty set $\mathcal{H}$ in Obj($\mathbf{C}$), we denote by $\mathcal{H}^\dagger : \mathbf{C} \to \mathbf{C}$ the functor sending each set $X \in \mathrm{Obj}(\mathbf{C})$ into $X \times \mathcal{H}$, and every map $f : X \to Y$ into

$$\mathcal{H}^\dagger(f) = f \times \mathrm{id}_{\mathcal{H}} : X \times \mathcal{H} \to Y \times \mathcal{H}. \tag{3.1}$$

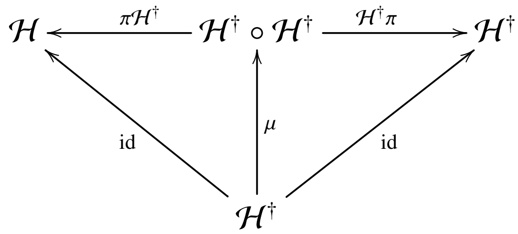
Notation. For every endofunctor $\mathcal{H}^\dagger : \mathbf{C} \to \mathbf{C}$, we denote by π the natural transformation $\mathcal{H}^\dagger \xrightarrow{\pi} \mathrm{Id}$ such that, for each set X, the map $\pi_X : X \times \mathcal{H} \to X$ is the canonical projection; where $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ denotes the identity endofunctor. Likewise, $\mathcal{H}^\dagger \xrightarrow{\mu} \mathcal{H}^\dagger \circ \mathcal{H}^\dagger$ is the natural transformation

$$\mu_X =: X \times \mathcal{H} \to X \times \mathcal{H} \times \mathcal{H} \tag{3.2}$$

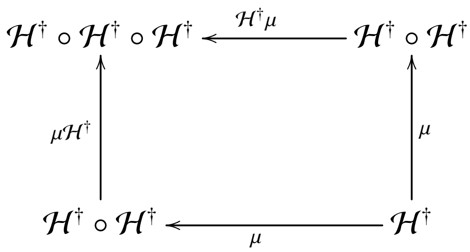
that sends each $(x, v) \in X \times \mathcal{H}$ into $(x, v, v) \in X \times \mathcal{H} \times \mathcal{H}$.

Proposition 3.1. *Let* $\mathbf{C}$ *be a subcategory of* **Set** *being stable under Cartesian products. For every nonempty set* $\mathcal{H} \in \mathrm{Obj}(\mathbf{C})$*, the endofunctor* $\mathcal{H}^\dagger : \mathbf{C} \to \mathbf{C}$ *together with both natural transformations* π *and* μ *form a comonad* $(\mathcal{H}^\dagger, \pi, \mu)$*.*

Proof. We show that the following diagrams commute.



(3.3)



(3.4)

Let X be a set and (x, v) any member of $\mathcal{H}^\dagger(X) = X \times \mathcal{H}$. By straightforward computations we obtain

$$\pi_{X\times\mathcal{H}}(\mu_X(x, v)) = \pi_{X\times\mathcal{H}}(x, v, v) = (x, v);$$

accordingly, $\pi\mathcal{H}^\dagger \circ \mu = \text{id}$. The proofs for the right triangle and quadrangle (3.4) are similar. □

Definition 3.1. Let **C** be a subcategory of **Set** being stable under Cartesian products. For each set $\mathcal{H} \in \text{Obj}(\mathbf{C})$ with cardinality greater than 1, a co-universal $\mathcal{H}$-extension of **C** with *hidden parameters* is the category $\mathbf{C}_\mathcal{H}$ defined as follows.

1. The object-classes of both $\mathbf{C}_\mathcal{H}$ and **C** are the same.
2. For each couple of objects X and Y, the set $\text{Hom}_{\mathbf{C}_\mathcal{H}}(X, Y)$ consists of all maps from $\mathcal{H}^\dagger(X) = X \times \mathcal{H}$ into Y such that there is $\breve{f} \in (\text{Hom}_\mathbf{C}(X, Y))^\mathcal{H}$ that satisfies the relation
$$\forall \alpha \in \mathcal{H} : \; f(x, \alpha) = \breve{f}_\alpha(x).$$
3. The composition $f \star g$ of two $\mathbf{C}_\mathcal{H}$-morphisms $g \in \text{Hom}_{\mathbf{C}_\mathcal{H}}(X, Y)$ and $f \in \text{Hom}_{\mathbf{C}_\mathcal{H}}(Y, Z)$ is given by
$$f \star g = f \circ \mathcal{H}^\dagger(g) \circ \mu_X \tag{3.5}$$
4. The identity associated to each $\mathbf{C}_\mathcal{H}$-object X is the projection
$$\pi_X : \mathcal{H}^\dagger(X) = X \times \mathcal{H} \to X.$$

◁

Notation. As in the preceding definition, for every co-universal $\mathcal{H}$-extension $\mathbf{C}_\mathcal{H}$ of a subcategory **C** of **Set**, we denote the morphism composition by the infix symbol $\star$.

Definition 3.2. Let **C** be a subcategory of **Set** such that there is the co-universal $\mathcal{H}$-extension $\mathbf{C}_\mathcal{H}$. We say a $\mathbf{C}_\mathcal{H}$-morphism $f : X \times \mathcal{H} \to Y$ to be π-factorizable whenever there is $f^* \in \text{Hom}_\mathbf{C}(X, Y)$ that satisfies the equation $f = f^* \circ \pi_X$. ◁

Lemma 3.1. *Let* **C** *be a subcategory of* **Set***, being stable under Cartesian products. For every set* $\mathcal{H} \in \mathrm{Obj}(\mathbf{C})$ *and each* $\mathbf{C}_{\mathcal{H}}$*-object* X*, the associated* $\mathbf{C}_{\mathcal{H}}$*-identity* π_X *is* π*-factorizable. In addition,* $\pi_X^* = \mathrm{id}_X$.

Proof. Setting $\pi_X^* = \mathrm{id}_X$, the relation $\pi_X = \mathrm{id}_X \circ \pi_X$ leads to $\pi_X = \pi_X^* \circ \pi_X$. □

Lemma 3.2. *A* $\mathbf{C}_{\mathcal{H}}$*-morphism* $f(x, \alpha) = \check{f}_\alpha(x)$ *is* π*-factorizable if and only if* $\check{f} \in \mathrm{Hom}_{\mathbf{C}}(X, Y)^{\mathcal{H}}$ *is constant.*

Proof. Assume $\check{f}$ to be a constant map, therefore the value of $f(x, \alpha)$ depends only on x. Thus, setting $f^*(x) = \check{f}_\alpha(x)$, for every $(x, \alpha) \in X \times \mathcal{H}$, the relation $f = f^* \circ \pi_X$ holds. The proof for the converse implication is similar. □

Lemma 3.3. *The composition of* π*-factorizable morphisms is again* π*-factorizable.*

Proof. Let $f \circ \pi_X : X \times \mathcal{H} \to Y$ and $g \circ \pi_Y : Y \times \mathcal{H} \to Z$ be two π-factorizable morphisms. According to (3.5)

$$(g \circ \pi_Y) \star (f \circ \pi_X) = (g \circ \pi_Y) \circ \mathcal{H}^\dagger(f \circ \pi_X) \circ \mu_X = \\ (g \circ \pi_Y) \circ ((f \circ \pi_X) \times \mathrm{id}_{\mathcal{H}}) \circ \mu_X = g \circ f \circ \pi_X \quad (3.6)$$

therefore $(g \circ \pi_Y) \star (f \circ \pi_X) = (g \circ f) \circ \pi_X$ is π-factorizable. □

Definition 3.3. Let **C** be any subcategory of **Set**, being stable under Cartesian products. For each set $\mathcal{H}$ with cardinality greater than 1, and each $\alpha \in \mathcal{H}$, we define the map $\Gamma_{\alpha,A} : \mathrm{Mor}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Mor}(\mathbf{C})$ as follows. For every couple $\mathbf{C}_{\mathcal{H}}$-objects X and Y, and each $f \in \mathrm{Hom}_{\mathbf{C}_{\mathcal{H}}}(X, Y)$:

$$\Gamma_{\alpha,A}(f) = \begin{cases} f^* \text{ if } f = f^* \circ \pi_X \text{ is } \pi\text{-factorizable} \\ \check{f}_\alpha \text{ where } \check{f} \in (\mathrm{Hom}_{\mathbf{C}}(X, Y))^{\mathcal{H}}_{\geq 2} \text{ otherwise.} \end{cases} \quad (3.7)$$

being $\check{f}$ the map such that $\forall (x, \alpha) \in X \times \mathcal{H}$: $\check{f}_\alpha(x) = f(x, \alpha)$. ◁

To agree with Lemma 3.2, in the former definition, when $f = f^* \circ \pi_X$ is π-factorizable, its image $\Gamma_{\alpha,A}(f)$ does not depend on the parameter α.

Proposition 3.2. *Let* $\mathbf{C}_{\mathcal{H}}$ *be a co-universal* $\mathcal{H}$*-extension of a subcategory* **C** *of* **Set** *with hidden parameters. For every* $\alpha \in \mathcal{H}$*, the map* $\Gamma_{\alpha,A} : \mathrm{Mor}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Mor}(\mathbf{C})$ *preserves identities and morphism compositions.*

Proof. We have to show that, for every couple of morphisms $f : X \times \mathcal{H} \to Y$ and $g : Y \times \mathcal{H} \to Z$, and each $\alpha \in \mathcal{H}$, the map $\Gamma_{\alpha,A}$ satisfies the following relation.

$$\Gamma_{\alpha,A}(g \star f) = \Gamma_{\alpha,A}(g) \circ \Gamma_{\alpha,A}(f) \quad (3.8)$$

If both $f = f^* \circ \pi_X$ and $g = g^* \circ \pi_Y$ are π-factorizable, then

$$\begin{aligned}\Gamma_{\alpha,A}(g \star f) = \Gamma_{\alpha,A}((g^* \circ \pi_Y) \star (f^* \circ \pi_X)) = \\ \Gamma_{\alpha,A}((g^* \circ \pi_Y) \circ \mathcal{H}^\dagger(f^* \circ \pi_X) \circ \mu_X) = \\ \Gamma_{\alpha,A}((g^* \circ \pi_Y) \circ ((f^* \circ \pi_X) \times \mathrm{id}_{\mathcal{H}}) \circ \mu_X) = \\ \Gamma_{\alpha,A}(g^* \circ f^* \circ \pi_X)) = g^* \circ f^* = \Gamma_{\alpha,A}(g) \circ \Gamma_{\alpha,A}(f)\end{aligned} \quad (3.9)$$

Thus, $\Gamma_{\alpha,A}$ preserves the composition of π-factorizable morphisms.

For non-π-factorizable morphisms, the expression $(g \star f)(x,\alpha)$ can be written explicitly as follows.

$$\begin{aligned}\forall (x,\alpha) \in X \times \mathcal{H} : \quad (g \star f)(x,\alpha) = \\ (g \circ \mathcal{H}^\dagger(f) \circ \mu_X)(x,\alpha) = (g \circ (f \times \mathrm{id}_{\mathcal{H}}) \circ \mu_X)(x,\alpha) = \\ (g(f(x,\alpha),\alpha) = \breve{g}_\alpha(\breve{f}_\alpha(x)) = (\breve{g}_\alpha \circ \breve{f}_\alpha)(x);\end{aligned} \quad (3.10)$$

and by definition, $\Gamma_{\alpha,A}(f) = \breve{f}_\alpha$ and $\Gamma_{\alpha,A}(g) = \breve{g}_\alpha$; therefore

$$\Gamma_{\alpha,A}(g \star f) = (g \star f)\breve{}_\alpha = \breve{g}_\alpha \circ \breve{f}_\alpha = \Gamma_{\alpha,A}(g) \circ \Gamma_{\alpha,A}(f); \quad (3.11)$$

hence $\Gamma_{\alpha,A}$ also preserves the composition of non-π-factorizable morphisms.

If $g = g^* \circ \pi_Y$ is π-factorizable and f is not, the same procedure yields

$$\forall (x,\alpha) \in X \times \mathcal{H} : \quad (g \star f)(x,\alpha) = (g(f(x,\alpha),\alpha) = g^*(\breve{f}_\alpha(x)); \quad (3.12)$$

and this equation leads to (3.11). The proof when f is π-factorizable and g is not, is similar.

It remains to be shown that $\Gamma_{\alpha,A}$ preserves identities. According to Lemma 3.1 and equation (3.7), $\Gamma_{\alpha,A}(\pi_X) = \mathrm{id}_X$. □

Corollary 3.1. *With the same assumptions as in Proposition 3.2, for every fixed $\alpha \in \mathcal{H}$, the identity* $\mathrm{Id} : \mathrm{Obj}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Obj}(\mathbf{C})$ *and the map* $\Gamma_{\alpha,A} : \mathrm{Mor}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Mor}(\mathbf{C})$ *form a functor* $\Gamma_\alpha = (\mathrm{Id}, \Gamma_{\alpha,A})$.

Proof. By definition, the object classes of $\mathbf{C}_{\mathcal{H}}$ and $\mathbf{C}$ are the same; hence the identity can be the object map of Γ_α. By Proposition 3.2 the map Γ_α, A preserves identities and morphism composition. □

Notation. For every subcategory $\mathbf{C}$ of **Set** being stable under Cartesian products, and each set $\mathcal{H} \in \mathrm{Obj}(\mathbf{C})$ with cardinality greater than 1, the expression

$$F_{\mathcal{H},A} : \mathrm{Mor}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Mor}(\mathbf{C}[\mathcal{H}])$$

denotes the map such that, for each pair X and Y in $\mathrm{Obj}(\mathbf{C}_{\mathcal{H}})$ and every $f \in \mathrm{Hom}_{\mathbf{C}_{\mathcal{H}}}(X,Y)$:

$$F_{\mathcal{H},A}(f) = \begin{cases}\Gamma_\alpha(f) \text{ if } f \text{ is } \pi\text{-factorizable} \\ \coprod_{\alpha \in \mathcal{H}}\{\Gamma_\alpha(f)\} = \coprod_{\alpha \in \mathcal{H}}\{\breve{f}_\alpha\} \text{ otherwise.}\end{cases} \quad (3.13)$$

where Γ_α is the functor defined in Corollary 3.1.

Theorem 3.1 (Main)**.** *For every subcategory* **C** *of* **Set** *being stable under Cartesian products, and each set* $\mathcal{H} \in \mathrm{Obj}(\mathbf{C})$ *with cardinality greater than* 1*, the following statements hold.*

1) *The identity* $\mathrm{Id} : \mathrm{Obj}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Obj}(\mathbf{C}[\mathcal{H}])$ *together with the arrow map*

$$F_{\mathcal{H},A} : \mathrm{Mor}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Mor}(\mathbf{C}[\mathcal{H}])$$

form an isomorphism $F_{\mathcal{H}} = (\mathrm{Id}, F_{\mathcal{H},A})$ *between both categories* $\mathbf{C}_{\mathcal{H}}$ *and* $\mathbf{C}[\mathcal{H}]$.

2) *If* **D** *is a subcategory of* **Set***, being stable under Cartesian products such that* $\mathcal{H}$ *belongs to* $\mathrm{Obj}(\mathbf{D})$*, then every functor* $T : \mathbf{C} \to \mathbf{D}$ *gives rise to another one* $T^{\natural}_{\mathcal{H}} : \mathbf{C}_{\mathcal{H}} \to \mathbf{D}_{\mathcal{H}}$*, having the same object map as* T*, which satisfies the following relation.*

$$\forall f \in \mathrm{Mor}(\mathbf{C}_{\mathcal{H}}) : \quad \Gamma_\alpha \circ T^{\natural}_{\mathcal{H}}(f) = T \circ \Gamma_\alpha(f) \tag{3.14}$$

3) *With the same conditions as in the preceding statement, if for every* $\alpha \in \mathcal{H}$*, the* **C***-morphism* $X_1 \xrightarrow{\breve{\sigma}_\alpha} T(X_2)$ *is a* T*-universal arrow, then* $X_1 \xrightarrow{\sigma} T^{\natural}_{\mathcal{H}}(X_2)$ *is a* $T^{\natural}_{\mathcal{H}}$*-universal one; where* σ *denotes the* $\mathbf{C}_{\mathcal{H}}$*-morphism* $\sigma : X_1 \times \mathcal{H} \to T^{\natural}_{\mathcal{H}}(X_2)$ *such that,* $\forall (x, \alpha) \in X_1 \times \mathcal{H}$*:* $\sigma(x, \alpha) = \breve{\sigma}_\alpha(x)$.

4) *With the same assumptions as in Statement 2), every* $T^{\natural}$*-algebra (co-algebra) is the extension with hidden parameters of an ordinary* $T^{\natural}_{\mathcal{H}}$*-algebra (co-algebra).*

Proof.

1) We have to show that $F_{\mathcal{H}}$ is a functor. For every object X, the $\mathbf{C}_{\mathcal{H}}$-identity is $\pi_X : X \times \mathcal{H} \to X$. According to Proposition 3.2, its image under $F_{\mathcal{H}}$ is $\Gamma_\alpha(\pi_X) = \mathrm{id}_X$. Thus, $F_{\mathcal{H}}$ preserves identities.

To show that $F_{\mathcal{H}}$ preserves morphism composition, let $f = f^* \circ \pi_X : X \times \mathcal{H} \to Y$ and $g = g^* \circ \pi_Y : Y \times \mathcal{H} \to Z$ be two π-factorizable morphisms. By equation (3.13) and taking into account Lemma 3.3,

$$\begin{aligned} F_{\mathcal{H}}(g \star f) = \Gamma_\alpha(g \star f) = \\ \Gamma_\alpha(g^* \circ \pi_Y \circ \mathcal{H}^{\dagger}(f^* \circ \pi_X) \circ \mu_X) = \Gamma_\alpha(g^* \circ f^* \circ \pi_X) = \\ g^* \circ f^* = \Gamma_\alpha(g) \circ \Gamma_\alpha(f) = \Gamma_\alpha(g) \diamond \Gamma_\alpha(f); \end{aligned} \tag{3.15}$$

therefore

$$F_{\mathcal{H}}(g \star f) = \Gamma_\alpha(g \star f) = \Gamma_\alpha(g) \diamond \Gamma_\alpha(f) = F_{\mathcal{H}}(g) \diamond F_{\mathcal{H}}(f). \tag{3.16}$$

If f and g are two non-π-factorizable morphisms, by definition,

$$\begin{aligned} (g \star f)(x, \alpha) = (g \circ \mathcal{H}^{\dagger}(f) \circ \mu_X)(x, \alpha) = \\ (g \circ (f \times \mathrm{id}_{\mathcal{H}}) \circ \mu_X)(x, \alpha) = g(f(x, \alpha), \alpha) \end{aligned} \tag{3.17}$$

Let $\breve{f} \in (\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}$ and $\breve{g} \in (\mathrm{Hom}_{\mathbf{C}}(Y,Z))^{\mathcal{H}}$ be the maps such that

$$\begin{cases} \forall (x,\alpha) \in X \times \mathcal{H}: & \breve{f}_\alpha(x) = f(x,\alpha) \\ \forall (y,\alpha) \in Y \times \mathcal{H}: & \breve{g}_\alpha(x) = g(y,\alpha). \end{cases} \tag{3.18}$$

These relations together with (3.17) lead to

$$\forall x \in X : (g \star f)(x) = g((f(x,\alpha),\alpha) = \breve{g}_\alpha(\breve{f}_\alpha(x)) = (\breve{g}_\alpha \circ \breve{f}_\alpha)(x), \tag{3.19}$$

for every fixed $\alpha \in \mathcal{H}$. Consequently, by virtue of (2.4) and (3.13),

$$F_{\mathcal{H}}(g \star f) = \coprod_{\alpha\in\mathcal{H}} \{\breve{g}_\alpha \circ \breve{f}_\alpha\} = \Big(\coprod_{\alpha\in\mathcal{H}} \{\breve{g}_\alpha\}\Big) \diamond \Big(\coprod_{\alpha\in\mathcal{H}} \{\breve{f}_\alpha\}\Big) = \\ \Big(\coprod_{\alpha\in\mathcal{H}} \{\Gamma_\alpha(g)\}\Big) \diamond \Big(\coprod_{\alpha\in\mathcal{H}} \{\Gamma_\alpha(f)\}\Big) = F_{\mathcal{H}}(g) \diamond F_{\mathcal{H}}(f). \tag{3.20}$$

If $g = g^* \circ \pi_Y$ is π-factorizable and f is not, the same procedure yields,

$$F_{\mathcal{H}}(g \star f) = \coprod_{\alpha\in\mathcal{H}} \{g^* \circ \breve{f}_\alpha\} = g^* \diamond \Big(\coprod_{\alpha\in\mathcal{H}} \{\breve{f}_\alpha\}\Big) = \\ \Gamma_\alpha(g) \diamond \Big(\coprod_{\alpha\in\mathcal{H}} \{\Gamma_\alpha(f)\}\Big) = F_{\mathcal{H}}(g) \diamond F_{\mathcal{H}}(f). \tag{3.21}$$

By the same method, we can build the proof when f is π-factorizable and g is not.

Since $F_{\mathcal{H}}$ preserves identities and morphism composition, it is a functor.

To be an isomorphism, $F_{\mathcal{H}} : \mathbf{C}_{\mathcal{H}} \to \mathbf{C}[\mathcal{H}]$ must be full, faithful, and bijective on objects. By definition, the object-classes of $\mathbf{C}$, $\mathbf{C}_{\mathcal{H}}$, and $\mathbf{C}[\mathcal{H}]$ are the same. Because the object map Id of $F_{\mathcal{H}}$ is the identity, $F_{\mathcal{H}}$ is bijective on objects.

It remains to be shown that $F_{\mathcal{H}}$ is full and faithful. The class $\mathrm{Mor}(\mathbf{C}[\mathcal{H}])$ consists of the ordinary maps in $\mathrm{Mor}(\mathbf{C})$ together with the coproduct class

$$\mathrm{Cprd}(\mathbf{C},\mathcal{H}) = \Big\{ \coprod_{\alpha\in\mathcal{H}} \{\breve{h}_\alpha\} \Big| \breve{h} \in (\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}_{\geq 2}) \wedge (X,Y) \in \mathrm{Obj}(\mathbf{C}) \times \mathrm{Obj}(\mathbf{C}) \Big\}$$

For every map $f : X \to Y$ in $\mathrm{Mor}(\mathbf{C})$ there is the preimage $F^{-1}_{\mathcal{H}}(f) = f \circ \pi_X$, because $F_{\mathcal{H}}(f \circ \pi_X) = \Gamma_\alpha(f \circ \pi_x) = f$. Likewise, for each $\mathbf{C}[\mathcal{H}]$-morphism $\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}$ lying in $\mathrm{Cprd}(\mathbf{C},\mathcal{H}) \subseteq \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(X,Y)$, the preimage is the morphism $f : X \times \mathcal{H} \to Y$ that satisfies the relation $f(x,\alpha) = \breve{f}_\alpha(x)$, for each fixed $\alpha \in \mathcal{H}$ and every $x \in X$; hence $F_{\mathcal{H}}$ is full.

To see that $F_{\mathcal{H}}$ is faithful, we split the class $\mathrm{Mor}(\mathbf{C}_{\mathcal{H}})$ into the subclass $\mathbf{C}_{\mathcal{H},\pi}$ of π-factorizable morphisms and its complement $\complement_{\mathrm{Mor}(\mathbf{C}_{\mathcal{H}})}\mathbf{C}_{\mathcal{H},\pi}$. If the images of two π-factorizable morphisms $f : X \times \mathcal{H} \to Y$ and $f : X \times \mathcal{H} \to Y$ are the same, then $\Gamma_\alpha(f) = \Gamma_\alpha(g)$; so then $f = \Gamma_\alpha(f) \circ \pi_X = \Gamma_\alpha(g) \circ \pi_X = g$.

Since the image of every π-factorizable morphisms belongs to $\mathbf{C}$, we only have to show that, the restriction of $F_{\mathcal{H}}$ to each homset in $\complement_{\mathrm{Mor}(\mathbf{C}_{\mathcal{H}})}\mathbf{C}_{\mathcal{H},\pi}$ is also injective. Let $f : X \times \mathcal{H} \to Y$ and $g : X \times \mathcal{H} \to Y$ be two morphisms with the same image $\coprod_{\alpha\in\mathcal{H}}\{\breve{h}_\alpha\}$. By definition, for every $(x, \alpha) \in X \times \mathcal{H}$: $f(x, \alpha) = \Gamma_\alpha(f)(x) = \breve{h}_\alpha(x) = \Gamma_\alpha(g)(x) = g(x, \alpha)$; therefore $f = g$. Finally, the image under $F_{\mathcal{H}}$ of each π-factorizable morphism f belongs to $\mathrm{Mor}(\mathbf{C})$, while the image of every non-π-factorizable one g lies in $\mathrm{Cprd}(\mathbf{C}, \mathcal{H})$. Since both sets are disjoint, $F_{\mathcal{H}}(f) \neq F_{\mathcal{H}}(g)$.

2) According to the preceding statement, there is the isomorphism $F_{\mathcal{H}} : \mathbf{C}_{\mathcal{H}} \to \mathbf{C}[\mathcal{H}]$; hence we can define $T^{\natural}_{\mathcal{H}}$ by

$$T^{\natural}_{\mathcal{H}} = F^{-1}_{\mathcal{H}} \circ T^{*} \circ F_{\mathcal{H}} \tag{3.22}$$

where $T^{*} : \mathbf{C}[\mathcal{H}] \to \mathbf{D}[\mathcal{H}]$ is the extension of T defined in Theorem 2.1. Taking into account (3.13), every π-factorizable morphism $f \in \mathrm{Mor}(\mathbf{C}_{\mathcal{H}})$ satisfies the equation,

$$\Gamma_\alpha \circ T^{\natural}_{\mathcal{H}}(f) = \Gamma_\alpha \circ F^{-1}_{\mathcal{H}} \circ T^{*} \circ F_{\mathcal{H}}(f) = T^{*} \circ \Gamma_\alpha(f) \tag{3.23}$$

because $\Gamma_\alpha = F_{\mathcal{H}}$. Since f is π-factorizable, $\Gamma_\alpha(f) \in \mathrm{Mor}(\mathbf{C})$, hence $T^{*} \circ \Gamma_\alpha(f) = T \circ \Gamma_\alpha(f)$. Thus, the former equation leads to

$$\Gamma_\alpha \circ T^{\natural}_{\mathcal{H}}(f) = T \circ \Gamma_\alpha(f) \tag{3.24}$$

For each non-π-factorizable morphism $f : X \times \mathcal{H} \to Y$,

$$\Gamma_\alpha \circ T^{\natural}_{\mathcal{H}}(f) = \Gamma_\alpha \circ F^{-1}_{\mathcal{H}} \circ T^{*} \circ F_{\mathcal{H}}(f) = \Gamma_\alpha \circ F^{-1}_{\mathcal{H}} \circ T^{*}\Big(\coprod_{\beta\in\mathcal{H}}\{\breve{f}_\beta\}\Big) = \Gamma_\alpha \circ F^{-1}_{\mathcal{H}}\Bigg(\coprod_{\beta\in\mathcal{H}}\{T(\breve{f}_\beta)\}\Bigg) = \Gamma_\alpha(h) \tag{3.25}$$

where $h : T(X) \times \mathcal{H} \to T(Y)$ is the map defined by

$$\forall (x, \alpha) \in T(X) \times \mathcal{H} : \quad h(x, \alpha) = T(\breve{f}_\alpha)(x).$$

Thus, $\Gamma_\alpha(h) = \breve{h}_\alpha = T(\breve{f}_\alpha) = T(\Gamma_\alpha(f))$. This relation and equation (3.25) lead to equation (3.14).

3) The image of σ under $F_{\mathcal{H}}$ is $\coprod_{\alpha\in\mathcal{H}}\{\sigma_\alpha\}$. Since $T^{\natural}_{\mathcal{H}} = F^{-1}_{\mathcal{H}} \circ T^{*} \circ F_{\mathcal{H}}$ and $F_{\mathcal{H}}$ is a category isomorphism, statement **3)** is a consequence of Theorem 2.1.

4) If (X, σ_X) is a $T^{\natural}_{\mathcal{H}}$-algebra, for every $\alpha \in \mathcal{H}$, its image $\Gamma_\alpha(X, \sigma_X) = (\Gamma_\alpha(X), \Gamma_\alpha(\sigma_X))$ under Γ_α is a T-algebra. By definition, every set $X \in \mathrm{Obj}(\mathbf{C})$ remains unaltered under Γ_α. Accordingly, $(\Gamma_\alpha(X), \Gamma_\alpha(\sigma_X)) = (X, \Gamma_\alpha(\sigma_X))$. In addition, although $\sigma_X : T^{\natural}_{\mathcal{H}}(X) \times \mathcal{H} \to X$ is a $\mathbf{C}_{\mathcal{H}}$-morphism, its image under Γ_α is an ordinary map. According to statement 2), and taking into account (3.7),

$$\Gamma_\alpha\left(T^{\natural}_{\mathcal{H}}(X) \xrightarrow{\sigma_X} X\right) = \Gamma_\alpha \circ T^{\natural}_{\mathcal{H}}(X) \xrightarrow{\Gamma_\alpha(\sigma_X)} \Gamma_\alpha(X) =$$
$$T \circ \Gamma_\alpha(X) \xrightarrow{\Gamma_\alpha(\sigma_X)} \Gamma_\alpha(X) = T(X) \xrightarrow{\Gamma_\alpha(\sigma_X)} X \quad (3.26)$$

therefore, $(X, \Gamma_\alpha(\sigma_X))$ is a T-algebra, where $\Gamma_\alpha(\sigma_X)$ is either the image of α under the map $\breve{\sigma}_X \in (\mathrm{Hom}_{\mathbf{C}}(T(X), X))^{\mathcal{H}}$ whenever σ_X is not π-factorizable, or the map σ^*_X such that $\sigma_X = \sigma^*_X \circ \pi_X$ otherwise. Likewise, if $f : (X, \sigma_X) \to (Y, \sigma_Y)$ is a morphism between two $T^{\natural}_{\mathcal{H}}$-algebras, the following quadrangle commutes.

$$\begin{array}{ccc} T^{\natural}_{\mathcal{H}}(X) & \xrightarrow{\sigma_X} & X \\ {\scriptstyle T^{\natural}_{\mathcal{H}}(f)}\downarrow & & \downarrow{\scriptstyle f} \\ T^{\natural}_{\mathcal{H}}(Y) & \xrightarrow[\sigma_Y]{} & Y \end{array} \quad (3.27)$$

Consequently, taking into account Statement 2), its image under Γ_α

$$\begin{array}{ccc} T(X) & \xrightarrow{\Gamma_\alpha(\sigma_X)} & X \\ {\scriptstyle T(\Gamma_\alpha(f))}\downarrow & & \downarrow{\scriptstyle \Gamma_\alpha(f)} \\ T(Y) & \xrightarrow[\Gamma_\alpha(\sigma_Y)]{} & Y \end{array} \quad (3.28)$$

is also commutative, and both $(X, \Gamma_\alpha(\sigma_X))$ and $(Y, \Gamma_\alpha(\sigma_Y))$ are ordinary T-algebras. The proof for co-algebras is the dual one.

□

Remark. The main application of the former result consists of considering most T-algebras (co-algebras) as restrictions or particular cases of $T^{\natural}_{\mathcal{H}}$-algebras (co-algebras) when we observe behavior changes. The members of $\mathcal{H}$ that work as parameters need not be ruled by the axioms of the extended constructs, and remain hidden until we observe either any anomalous event, or some behavior changes. In the following sections we expose two illustrative applications.

4. Bernoulli distribution with hidden parameters.

Probability spaces can be formalized as co-algebras. For instance, let $(\Omega, \mathcal{E}, P)$ be a probability space; where Ω is the set of outcomes, $\mathcal{E}$ the set of events, and $P : \mathcal{E} \to [0,1]$ the probability assignation. If $T : \mathbf{Set} \to \mathbf{Set}$ is the endofunctor sending each set into $[0,1]$, and every map $f : X \to Y$ into the identity id $: [0,1] \to [0,1]$, then $P : \mathcal{E} \to T(\mathcal{E}) = [0,1]$ gives rise to a co-algebra. A map $f : \mathcal{E}_1 \to \mathcal{E}_2$ is a morphism whenever the following quadrangle commutes.

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{P_1} & T(\mathcal{E}_1) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle T(f)=\mathrm{id}} \\ \mathcal{E}_2 & \xrightarrow[P_2]{} & T(\mathcal{E}_2) \end{array}$$

We can interpret these co-algebras as restrictions of those with hidden parameters, such that the probability assignations P_1 and P_2 depend on some parameter set $\mathcal{H}$. The following paragraphs illustrate these ideas.

Let X be a random variable, with Bernoulli distribution, like tossing a coin n-times. Let (T, P) be the associated co-algebra, where P denotes the probability assignation. Let $S = f_1, f_2, f_3 \dots f_n$ be the observed relative frequency sequence of the event $X = 1$ (success) in some experiment. Suppose that the sequence S converges in probability to $\frac{1}{2}$, and the relative frequencies satisfy the relation $\forall n \in \mathbb{N} : f_n \leq \frac{1}{2}$. By the *weak law of large numbers* we know that $p = q = \frac{1}{2}$ and both events (success and failure) are equiprobable. Nevertheless, the relation $\forall n \in \mathbb{N} : f_n \leq \frac{1}{2}$ leads to $P(f_n \leq \frac{1}{2}) = 1$. This relation is not a consequence of probability laws. By contrast, it does not satisfy the expected symmetry in equiprobable situations. We can interpret this fact introducing hidden parameters as follows.

We can consider (T, P) as a particular case of an extension $(T^{\natural}_{\mathcal{H}}, \tilde{P})$ with a hidden parameter set $\mathcal{H} = \{\tau, \omega\}$, where the probability assignation is a $\mathbf{Set}_{\mathcal{H}}$-morphism $\tilde{P} : X \times \mathcal{H} \to T(X) = [0, 1]$ defined as follows.

$$\tilde{P}(X, \alpha) = \begin{cases} \frac{1}{2} \text{ if } (X, \alpha) = (0, \tau) \\ \frac{1}{2} \text{ if } (X, \alpha) = (1, \tau) \\ 1 \text{ if } (X, \alpha) = (0, \omega) \\ 0 \text{ if } (X, \alpha) = (1, \omega) \end{cases} \tag{4.1}$$

Now, suppose that

$$\forall n \in \mathbb{N} : \quad \alpha = \begin{cases} \omega \text{ if } n = 1 \\ \tau \text{ if } n > 1 \text{ and } f_{n-1} < \frac{1}{2} \\ \omega \text{ if } n > 1 \text{ and } f_{n-1} = \frac{1}{2} \end{cases} \tag{4.2}$$

With these conditions the relative frequency sequence of the event $X = 1$ converges in probability to $\frac{1}{2}$ and keeps always less than or equal to $\frac{1}{2}$. Notice that the parameter α takes the value ω whenever the event $f_n = \frac{1}{2}$ occurs; otherwise keeps equal to τ.

In the former example, we can see that hidden parameters correspond to “events” or “situations” that can occur in real world phenomena. This example is artificial, but there are natural random phenomena whose probability assignation can be modified by hidden parameters. For instance, the frequency under which a word “w” occurs increases its probability occurrence. However, in smart text, under excessive repetition the probability occurrence of ”w” can vanish. Academic style, smartness, and word repetition can be regarded as hidden parameters that modify the occurrence probability of any word.

5. Structured Languages

As in (Palomar Tarancón, 2011), for each nonempty object-class $\mathbf{C}$, we denote by $\mathbf{C}^{\curlyvee}$ the generic object of $\mathbf{C}$. For instance, if $\mathbf{C}$ is the set $\{n \in \mathbb{N} \mid n \equiv 1 \,(\bmod 2)\}$, then $\mathbf{C}^{\curlyvee}$ denotes the concept of odd positive integer. To avoid any exception, we apply the same operator to singletons or one-member classes. The generic object of any singleton $\{O\}$ coincides with its unique member; hence

$$\{O\}^{\curlyvee} = O. \tag{5.1}$$

Definition 5.1. A predicate $P(X) \in \mathbf{Pr}$ is self-contradictory provided that $\neg P(X)$ is a tautology. ◁

It is straightforward consequence of the preceding definition that if $P(X)$ is a tautology, its negation $\neg P(X)$ is self-contradictory. If $P(X)$ is not self-contradictory there is at least one object O such that $P(O)$ is true; otherwise $\neg P(X)$ would be true for every value of X, hence a tautological predicate.

In this section, $\mathbf{Pr}$ denotes a predicate class of higher-order logic, being stable under conjunctions, disjunctions and negations. Likewise, $\mathbf{Mc}(\mathbf{Pr})$ denotes an object class satisfying the following axioms.

Axiom 5.1. *If a predicate $P(X) \in \mathbf{Pr}$ is neither self-contradictory nor tautological, the class $\mathbf{Mc}(\mathbf{Pr})$ contains the generic object $\{O \mid P(O)\}^{\curlyvee}$.*

Axiom 5.2. *For every $O \in \mathbf{Mc}(\mathbf{Pr})$ there is $P(X) \in \mathbf{Pr}$ such that*

$$\{Q \in \mathbf{Mc}(\mathbf{Pr}) \mid P(Q)\} = \{O\}.$$

Definition 5.2. An attributive definition for a member O of $\mathbf{Mc}(\mathbf{Pr})$ is any predicate $P(X) \in \mathbf{Pr}$ such that $O = \{Q \in \mathbf{Mc}(\mathbf{Pr}) \mid P(Q)\}^{\curlyvee}$. If the class $\{Q \in \mathbf{Mc}(\mathbf{Pr}) \mid P(Q)\}$ is a singleton, we say $P(X)$ to be a strictly attributive definition of O. ◁

Remark. In natural languages, most words denote generic objects of equivalence classes. For instance, the word "polygon" denotes a class that contains "triangles" and "quadrangles among others. Each of these words again denotes some object class. Attributive definitions consist of an attribute or property that is stated by a predicate $P(X)$. The defined object O is the generic one of the class that satisfies $P(X)$. Thus, if O_1 is a concretion of O obtained by adding another property $Q(X)$, that is, if O_1 is the generic object of the class $\{R \mid P(R) \wedge Q(R)\}$, then $P(X) \wedge Q(X) \Rightarrow P(X)$.

Lemma 5.1. *Each predicate $P(X) \in \mathbf{Pr}$ that is neither tautological nor self-contradictory, gives rise to a strictly attributive definition for some object $O \in \mathbf{Mc}(\mathbf{Pr})$.*

Proof. Let $P^*(Y, P(X))$ denote the predicate

$$\text{“ } Y \text{ is the generic object of the class} \quad \mathbf{C} = \{O \in \mathbf{Mc}(\mathbf{Pr}) \mid P(O)\}.\text{”}$$

The class $\mathbf{C}$ is nonempty because, by hypothesis, $P(X)$ is not self-contradictory (see Definition 5.1). According to Axiom 5.1 there is the generic object $\mathbf{C}^{\curlyvee}$ in $\mathbf{Mc}(\mathbf{Pr})$, besides, taking into account (5.1),

$$\{O \in \mathbf{Mc}(\mathbf{Pr}) \mid P^*(O, P(X))\}^{\curlyvee} = \{\mathbf{C}^{\curlyvee}\}^{\curlyvee} = \mathbf{C}^{\curlyvee}.$$

Consequently, it is a strictly definition. □

Definition 5.3. The class $\mathbf{Mc}(\mathbf{Pr})$ can be enriched with an order relation $\preceq$ such that, between every couple of objects O_1 and O_2, the relation $O_1 \preceq O_2$ holds whenever there are two attributive definitions $P_{O_1}(X)$ and $P_{O_2}(X)$ for O_1 and O_2, respectively, such that $P_{O_1}(X) \Rightarrow P_{O_2}(X)$. ◁

Enriched with the relation $\preceq$, the class $\mathbf{Mc}(\mathbf{Pr})$ satisfies the structure of a category $\mathbf{Mc}(\mathbf{Pr}, \preceq)$ such that, for every couple of objects O_1 and O_2, the set $\mathrm{Hom}_{\mathbf{Mc}(\mathbf{Pr},\preceq)}(O_1, O_2)$ either is empty or it is the singleton $\{O_1 \preceq O_2\}$. From now on, we assume that the category $\mathbf{Mc}(\mathbf{Pr}, \preceq)$ satisfies the following axiom.

Axiom 5.3. *The object-class of* $\mathbf{Mc}(\mathbf{Pr}, \preceq)$ *contains with each subset* $\{O_i \mid i \in I\}$ *its coproduct* $\coprod_{i \in I} O_i$*; where* I *is any nonempty index set.*

Notation. For each phrase W in any meaningful language, we denote by $\|W\|$ the meaning associated to W.

Remark. Let $P_1(X)$, $P_2(X)$, and $P(X)$ be attributive definitions for O_1, O_2, and $O_1 \coprod O_2$, respectively. According to the definition of $\preceq$, the following relations are true: $P_1(X) \Rightarrow P(X)$ and $P_2(X) \Rightarrow P(X)$. Thus, $P(X)$ is the more restrictive definition that both objects O_1 and O_2 satisfy. In other words, $O_1 \coprod O_2$ is the more concrete abstraction of both objects O_1 and O_2. For instance,

$$\|\textit{Large positive integer}\| \coprod \|\textit{small positive integer}\| = \|\textit{positive integer}\|.$$

Notation. For every object O in $\mathbf{Mc}(\mathbf{Pr})$ the expression $|O|$ denotes the predicate class $\{P(X) \in \mathbf{Pr} \mid P(O)\}$.

Lemma 5.2. *For every object* $O \in \mathrm{Ob}\,(\mathbf{Mc}(\mathbf{Pr}, \preceq))$ *and each predicate* $Q(X) \in |O|$*, the statement*

$$\forall P(X) \in |O| : \quad Q(X) \Rightarrow P(X) \tag{5.2}$$

is true if and only if $Q(X)$ *is a strictly attributive definition for* O*.*

Proof. First assume $Q(X)$ to be a strictly attributive definition for O, and let $P(X)$ be a member of $|O|$. Suppose that (5.2) is false; hence there is O_1 such that the conjunction $Q(O_1) \wedge (\neg P(O_1))$ is true. Since $Q(X)$ is a strictly attributive definition for O, this relation leads to $O = O_1$ because, by Definition 5.2, the set $\{X \mid Q(X)\}$ must be a singleton. Consequently, these relations lead to $\neg P(O)$, which contradicts the initial assumption $P(X) \in |O|$.

Now suppose that (5.2) holds, and let $Q_1(X)$ be a strictly attributive definition for O. As we have just seen, $Q_1(X) \Rightarrow Q(X)$. Since O must satisfy its own definition $Q_1(X) \in |O|$. As a consequence of (5.2) this membership relation leads to $Q(X) \Rightarrow Q_1(X)$; consequently $Q_1(X) \Leftrightarrow Q(X)$, and $Q(X)$ is also an attributive definition for O. □

Definition 5.4. From now on, we term structured language on a category $\mathbf{Mc}(\mathbf{Pr}, \prec)$ each 4–tuple $\mathfrak{L} = (A, A^*, A^{**}, M)$ such that,

1. The set A is a finite collection of symbols (alphabet).
2. The set A^* is a partial (syntactic) free-monoid generated by A. We term "word" each member of A^*.
3. The set A^{**} is a partial free-monoid generated by A^*. We say each member of A^{**} to be a phrase.

4. The symbol M denotes a nonempty subset of A^{**} each of its members has a meaning lying in **Mc**(**Pr**). The set A^* contains words denoting the concepts of conjunction, disjunction, and negation. In addition, M is stable under conjunctions, disjunctions and negations.
5. The set M contains with each subset $\{W_i \mid i \in I\}$ a phrase the meaning of which is the coproduct $\coprod_{i \in I} \|W_i\|$, (see Axiom 5.3). ◁

The members of M can be also single words because each meaningful word can be regarded as a one-word phrase. As usual, we term sentence each meaningful phrase. Likewise, statements are truth-valued sentences.

Notation. For each structured language $\mathfrak{L} = (A, A^*, A^{**}, M)$, we denote by $\perp^{\Upsilon}$ a variable ranging over all phrases in A^{**}. This notation allows us to write patterns obtained from any phrase. For instance, consider a phrase $W = w_1 w_2 \dots w_i w_{i+1} \dots w_{i+j} \dots w_n$, where the w_i are the involved words. Substituting the sub-phrase $V = w_i w_{i+1} \dots w_{i+j}$ by $\perp^{\Upsilon}$, we obtain the pattern $W_V(\perp^{\Upsilon})$ that sends each phrase $U = u_1, u_2 \dots u_k \in A^{**}$ into

$$W_V(U) = w_1 w_2 \dots u_1 u_2 \dots u_k w_{j+1} \dots w_n.$$

For instance, let W be the phrase

We can evaluate the area of every polygon.

If we substitute the one-word phrase "*polygon*" by $\perp^{\Upsilon}$, we obtain the pattern

$$W_V(\perp^{\Upsilon}) = \textit{We can evaluate the area of every } \perp^{\Upsilon}.$$

The subscript V in the expression W_V denotes V to be the sub-phrase that we substitute by the variable $\perp^{\Upsilon}$. If U = "*regular triangle,*" then

$$W_V(\textit{regular triangle}) = \textit{We can evaluate the area of every regular triangle.}$$

Definition 5.5. Let $\mathfrak{L} = (A, A^*, A^{**}, M)$ be a structured language. A pattern $W_V(\perp^{\Upsilon})$ is continuous provided that for every couple U_1 and U_2 of phrases in M the following conditions hold.

1. If both relations $W_V(U_1) \in M$ and $\|U_2\| \preceq \|U_1\|$ are true, then $W_V(U_2) \in M$.
2. Let $\mathbf{D} = \{U_i \mid i \in I\} \subseteq M$ be a subset with cardinality greater than 1. If a phrase $R \in M$ denotes the object $\coprod_{i \in I} \|U_i\|$, and for every $i \in I$: $W_V(U_i) \in M$, then $W_V(R) \in M$. ◁

Example 5.1. Let $W_V(\perp^{\Upsilon})$ be the English pattern "*The area of every* $\perp^{\Upsilon}$ *is finite.*" Let U_1 denote the word "*triangle*" and U_2 the phrase "*regular triangle.*" If M denotes the class of meaningful English sentences, then the phrase $W_V(U_1)$= *"The area of every triangle is finite"* belongs to M. Likewise, the relation $\|U_2\| \preceq \|U_1\|$ holds because if $\|U_2\|$ is a regular triangle, it is also a triangle. Indeed, $W_V(U_2) \in M$. Finally, $\|U_1\| \coprod \|U_2\| = \|U_1\|$, and by assumption, $W_V(U_1) \in M$.

Since the conjunction of a set of phrases is again a phrase, it is a straightforward consequence that the conjunction of a set of patterns is again a pattern. By definition, there is some symbol or word in each structured language that denotes conjunction. From now on, we denote by the symbol $\mathring{\wedge}$ the conjunction in any structured language. Thus, if the considered language is the English one, $\mathring{\wedge}$ stands for the word "and".

Proposition 5.1. *The conjunction of a set of continuous patterns is again continuous.*

Proof. Let $\mathfrak{L} = (A, A^*, A^{**}, M)$ be a structured language. Let $\mathbf{P} = \{W_{V_i}(\perp^{\curlyvee}) \mid i \in I\}$ a set of patterns in $\mathfrak{L}$ and $\mathbf{P}(\perp^{\curlyvee}) = \mathring{\bigwedge}_{i \in I} W_{V_i}(\perp^{\curlyvee})$ the conjunction of all members of $\mathbf{P}$. Let $U_0 \in M$ and $U_1 \in M$ be two phrases such that $\|U_1\| \preceq \|U_0\|$ and

$$\forall i \in I : \quad W_{V_i}(U_0) \in M \tag{5.3}$$

By continuity,

$$\forall i \in I : \quad W_{V_i}(U_1) \in M \tag{5.4}$$

hence, taking into account Definition 5.4, $\mathbf{P}(U_0) \in M$ and $\mathbf{P}(U_1) \in M$. □

Theorem 5.1. *For every continuous pattern $W_V(\perp^{\curlyvee})$ the following statements are true.*

1. *There is a $\preceq$-maximum element in the class*

$$\mathbf{W} = \{\|U\| \mid W_V(U) \in M)\}.$$

2. *If $\|U\|$ is the $\preceq$-maximum element of* $\mathbf{W}$*, the predicate*

$$P(X) = X \textit{ is the maximum element of } \mathbf{W}$$

is a strictly attributive definition of $\|U\|$, whenever $P(X) \in$ **Pr**.

Proof.

1. If every element in a chain $\|U_1\| \preceq \|U_2\| \preceq \cdots \preceq \|U_n\|$ lies in $\mathbf{W}$, by Definition 5.5, so does its upper bound $\coprod_{0<i\leq n} \|U_i\|$. Thus, $\mathbf{W}$ satisfies the conditions of Zorn's Lemma. Accordingly, there is, at least, one $\preceq$-maximal element $\|U_1\|$ in $\mathbf{W}$.
   To see that $\|U_1\|$ is the maximum element of $\mathbf{W}$, let $\|U\| \in \mathbf{W}$ be any member. By virtue of both Definition 5.4 and Definition 5.5, there is a phrase R in M such that $\|R\| = \|U\| \coprod \|U_1\|$; hence there are the $\mathbf{Mc}(\mathbf{Pr})$-morphisms $\|U_1\| \preceq \|R\|$ and $\|U\| \preceq \|R\|$. Since $\|U_1\|$ is maximal these relations lead to $\|R\| = \|U_1\|$ and $\|U\| \preceq \|R\| = \|U_1\|$. Accordingly, $\|U_1\|$ is comparable with every member of $\mathbf{W}$.
2. It is a straightforward consequence of the maximum-element uniqueness.

□

Definition 5.6. Let $\mathfrak{L} = (A, A^*, A^{**}, M)$ be a structured language. A pattern class $\mathbf{Pt} = \{W_{i,V_i}(\perp^{\curlyvee}) \mid i \in I\}$ is compatible provided that there is at least one phrase U in M such that, for every $i \in I$: $W_{i,V_i}(U) \in M$. ◁

Recall that, by virtue of statement 4) in Definition 5.4, the conjunction of all phrases in **Pt** again belongs to M.

Notation. By $\in^\partial$ we denote the "sub-phrase/phrase" relationship. For instance, if

$$W = w_1 w_2 \dots w_i w_{i+1} \dots w_{i+j} \dots w_n$$

is a phrase, the following expression denotes the word sequence $w_i w_{i+1} \dots w_{i+j}$ to be a sub-phrase.

$$w_i w_{i+1} \dots w_{i+j} \in^\partial w_1 w_2 \dots w_i w_{i+1} \dots w_{i+j} \dots w_n.$$

From now on, for each phrase set A^{**} and every $V \in A^{**}$, the expression A^{**}_V denotes the subset $A^{**}_V = \{W \in A^{**} \mid V \in^\partial W\}$. Likewise, $[A^{**}, M]$ denotes the phrase-set collection

$$[A^{**}, M] = \bigcup_{V \in M} \{X \subseteq A^{**}_V \mid V \in X\} \tag{5.5}$$

Finally, for every couple of phrases V_1 and V_2, the expression $\langle V_1 \leftrightarrows V_2 \rangle : M \to M$ denotes the result of substituting each occurrence of the sub-phrase V_1 in W by one of V_2. If W does not contain any occurrence of V_1, then $\langle V_1 \leftrightarrows V_2 \rangle W = W$. Likewise, the infix operator $\leftrightarrows$ can be used to obtain patterns; for instance $\langle V_1 \leftrightarrows \perp^\curlyvee \rangle W = W_{V_1}(\perp^\curlyvee)$.

Notation. From now on, for each $V \in M$ and every $X \subseteq A^{**}_V$, the expression $\mathrm{Pat}(X)$ denotes the pattern class defined as follows.

$$\mathrm{Pat}(V, X) = \{\langle V \leftrightarrows \perp^\curlyvee \rangle W \mid W \in X\}$$

Proposition 5.2. *If $\mathfrak{L} = (A, A^*, A^{**}, M)$ is a structured language, for every $V \in M$, each subset E of A^{**}_V satisfies the following statements.*

1. *The pattern class $\mathrm{Pat}(V, E) = \{\langle V \leftrightarrows \perp^\curlyvee \rangle W \mid W \in E\}$ is compatible.*
2. *Let E_0 a nonempty subset of E. Let $\mathbf{U}_V(\perp^\curlyvee)$ and $\mathbf{V}_V(\perp^\curlyvee)$ be the conjunctions of the pattern classes $\mathrm{Pat}(V, E)$ and $\mathrm{Pat}(V, E_0)$, respectively. If both patterns $\mathbf{U}_V(\perp^\curlyvee)$ and $\mathbf{V}_V(\perp^\curlyvee)$ are continuous, the maximum elements $\|U\|$ and $\|U_0\|$ of the classes $\mathbf{W} = \{\|X\| \mid \mathbf{U}_V(X) \in M\}$ and $\mathbf{W}_0 = \{\|X\| \mid \mathbf{V}_V(X) \in M\}$ respectively, satisfy the relation $\|U\| \preceq \|U_0\|$.*

Proof. 1. By definition, for each $W \in E$: $W_V(V) = W$; hence

$$\forall W_V \in \mathrm{Pat}(V, E) : \; W_V(V) \in M.$$

2. Since $\mathrm{Pat}(V, E_0)$ is a subset of $\mathrm{Pat}(V, E)$, for each phrase P the relation $\mathbf{U}_V(P) \in M$ leads to $\mathbf{V}_V(P) \in M$; therefore $\|U\|$ belongs to $\mathbf{W}_0$. By assumption, $\|U_0\|$ is the maximum element of the class $\mathbf{W}_0$, then $\|U\| \preceq \|U_0\|$.

□

Lemma 5.3. *For every $E \in [A^{**}, M]$, there is a unique $V \in M$ such that $E \subseteq A^{**}_V$ and $V \in E$.*

Proof. It is a straightforward consequence of (5.5). □

Definition 5.7. For each structured language $\mathfrak{L} = (A, A^*, A^{**}, M)$, the expression $\mathbf{Ph}(\mathfrak{L})$ denotes the small category the object class of which is

$$\mathrm{Ob}\,(\mathbf{Ph}(\mathfrak{L})) = [A^{**}, M]$$

For every pair of objects E_1 and E_2, the homset $\mathrm{Hom}_{\mathbf{Ph}(\mathfrak{L})}(E_1, E_2)$ consists of each map $f : E_1 \to E_2$ that satisfies the following condition.

$$\forall W \in E_1 : \quad \langle V_1 \leftrightarrows V_2 \rangle W = f(W) \tag{5.6}$$

where V_1 and V_2 are members of M such that $E_1 \subseteq A^{**}_{V_1}$ and $E_2 \subseteq A^{**}_{V_2}$ ◁

Recall that, by virtue of Lemma 5.3, for every $\mathbf{Ph}(\mathfrak{L})$-object E, there is $V \in M$ such that $E \subseteq A^{**}_V$ and $V \in E$.

The map $\mathfrak{T}_O : \mathrm{Ob}\,(\mathbf{Ph}(\mathfrak{L})) \to \mathrm{Ob}\,(\mathbf{Ph}(\mathfrak{L}))$ sending each set $E \in A^{**}_V$ into the singleton $\mathfrak{T}_O(E) = \{V\}$ is the object-map for an endofunctor $\mathfrak{T} : \mathbf{Ph}(\mathfrak{L}) \to \mathbf{Ph}(\mathfrak{L})$ that sends each morphism $f \in \mathrm{Hom}(E_1, E_2)$ into the map $\mathfrak{T}(f) : \{V_1\} \to \{V_2\}$ such that $V_1 \mapsto V_2$. Indeed, this map definition satisfies the condition $\langle V_1 \leftrightarrows V_2 \rangle V_1 = V_2$. We denote this endofunctor by $\mathfrak{T}$.

Proposition 5.3. *Let $\mathfrak{L} = (A, A^*, A^{**}, M)$ a structured language. Let V_1 and V_2 two members of M. If two $\mathfrak{T}$-algebras (E_1, σ_1) and (E_2, σ_2) satisfy the following hypotheses*

1. *There is a morphism $f : (E_1, \sigma_1) \to (E_2, \sigma_2)$.*
2. *The sets E_1 and E_2 are subsets of $A^{**}_{V_1}$ and $A^{**}_{V_2}$, respectively. In addition, all members of both pattern classes*
$$\mathrm{Pat}(V_1, \sigma_1(V_1)) = \{\langle V_1 \leftrightarrows \perp^{\curlyvee} \rangle W \mid W \in \sigma_1(V_1)\}$$
and
$$\mathrm{Pat}(V_2, \sigma_2(V_2)) = \{\langle V_2 \leftrightarrows \perp^{\curlyvee} \rangle W \mid W \in \sigma_2(V_2)\}$$
are continuous.
3. *The objects $\|V_1\|$ and $\|V_2\|$ are the $\preceq$-maximum elements of the object classes $\mathbf{W}_1 = \{\|X\| \mid \mathbf{P}_1(X) \in M\}$ and $\mathbf{W}_2 = \{\|X\| \mid \mathbf{P}_2(X) \in M\}$, respectively; where*
$$\mathbf{P}_1(\perp^{\curlyvee}) = \mathring{\bigwedge}_{W(\perp^{\curlyvee}) \in \mathrm{Pat}\left(V_1, \sigma_1(V_1)\right)} W(\perp^{\curlyvee}),$$
and
$$\mathbf{P}_2(\perp^{\curlyvee}) = \mathring{\bigwedge}_{W(\perp^{\curlyvee}) \in \mathrm{Pat}\left(V_2, \sigma_2(V_2)\right)} W(\perp^{\curlyvee}),$$
respectively.

then the phrases V_1 and V_2 satisfy the relation $\|V_2\| \preceq \|V_1\|$.

Proof. By the definition of $\mathfrak{T}$, and taking into account hypothesis 2), the following relations are true.

$$\begin{cases} \mathfrak{T}(E_1) = \{V_1\} \\ \mathfrak{T}(E_2) = \{V_2\} \\ \sigma_1(V_1) \subseteq E_1 \\ \sigma_2(V_2) \subseteq E_2 \end{cases} \tag{5.7}$$

The existence of the morphism f leads to the relation

$$\forall W \in \sigma_1(V_1): \quad \langle V_1 \leftrightarrows V_2 \rangle W = f(W) \in \sigma_2(V_2) \tag{5.8}$$

therefore $\mathrm{Pat}(V_1, \sigma_1(V_1)) \subseteq \mathrm{Pat}(V_2, \sigma_2(V_2))$. By virtue of Proposition 5.2, this relation leads to $\|V_2\| \preceq \|V_1\|$. □

Remark. By the former proposition we know that $\|U_2\| \preceq \|U_1\|$; accordingly if $P_1(X)$ and $P_2(X)$ are attributive definitions of $\|U_1\|$ and $\|U_2\|$, respectively, the relation $P_2(X) \Rightarrow P_1(X)$ holds (see Definition 5.3). We can deduce this relation, simply, by knowing that substituting V_1 by V_2 in every member of the phrase set $\sigma_1(V_1)$ we obtain a subset of $\sigma_1(V_2)$. This property is a straightforward consequence of the $\mathbf{Ph}(\mathfrak{L})$-morphism definition. Thus, observing occurrences of some sub-phrases in two phrase sets we can find logical implications between attributive definitions of their meanings blindly, that is, without knowing what they mean. Nevertheless, several meanings can be assigned to the same phrase in natural languages or artificial ones, depending on the context, state, style, among other circumstances. Accordingly, contexts, states, styles work as hidden parameters in a set $\mathcal{H}$. Consequently, to apply the method arising from the preceding result, and to interpret sentences in a language properly, we must consider that each $\mathfrak{T}$-algebra (E, σ) is a particular case of a $\mathfrak{T}^{\natural}_{\mathcal{H}}$-algebra; where the members of $\mathcal{H}$ denote states, contexts, styles, frequencies and any other event modifying the meaning of any phrase.

6. Conclusion

Hidden parameters are handled implicitly in Computer Science and Linguistics. We can find noticeable instances almost in each subject. This is a very exciting research field. Theorem 3.1 is the bridge between structured sets, namely, algebras (co-algebras), and any set of hidden parameters that modify their behavior. For instance, the research of those relative frequency anomalies that can be interpreted as the action of hidden parameters is an open problem.

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Modeling Evolution by Evolutionary Machines: A New Perspective on Computational Theory and Practice

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Abstract

The main goal of this paper is the further development of the foundations of evolutionary computations, connecting classical ideas in the theory of algorithms and the contemporary state of art in evolutionary computations. To achieve this goal, we develop a general approach to evolutionary processes in the computational context, building mathematical models of computational systems, called evolutionary machines or automata. We introduce two classes of evolutionary automata: basic evolutionary automata and general evolutionary automata. Relations between computing power of these classes are explored. Additionally, several other classes of evolutionary machines are investigated, such as bounded, periodic and recursively generated evolutionary machines. Different properties of these evolutionary machines are obtained.

Keywords: Turing unorganized machines, evolutionary computation, evolutionary automata, evolutionary Turing machines, evolutionary finite automata, evolutionary inductive Turing machines.
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1. Introduction

The classical theory of algorithms has been developed under the influence of Alan Turing, who was one of the founders of theoretical computer science and whose model of computation, which is now called Turing machine, is the most popular in computer science. He also had many other ideas. In this report the National Physical Laboratory in 1948 (Turing, 1992), Turing proposed a new model of computation, which he called unorganized machines (u-machines). There were two types of u-machines: based on Boolean networks and based on finite state machines.

- A-type and B-type u-machines were Boolean networks made up of a fixed number of two-input NAND gates (neurons) and synchronized by a global clock. While in A-type u-machines the connections between neurons were fixed, B-type u-machines had modifiable

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switch type interconnections. Starting from the initial random configuration and applying a kind of genetic algorithm, B-type u-machines were supposed to learn which of their connections should be on and which off.

- P-type u-machines were tapeless Turing machines reduced to their finite state machine control, with an incomplete transition table, and two input lines for interaction: the pleasure and the pain signals.

Although Turing never formally defined a genetic algorithm or evolutionary computation, in his B-type u-machines, he predicted two areas at the same time: neural networks and evolutionary computation (more precisely, evolutionary artificial neural networks), while his P-type u-machines represent reinforcement learning. However, this work had no impact on these fields (Eberbach *et al.*, 2004), although these ideas are one of the (almost forgotten) roots of evolutionary computation.

Evolutionary computation theory is still very young and incomplete. Until recently, evolutionary computation did not have a theoretical model that represented practice in this domain. Even though there are many results on the theory of evolutionary algorithms (see, e.g., (Michalewicz & Fogel, 2004), (He & Yao, 2004), (Holland, 1975), (Rudolph, 1994), (Wolpert & Macready, 1997), (Koza, 1992, 1994; Koza *et al.*, 1999), (Michalewicz, 1996), very little has been known about expressiveness, or computational power, of evolutionary computation (EC) and its scalability. Of course, there are many results on the theory of evolutionary algorithms (again see, for instance, (Michalewicz & Fogel, 2004), (He & Yao, 2004), (Holland, 1975), (Rudolph, 1994), (Wolpert & Macready, 1997), (Koza, 1992, 1994; Koza *et al.*, 1999), (Michalewicz, 1996)). Studied in EC theoretical topics include convergence in the limit (elitist selection, Michalewiczs contractive mapping GAs, (1+1)-ES), convergence rate (Rechenbergs 1/5 rule), the Building Block analysis (Schema Theorems for GA and GP), best variation operators (No Free Lunch Theorem). However, these authors do not introduce automata models - rather they apply a high-quality mathematical apparatus to existing process models, such as Markov chains, etc. They also cover only some aspects of evolutionary computation like convergence or convergence rate, neglecting for example EC expressiveness, self-adaptation, or scalability. In other words, EC is not treated as a distinct and complete area with its own distinct model situated in the context of general computational models. This means that in spite of intensive usage of mathematical techniques, EC lacks more complete theoretical foundations. As a result, many properties of evolutionary processes could not be precisely studied or even found by researchers. Our research is aimed at filling this gap to define more precisely conditions under which evolutionary algorithms will work and will be superior compared to other optimization methods.

In 2005, the evolutionary Turing machine model was proposed to provide more rigorous foundations for EC (Eberbach, 2005). An evolutionary Turing machine is an extension of the conventional Turing machine, which goes beyond the Turing machine and belongs to the class of super-recursive algorithms (Burgin, 2005). In several papers, the authors studied and extended the ETM (evolutionary Turing machine) model to reflect cooperation and competition (Burgin & Eberbach, 2008), universality (Burgin & Eberbach, 2009*b*), self-evolution (Eberbach & Burgin, 2007), and expressiveness of evolutionary finite automata (Burgin & Eberbach, 2009*a*), (Burgin & Eberbach, 2012).

In this paper, we continue developing a general approach to evolutionary processes in the computational context constructing mathematical models of the systems, functioning of which is based on evolutionary processes, and study properties of such systems with the emphasis on their generative power. Two classes are introduced in Section 2: basic evolutionary automata and general evolutionary automata. Relations between computing power of these classes are explored in Section 3. Additionally, several other classes of evolutionary machines are investigated, such as bounded, periodic and recursively generated evolutionary machines. Different properties of these evolutionary machines are obtained. Section 4 contains conclusions and some open problems.

2. Modeling Evolution by Evolutionary Machines

Evolutionary algorithms describe artificial intelligence processes based on the theory of natural selection and evolution. Evolutionary computation is directed by evolutionary algorithms. In technical terms, an evolutionary algorithm is a probabilistic beam hill climbing search algorithm directed by the chosen fitness function. It means that the beam (population size) maintains multiple search points, hill climbing implies that only a current search point from the search tree is remembered and used for optimization (going to the top of the hill), and the termination condition very often is set to the optimum of the fitness function.

Let X be the representation space, also called the optimization space, for species (systems) used in the process of optimization and a fitness function $f : X \to \mathbb{R}^+$ is chosen, where $\mathbb{R}^+$ is the set of nonnegative real numbers.

Definition 2.1. *A generic evolutionary algorithm EA* can be represented as the collection $EA = (X, s, v, f, R, X[0], F)$ and described in the form of the functional equation (recurrence relation) R working in a simple iterative loop in discrete time t, defining generations $X[t]$, $(t = 0, 1, 2, 3, ...)$ with $X[t + 1] = s(v(X[t]))$, where

- $X[t] \subseteq X$ is a population under a representation consisting of one or more individuals from the set X (e.g., fixed binary strings for genetic algorithms (GAs), finite state machines for evolutionary programming (EP), parse trees for genetic programming (GP), vectors of reals for evolution strategies (ES)),
- s is a selection operator (e.g., truncation, proportional, tournament),
- v is a variation operator (e.g., variants and compositions of mutation and crossover),
- $X[0]$ is an initial population,
- $F \subseteq X$ is the set of final populations satisfying the termination condition (goal of evolution). The desirable termination condition is the optimum in X of the fitness function $f(x)$, which is extended to the fitness function $f(X[t])$ of the best individual in the population $X[t] \subseteq F$, where $f(x)$ typically takes values in the domain of nonnegative real numbers. In many cases, it is impossible to achieve or verify this optimum. Thus, another stopping criterion is used (e.g., the maximum number of generations, the lack of progress through several generations.).

The above definition is applicable to all typical EAs, including GA, EP, ES, GP. It is possible to use it to describe other emerging subareas like ant colony optimization, or particle swarm optimization. Of course, it is possible to think and implement more complex variants of evolutionary algorithms.

Evolutionary algorithms evolve population of solutions x, but they may be the subject of self-adaptation (like in ES) as well. For sure, evolution in nature is not static, the rate of evolution fluctuates, their variation operators are subject to slow or fast changes, and its goal (if it exists at all) can be a subject of modifications as well.

Formally, an evolutionary algorithm looking for the optimum of the fitness function violates some classical requirements of recursive algorithms. If its termination condition is set to the optimum of the fitness function, it may not terminate after a finite number of steps. To fit it to the conventional algorithmic approach, an artificial (or somebody can call it pragmatic) stop criterion has had to be added (see e.g., (Michalewicz, 1996), (Michalewicz & Fogel, 2004), (Koza, 1992, 1994; Koza *et al.*, 1999)). To remain recursive, i.e., to give some result after a finite number of steps, the evolutionary algorithm has to reach the set F of final populations satisfying the termination condition after a finite number of generations or to halt when no visible progress is observable. Usually this is a too restrictive condition, and naturally, in a general case, evolutionary algorithms form a special class of super-recursive algorithms.

To formalize the concept of an evolutionary algorithm in mathematically rigorous terms, we define a formal algorithmic model of evolutionary computation - an evolutionary automaton also called an evolutionary machine.

Let K be a class of automata working with words in an alphabet E. It means that the representation or optimization space X is the set E^* of all words in an alphabet E.

Definition 2.2. *A basic evolutionary K-machine* (BEM), also called *basic evolutionary K-automaton*, is a (possibly infinite) sequence $E = \{A[t]; t = 0, 1, 2, 3, ...\}$ of automata $A[t]$ from K each working on the population $X[t] \subseteq X (t = 0, 1, 2, 3, ...)$ where:

- the automaton $A[t]$ called a component, or more exactly, a level automaton, of E represents (encodes) a one-level evolutionary algorithm that works with the generation $X[t]$ of the population by applying the *variation operators* v and *selection operator* s;

- the zero generation $X[0]$ is given as input to E and is processed by the automaton $A[0]$, so that either $X[0]$ is the result of the whole computation by E when it satisfies the search condition or $A[0]$ generates/produces the first generation $X[1]$ as its output, which goes to the automaton $A[1]$;

- for all $t = 1, 2, 3, ...$, the generation $X[t + 1]$ is obtained by applying the variation operator v and selection operator s to the generation $X[t]$ and these operations are performed by the automaton $A[t]$, which receives $X[t]$ as its input; the generation $X[t+1]$ either is the result of the whole computation by E when it satisfies the search condition or it goes to the automaton $A[t + 1]$;

- the goal of the BEM E is to build a population Z satisfying the search condition.

The desirable search condition is the optimum of the fitness performance measure $f(x[t])$ of the best individual from the population $X[t]$. There are different modes of the EM functioning and different termination strategies. When the search condition is satisfied, then working in the recursive mode, the EM E halts (t stops to be incremented), otherwise a new input population $X[t+1]$ is generated by $A[t]$. In the inductive mode, it is not necessary to halt to give the result (cf. (Burgin, 2005)). When the search condition is satisfied and E is working in the inductive mode, the EM E stabilizes (the population $X[t]$ stops changing), otherwise a new input population $X[t+1]$ is generated by $A[t]$.

We denote the class of all basic evolutionary machines with level automata from K by BEAK.

Definition 2.3. *A general evolutionary K-machine* (GEM), also called *general evolutionary K-automaton*, is a (possibly infinite) sequence $E = \{A[t]; t = 0, 1, 2, 3, ...\}$ of automata $A[t]$ from K each working on generations $X[i] \subseteq X$ where:

- the automaton $A[t]$ called a component, or more exactly, a level automaton, of E represents (encodes) a one-level evolutionary algorithm that works with generations $X[i]$ of the population by applying the variation operators v and selection operator s;

- the zero generation $X[0] \subseteq X$ is given as input to E and is processed by the automaton $A[0]$, which generates/produces the first generation $X[1]$ as its output, which either is the result of the whole computation by E when it satisfies the search condition or it goes to the automaton $A[1]$;

- for all $t = 1, 2, 3, ...$, the automaton $A[t]$, which receives $X[i]$ as its input either from $A[t+1]$ or from $A[t-1]$, then $A[t]$ applies the variation operator v and selection operator s to the generation $X[t]$, producing the generation $X[t+1]$ as its output, which either is the result of the whole computation by E when it satisfies the search condition or it goes either to $A[t+1]$ or to $A[t-1]$. To perform such a transmission, the automaton A[t] uses one of the two techniques: transmission by the output and transmission by the state. In transmission by the output, the automaton $A[t]$ uses two more symbols u_{up} and u_{dw} in its output alphabet, giving one of these symbols as a part of its output in addition to the regular output $X[t+1]$. If this part of the output is u_{up}, then $A[t]$ sends the output generation $X[t+1]$ to $A[t+1]$. If the additional part of the output is u_{dw}, then $A[t]$ sends the output generation $X[t+1]$ to $A[t-1]$. In transmission by the state, the automaton $A[t]$ uses two more symbols u_{up} and u_{dw} as its final-transmission states. In these states the automaton $A[t]$ stops computing and performs the necessary transmission of the output - to the automaton $A[t+1]$ when the state is u_{up} and to the automaton $A[t-1]$ when the state is u_{dw}.

- the goal of the GEM E is to build a population Z satisfying the search condition.

We denote the class of all general evolutionary K-machines GEAK. As any basic evolutionary K-machine is also a general evolutionary K-machine, we have inclusion of classes $BEAK \subseteq GEAK$.

Let us consider some examples of evolutionary K-machines. An important class of evolutionary machines are evolutionary finite automata (Burgin & Eberbach, 2009*a*), (Burgin & Eberbach, 2012). Here K consists of finite automata.

Definition 2.4. *A basic (general) evolutionary finite automaton* (EFA) is a basic (general) evolutionary machine E in which all automata $A[t]$ are finite automata $G[t]$ each working on the population $X[t]$ in generations $t = 0, 1, 2, 3, ...$

We denote the class of all general evolutionary finite automata by GEFA. It is possible to take as K deterministic finite automata, which form the class DFA, or nondeterministic finite automata, which form the class NFA. This gives us four classes of evolutionary finite automata: BEDFA (GEDFA) of all deterministic basic (general) evolutionary finite automata and BENFA (GENFA) of all nondeterministic basic (general) evolutionary finite automata.

Evolutionary Turing machines (Burgin & Eberbach, 2008), (Eberbach, 2005) are form another important class of evolutionary machines.

Definition 2.5. *A basic (general) evolutionary Turing machine* (ETM) $E = \{T[t]; t = 0, 1, 2, 3, ...\}$ is a basic (general) evolutionary machine E in which all automata $A[t]$ are Turing machines $T[t]$ each working on population $X[t]$ in generations $t = 0, 1, 2, 3, ...$

Turing machines $T[t]$ as components of E perform multiple computations (Burgin, 1983). Variation and selection operators are recursive to allow performing level computation on Turing machines.

Definition 2.6. *A basic (general) evolutionary inductive Turing machine* (EITM) $EI = \{M[t]; t = 0, 1, 2, ...\}$ is a basic (general) evolutionary machine E in which all automata $A[t]$ are inductive Turing machines $M[t]$ (Burgin, 2005) each working on the population $X[t]$ in generations $t = 0, 1, 2, ...$

Simple inductive Turing machines are abstract automata (models of algorithms) closest to Turing machines. The difference between them is that a Turing machine always gives the final result after a finite number of steps and after this it stops or, at least, informs when the result is obtained. Inductive Turing machines also give the final result after a finite number of steps, but in contrast to Turing machines, inductive Turing machines do not always stop the process of computation or inform when the final result is obtained. In some cases, they do this, while in other cases they continue their computation and give the final result. Namely, when the content of the output tape of a simple inductive Turing machine forever stops changing, it is the final result.

Definition 2.7. *A basic (general) evolutionary inductive Turing machine* (EITM) $EI = \{M[t]; t = 0, 1, 2, ...\}$ has *order n* if all inductive Turing machines $M[t]$ have order less than or equal to n and at least, one inductive Turing machine $M[t]$ has order n.

We remind that inductive Turing machines with recursive memory are called inductive Turing machines of the first order (Burgin, 2005). The memory E is called n-inductive if its structure is constructed by an inductive Turing machine of the order n. Inductive Turing machines with n-inductive memory are called inductive Turing machines of the order $n + 1$. We denote the class of all evolutionary inductive Turing machines of the order n by $EITM_n$.

Definition 2.8. *A basic (general) evolutionary limit Turing machine* (ELTM) $EI = \{LTM[t]; t = 0, 1, 2, ...\}$ is a basic (general) evolutionary machine E in which all automata $A[t]$ are limit Turing machines $LTM[t]$ (cf. (Burgin, 2005)) each working on the population $X[t]$ in generations $t = 0, 1, 2, ...$

When the search condition is satisfied, then the ELTM EI stabilizes (the population $X[t]$ stops changing), otherwise a new input population $X[t+1]$ is generated by LT $M[t]$. We denote the class of all evolutionary limit Turing machines of the first order by ELTM.

Basic and general evolutionary K-machines from BEAK and GEAK are called unrestricted because sequences of the level automata $A[t]$ and the mode of the evolutionary machines functioning are arbitrary. For instance, there are unrestricted evolutionary Turing machines when K is equal to T and unrestricted evolutionary finite automata when K is equal to FA. However it is possible to consider only basic (general) evolutionary K-machines from BEAK (GEAK) in which sequences of the level automata have some definite type Q. Such machines are called Q-formed basic (general) evolutionary K-machines and their class is denoted by $BEAK^Q$ for basic machines and $GEAKQ$ for general machines. When the type Q contains all finite sequences, we have bounded basic (general) evolutionary K-machines. Some classes of bounded basic evolutionary K-machines are studied in (Burgin & Eberbach, 2010) for such classes K as finite automata, pushdown automata, Turing machines, or inductive Turing machines, i.e., such classes as bounded basic evolutionary Turing machines or bounded basic evolutionary finite automata. When the type Q contains all periodic sequences, we have periodic basic (general) evolutionary K-machines. Some classes of periodic basic evolutionary K-machines are studied in (Burgin & Eberbach, 2010) for such classes K as finite automata, push down automata, Turing machines, inductive Turing machines and limit Turing machines. Note that while in a general case, evolutionary automata cannot be codified by finite words, periodic evolutionary automata can be codified by finite words.

Another condition on evolutionary machines determines their mode of functioning or computation. Here we consider the following modes of functioning/computation.

1. The finite-state mode: any computation is going by state transition where states belong to a fixed finite set.
2. The bounded mode: the number of generations produced in all computations is bounded by the same number.
3. The terminal or finite mode: the number of generations produced in any computation is finite.
4. The recursive mode: in the process of computation, it is possible to reverse the direction of computation, i.e., it is possible to go from higher levels to lower levels of the automaton, and the result is defined after finite number of steps.
5. The inductive mode: the computation goes in one direction, i.e., without reversions, and if for some t, the generation $X[t]$ stops changing, i.e., $X[t] = X[q]$ for all $q > t$, then $X[t]$ is the result of computation.
6. The inductive mode with recursion: recursion (reversion) is permissible and if for some t, the generation $X[t]$ stops changing, i.e., $X[t] = X[q]$ for all $q > t$, then $X[t]$ is the result of computation.
7. The limit mode: the computation goes in one direction and the result of computation is the limit of the generations $X[t]$.
8. The limit mode with recursion: recursion (reversion) is permissible and the result of computation is the limit of the generations $X[t]$.

These modes are complementary to the three traditional modes of computing automata: computation, acceptation and decision/selection (Burgin, 2010). Existence of different modes of computation shows that the same algorithmic structure of an evolutionary automaton/machine E provides for different types of evolutionary computations. We see that only general evolutionary machines allow recursion. In basic evolutionary machines, the process of evolution (computation) goes strictly in one direction. Thus, general evolutionary machines have more possibilities than basic evolutionary machines and it is interesting to relations between these types of evolutionary machines. This is done in the next section. Note that utilization of recursive steps in evolutionary machines provides means for modeling reversible evolution, as well as evolution that includes periods of decline and regression.

3. Computing and Accepting Power of Evolutionary Machines

As we know from the theory of automata and computation, it is proved that different automata or different classes of automata are equivalent. However there are different kinds of equivalence. Here we consider two of them: functional equivalence and linguistic equivalence.

Definition 3.1. (Burgin, 2010)

a. Two automata A and B are functionally equivalent if given the same input, they give the same output.
b. Two classes of automata A and B are functionally equivalent if for any automaton from A, there is a functionally equivalent automaton from B and vice versa.

For instance, it is proved that deterministic and nondeterministic Turing machines are functionally equivalent (cf., for example, (Hopcroft *et al.*, 2001)). Similar results are true for evolutionary automata.

Theorem 3.1. *(Burgin & Eberbach, 2010) For any basic n-level evolutionary finite automaton E, there is a finite automaton AE functionally equivalent to E.*

Here we study relations between basic and general evolutionary machines, assuming that all these machines work in the terminal mode.

Let $P : X \times U \to N$ be a function such that

$$U = \{u_{1,up}, u_{1,dw}, u_{2,up}, u_{2,dw}, ..., u_{k,up}, u_{k,dw}, ...\},$$

$$P(x, u_{k,up}) = k + 1$$

and

$$P(x, u_{k,dw}) = k - 1$$

for any x from optimization space $X = E^*$.

Definition 3.2. (Burgin, 2010) The P-conjunctive parallel composition $\wedge_P A_i$ of the algorithms/automata A_i $(i = 1, 2, 3, ..., n)$ is an algorithm/automaton D such that the result of application of D to any input u is equal to $A_i(u)$ when $P(u) = i$.

This concept allows us to show in a general case of the terminal mode that basic and general evolutionary machines are equivalent.

Theorem 3.2. *If a class K is closed with respect to P-conjunctive parallel composition, then for any general evolutionary K-machine, there is a functionally equivalent basic evolutionary K-machine.*

Proof. Let us consider an arbitrary general evolutionary K-machine $E = \{A[t]; t = 0, 1, 2, 3, ...\}$. We correspond the evolutionary system $H = \{C[t]; t = 0, 1, 2, 3, ...\}$ to the K-machine E. Each component $C[t]$ in H is a system that consists of the automata $C_0[t], C_1[t], C_2[t], C_3[t], ..., C_t[t]$ such that for all $k = 0, 1, 2, 3, ..., t$, the automaton $C_k[t]$ is a copy of the automaton $A[k]$ and it uses the elements $u_{k,up}$ and $u_{k,dw}$ instead of the elements u_{up} and u_{dw} employed by $A[k]$.

The system H has the same search condition as the evolutionary K-machine E and functions in the following way. The zero generation $X[0] \subseteq X$ is given as input to the automaton $C_0[0]$, which is a copy of the automaton $A[0]$ and is processed by the automaton $C_0[0]$, which generates/produces the first generation $X[1]$ as its output. Then $X[1]$ either is the result of the whole computation by H when it satisfies the search condition or it goes to the automaton $C_1[1]$ as its input. In the general case, for all $t = 1, 2, 3, ...$ and $k = 1, 2, 3, ..., t$, the automaton $C_k[t]$ receives $X[t]$ as its input either from $C_{k+1}[t-1]$ when the automaton $A[k]$ receives its input from $A[k+1]$ or from $C_{k-1}[t-1]$ when the automaton $A[k]$ receives its input from $A[k-1]$. Then $C_k[t]$ applies the variation operator v and selection operator s to the generation $X[t]$ and producing the generation $X[t+1]$. Then either this generation is the result of the whole computation by H when it satisfies the search condition or $C_k[t]$ sends this generation either to $C_{k+1}[t+1]$ when the automaton $A[k]$ sends its output to $A[k+1]$ or to $C_{k-1}[t+1]$ when the automaton $A[k]$ sends its output to $A[k-1]$.

In such a way, the system H simulates functioning of the general evolutionary K-machine $E = \{A[t]; t = 0, 1, 2, 3, ...\}$. Let us prove this by induction on the number of steps that the K-machine E is making.

The base of induction:

Making the first step, the K-machine E receives is the zero generation $X[0] \subseteq X$ as its input, processes it by the first automaton $A[0]$ producing the first generation $X[1]$, which either is the result of the whole computation by E when it satisfies the search condition or it goes to the automaton $A[1]$.

Making the first step, the system H receives the zero generation $X[0] \subseteq X$ as its input, processes it by the first automaton $C_0[0]$ producing the first generation $Z[1]$, which either is the result of the whole computation by H when it satisfies the search condition or it goes to the automaton $C_1[1]$. Because the system H has the same search condition as the evolutionary K-machine E, $C_0[0]$ is a copy of the automaton $A[0]$, while $C_1[1]$ is a copy of the automaton $A[1]$, we have the equality $Z[1] = X[1]$ and the first step of the system H exactly simulates the first step of the K-machine E.

The general step of induction:

We suppose that making $n-1$ steps the system H exactly simulates $n-1$ steps of the K-machine E. It means that making $n-1$ steps, both systems E and H produce the same n-th generation $X[n]$ using automata $A[r]$ $(r \leqslant n-1)$ and $C_r[n-1]$, correspondingly, and this output either is the result of the whole computation by E and by H when it satisfies the search condition or it goes either to the automaton $A[r + 1]$ or to the automaton $A[r - 1]$ in E and either to the automaton $C_{r+1}[n]$ or to the automaton $C_{r-1}[n]$ in H.

Then the automaton $A[r+1]$ (or $A[r-1]$) in E produces the next generation $X[n+1]$, applying the variation operator v and selection operator s to the generation $X[n]$ and producing the next generation $X[n + 1]$. When the automaton $A[r + 1]$ in E produces the next generation $X[n + 1]$, then either this generation is the result of the whole computation by E when it satisfies the search condition or $A[r + 1]$ sends this generation either to $A[r + 2]$ or to $A[r]$. When the automaton $A[r - 1]$ in E produces the next generation $X[n + 1]$, then either this generation is the result of the whole computation by E when it satisfies the search condition or $A[r - 1]$ sends this generation either to $A[r - 2]$ or to $A[r]$.

At the same time, the automaton $C_{r+1}[n]$ (or $C_{r-1}[n]$) applies the variation operator v and selection operator s to the generation $X[n]$ and producing the generation $Z[n + 1]$. When the automaton $C_{r+1}[n]$ in H produces the next generation $Z[n + 1]$, then either this generation is the result of the whole computation by H when it satisfies the search condition or $C_{r+1}[n]$ sends this generation either to $C_{r+2}[n + 1]$ or to $C_r[n + 1]$. When the automaton $C_{r-1}[n]$ in H produces the next generation $Z[n + 1]$, then either this generation is the result of the whole computation by H when it satisfies the search condition or $C_{r-1}[n]$ sends this generation either to $C_{r-2}[n]$ or to $C_r[n]$.

Because system H has the same search condition as the evolutionary K-machine E, $C_{r+1}[n]$ is a copy of the automaton $A[r + 1]$, while $C_{r-1}[n]$ is a copy of the automaton $A[r - 1]$, we have the equality $Z[n + 1] = X[n + 1]$ and the n-th step of the system H exactly simulates the n-th step of the K-machine E.

Now it is possible to conclude that the system H exactly simulates functioning of the K-machine E. However, the system H is not an evolutionary K-machine. So we need to build a basic evolutionary K-machine B equivalent to H. We can do this using P -conjunctive parallel composition. This composition allows us for all $t = 0, 1, 2, 3, ...$, to substitute each system $\{C_0[t], C_1[t], C_2[t], C_3[t], ..., C_t[t]\}$ by an automaton $B[t]$ from K, which by the definition of function P and P-conjunctive parallel composition, works exactly as this system. Then by construction of the system H, $B = \{B[t]; t = 0, 1, 2, 3, ...\}$ is a basic evolutionary K-machine B equivalent to H. Theorem is proved. □

Corollary 3.1. *If a class K is closed with respect to P-conjunctive parallel composition, then classes GEAK and BEAK are functionally equivalent.*

The class T of all Turing machines is closed with respect to P-conjunctive parallel composition (Burgin, 2010). Thus, Theorem 3.2 implies the following result.

Corollary 3.2. *Classes GEAT of all general evolutionary Turing machines and BEAT of all basic evolutionary Turing machines are functionally equivalent.*

The class IT of all inductive Turing machines is closed with respect to P-conjunctive parallel composition (Burgin, 2010). Thus, Theorem 3.2 implies the following result.

Corollary 3.3. *Classes GEAIT of all general evolutionary inductive Turing machines and BEAIT of all basic evolutionary inductive Turing machines are functionally equivalent.*

Corollary 3.4. *Classes $GEAIT_n$ of all general evolutionary inductive Turing machines of order n and $BEAIT_n$ of all basic evolutionary inductive Turing machines of order n are functionally equivalent.*

The same is true for evolutionary limit Turing machines.

Corollary 3.5. *Classes GEALT of all general evolutionary limit Turing machines and BEALT of all basic evolutionary limit Turing machines are functionally equivalent.*

Definition 3.3. (Burgin, 2010)

a. Two automata A and B are linguistically equivalent if they accept (generate) the same language.
b. Two classes of automata A and B are linguistically equivalent if they accept (generate) the same class of languages.

For instance, it is proved that deterministic and nondeterministic finite automata are linguistically equivalent (cf,. for example (Hopcroft *et al.*, 2001)). It is proved that functional equivalence is stronger than linguistic equivalence (Burgin, 2010).

Because P-conjunctive parallel composition of the level automata in an evolutionary automaton allows the basic evolutionary K-machine to choose automata for data transmission, it is possible to prove the following results.

Theorem 3.3. *If a class K is closed with respect to P-conjunctive parallel composition, then for any general evolutionary K-machine, there is a linguistically equivalent basic evolutionary K-machine.*

Proof. Let us consider an arbitrary general evolutionary K-machine $E = \{A[t]; t = 0, 1, 2, 3, ...\}$. Then by Theorem 3.2, there is a basic evolutionary K-machine L that is functionally equivalent to E. As it is proved in (Burgin, 2010), functional equivalence implies linguistic equivalence. So, the K-machine L is linguistically equivalent to the K-machine E. Theorem is proved. □

Corollary 3.6. *If a class K is closed with respect to P-conjunctive parallel composition, then classes GEAK and BEAK are linguistically equivalent.*

The class T of all Turing machines is closed with respect to P-conjunctive parallel composition (Burgin, 2010). Thus, Theorem 3.3 implies the following result.

Corollary 3.7. *Classes GEAT of all general evolutionary Turing machines and BEAT of all basic evolutionary Turing machines are linguistically equivalent.*

The class IT of all inductive Turing machines is closed with respect to P-conjunctive parallel composition (Burgin, 2010). Thus, Theorem 3.3 implies the following results.

Corollary 3.8. *Classes GEAIT of all general evolutionary inductive Turing machines and BEAIT of all basic evolutionary inductive Turing machines are linguistically equivalent.*

Corollary 3.9. *Classes $GEAIT_n$ of all general evolutionary inductive Turing machines of order n and $BEAIT_n$ of all basic evolutionary inductive Turing machines of order n are linguistically equivalent.*

The same is true for evolutionary limit Turing machines.

Corollary 3.10. *Classes GEALT of all general evolutionary limit Turing machines and BEALT of all basic evolutionary limit Turing machines are linguistically equivalent.*

Obtained results allow us to solve the following problem formulated in (Burgin & Eberbach, 2010).

Problem 3.1. *Are periodic evolutionary finite automata more powerful than finite automata?*

To solve it, we need additional properties of periodic evolutionary finite automata.

Theorem 3.4. *Any general (basic) periodic evolutionary finite automaton F with the period $k > 1$ is functionally equivalent to a periodic evolutionary finite automaton E with the period* 1.

Proof. Let us consider an arbitrary basic periodic evolutionary finite automaton $E = \{A[t]; t = 0, 1, 2, 3, ...\}$. By the definition of basic periodic evolutionary automata (cf. Section 2), the sequence $\{A[t]; t = 0, 1, 2, 3, ...\}$ of finite automata $A[t]$ is either finite or periodic, i.e., there is a finite initial segment of this sequence such that the whole sequence is formed by infinite repetition of this segment. Note that finite sequences are also treated as periodic (Burgin & Eberbach, 2010). When the sequence $\{A[t]; t = 0, 1, 2, 3, ...\}$ of automata $A[t]$ from K is finite, then by Theorem 3.2, the evolutionary machine E is functionally equivalent to a finite automaton AE. By the definition of periodic evolutionary automata, AE is a periodic evolutionary finite automaton with the period 1. Thus, in this case, theorem is proved.

Now let us assume that the sequence $\{A[t]; t = 0, 1, 2, 3, ...\}$ of automata $A[t]$ is infinite. In this case, there is a finite initial segment $H = \{A[t]; t = 0, 1, 2, 3, ..., n\}$ of this sequence such that the whole sequence is formed by infinite repetition of this segment H. By the definition of bounded basic evolutionary automata (cf. Section 2), H is a basic n-level evolutionary finite automaton. Then by Theorem 3.1 from (Burgin & Eberbach, 2010), there is a finite automaton AH functionally equivalent to H. Thus, the evolutionary machine E is functionally equivalent to the basic periodic evolutionary finite automaton $B = \{B[t]; t = 0, 1, 2, 3, ...\}$ in which all automata $B[t] = AH$ for all $t = 0, 1, 2, 3, ...$ Thus, B is a basic periodic evolutionary finite automaton with the period 1. This concludes the proof for basic periodic evolutionary finite automata.

Now let us consider an arbitrary general periodic evolutionary finite automaton $E = \{A[t]; t = 0, 1, 2, 3, ...\}$. By the definition of general periodic evolutionary automata (cf. Section 2), the sequence $\{A[t]; t = 0, 1, 2, 3, ...\}$ of finite automata $A[t]$ is either finite or periodic, i.e., there is a finite initial segment of this sequence such that the whole sequence is formed by infinite repetition of this segment.

At first, we show that when the sequence $\{A[t]; t = 0, 1, 2, 3, ...\}$ of automata $A[t]$ from K is finite, i.e., $E = \{A[t]; t = 0, 1, 2, 3, ..., n\}$, then the evolutionary machine E is functionally equivalent to a finite automaton AE . It is possible to assume that the automata $A[t]$ use transmission by the output when the automaton $A[t]$ uses two more symbols u_{up} and u_{dw} in its output alphabet, giving one of these symbols in its output in addition to the regular output $X[t+1]$, i.e., the output has the form (w, u_{up}) or (w, u_{dw}). If the second part of the output is u_{up} , then $A[t+1]$ sends the output generation $X[t+1]$ to $A[t+1]$. If the second part of the output is u_{dw}, then $A[t+1]$ sends the output generation $X[t+1]$ to $A[t-1]$.

We change all automata $A[t]$ to the automata $C[t]$ in the following way. If $\{q_0, q_1, q_2, ..., q_k\}$ is the set of all states of the automaton $A[t]$, then we take $\{q_{t,0}, q_{t,1}, q_{t,2}, ..., q_{t,k}\}$ as the set of all states of the automaton $C[t]$ $(t = 0, 1, 2, ..., n)$ and in the transition rules of $C[t]$, we change each q_l to $q_{t,l}$. In addition, we change the symbols u_{up} and u_{dw} to the symbols $u_{t,up}$ and $u_{t,dw}$ in the alphabet and in the transition rules of $C[t]$.

By construction, the new system $AE = \{C[t]; t = 0, 1, 2, 3, ..., n\}$ is a finite automaton functionally equivalent to the general periodic evolutionary finite automaton $E = \{A[t]; t = 0, 1, 2, 3, ..., n\}$. Then by the definition of periodic evolutionary automata (cf. Section 2), the automaton AE is a general periodic evolutionary finite automaton with the period 1. Thus, in the finite case, theorem is proved.

Now let us assume that the sequence $\{A[t]; t = 0, 1, 2, 3, ...\}$ of automata $A[t]$ is infinite. In this case, there is a finite initial segment $H = \{A[t]; t = 0, 1, 2, 3, ..., n\}$ of this sequence such that the whole sequence is formed by infinite repetition of this segment H. By the definition of bounded general evolutionary automata (cf. Section 2), H is a general n-level evolutionary finite automaton. Then as we have already proved, there is a finite automaton AH functionally equivalent to H. Thus, the evolutionary machine E is functionally equivalent to the general periodic evolutionary finite automaton $B = \{B[t]; t = 0, 1, 2, 3, ...\}$ in which all automata $B[t] = AH$ for all $t = 0, 1, 2, 3, ...$ Thus, B is a general periodic evolutionary finite automaton with the period 1. This concludes the proof for general periodic evolutionary finite automata. Theorem is proved. □

Functional equivalence implies linguistic equivalence (Burgin, 2010). Thus, Theorem 3.4 implies the following result.

Corollary 3.11. *Any general (basic) periodic evolutionary finite automaton F with the period $k > 1$ is linguistically equivalent to a periodic evolutionary finite automaton E with the period* 1.

As a periodic evolutionary finite automaton F with the period 1 consists of multiple copies of the same finite automaton, we have the following results.

Theorem 3.5. *Any basic periodic evolutionary finite automaton F is linguistically equivalent to a finite automaton.*

Proof. By Theorem 3.4, any basic periodic evolutionary finite automaton F with the period $k > 1$ is functionally equivalent to a basic periodic evolutionary finite automaton E with the period 1. It means that all levels in the evolutionary finite automaton E are copies of the same finite automaton. As a finite automaton accepts (computes) a regular language (Hopcroft *et al.*, 2001), the language of the evolutionary finite automaton E is also regular. As the evolutionary finite automaton F is linguistically equivalent to the automaton E, the language L of the evolutionary finite automaton F is also regular. Then there is a finite automaton D that accepts (computes) L (Hopcroft *et al.*, 2001). Thus, the evolutionary finite automaton F is linguistically equivalent to the finite automaton D. Theorem is proved. □

Corollary 3.12. *Basic periodic evolutionary finite automata have the same accepting power as finite automata.*

Theorem 3.6. *Any general periodic evolutionary finite automaton E is equivalent to a one-dimensional cellular automaton.*

Proof. By Theorem 3.4, any general periodic evolutionary finite automaton G with the period $k > 1$ is functionally equivalent to a general periodic evolutionary finite automaton E with the period 1. By definition, E is a sequence of copies of the same finite automaton, which each of them is connected with two its neighbors, and this is exactly a one-dimensional cellular automaton (Trahtenbrot, 1974).

At the same time, taking a finite automaton A with a feedback that connects the automaton output with the automaton input, we see that A can simulate a periodic evolutionary finite automaton E with the period 1 because in E all level automata are copies of the same finite automaton. □

In the theory of cellular automata, it is proved that for any Turing machine T, there is a cellular automaton functionally equivalent to T (Trahtenbrot, 1974). Thus, Theorem 3.6 implies the following result.

Corollary 3.13. *General periodic evolutionary finite automata have the same accepting power as Turing machines.*

Consequently, we have the following result.

Corollary 3.14. *General periodic evolutionary finite automata have more accepting power than basic periodic evolutionary finite automata and than finite automata.*

Note that we cannot apply Theorem 3.2 to periodic evolutionary finite automata because the general evolutionary machine constructed in the proof of this theorem is not periodic.

These results also allow us to solve Problem 4 from (Burgin & Eberbach, 2010).

Problem 3.2. *What class of languages is generated/accepted by periodic evolutionary finite automata?*

Namely, we have the following results.

Corollary 3.15. *The class of languages generated/accepted by basic periodic evolutionary finite automata coincides with regular languages.*

Corollary 3.16. *The class of languages generated/accepted by general periodic evolutionary finite automata coincides with recursively enumerable languages.*

Note that for unrestricted evolutionary finite automata results of Theorems 3.5, 3.6 and their corollaries are not true. Namely, we have the following result.

Theorem 3.7. *The class GEAFA of general unrestricted evolutionary finite automata and the class BEAFA of basic unrestricted evolutionary finite automata have the same accepting power.*

Proof. Indeed, as it is demonstrated in (Eberbach & Burgin, 2007), basic unrestricted evolutionary finite automata can accept any formal language. In particular, they accept any language that general unrestricted evolutionary finite automata accept. As general unrestricted evolutionary finite automata are more general than basic unrestricted evolutionary finite automata, the class of the languages accepted by the former automata is, at least, as big as the class of the languages accepted by the latter automata. Thus, these classes coincide, which means that the class of all general unrestricted evolutionary finite automata and the class of all basic unrestricted evolutionary finite automata have the same accepting power. □

The results from this paper show that in some cases, general evolutionary machines are more powerful than basic evolutionary machines, e.g., for all periodic evolutionary finite automata, while in other cases, it is not true, e.g., for all evolutionary finite automata, general and basic evolutionary finite automata have the same computing power. There are similar results in the theory of classical automata and algorithms. For instance, deterministic and nondeterministic finite automata have the same accepting power. Deterministic and nondeterministic Turing machines have the same accepting power. However, nondeterministic pushdown automata have more accepting power than deterministic pushdown automata.

4. Conclusion

We started our paper with a description of Turings unorganized machines (u-machines) that were supposed to work under the control of some kind of genetic algorithms (note that Turing never formally defined a genetic algorithm or evolutionary computation). This was our inspiration. However, our evolutionary machines are closely related to conventional Turing machines, as well as to the subsequent definitions of genetic algorithms from 1960-80s. This means that level automata of evolutionary machines are finite automata, pushdown automata or Turing machines rather than more primitive NAND logic gates of u-machines. We have introduced several classes of evolutionary machines, such as bounded, periodic and recursively generated evolutionary machines, and studied relations between these classes, giving an interpretation of how modern u-machines could be formalized and how plentiful their computations and types are. Of course, we will never know whether Turing would accept our definitions of evolutionary automata and formalization of evolutionary computation.

In this paper, we introduced two fundamental classes of evolutionary machines/automata: general evolutionary machines and basic evolutionary machines, exploring relations between these classes. Problems of generation of evolutionary machines/automata by automata from a given class are also studied. Examples of such evolutionary machines are evolutionary Turing machines generated by Turing machines and evolutionary inductive Turing machines generated by inductive Turing machines.

There are open problems important for the development of EC foundations.

Problem 4.1. *Can an inductive Turing machine of the first order simulate an arbitrary periodic evolutionary inductive Turing machine of the first order?*

Problem 4.2. *Are there necessary and sufficient conditions for general evolutionary machines to be more powerful than basic evolutionary machines?*

In (Burgin, 2001), topological computations are introduced and studied. This brings us to the following problem.

Problem 4.3. *Study topological computations for evolutionary machines.*

As we can see from results of this paper, in some cases general evolutionary machines are more powerful than basic evolutionary machines, e.g., for all evolutionary finite automata, while in other cases, it is not true, e.g., for all periodic evolutionary machines.

Note that the approach presented in this paper has an enormous space to grow. First of all, similar to natural evolution, our evolutionary automata/machines are not static, i.e., we cover the case of evolution of evolution (currently explored in a very limited way in evolution strategies by changing the σ parameter in mutation). Secondly, our evolutionary finite automata cover already both evolutionary algorithms (i.e., genetic algorithms, evolutionary programming, evolution strategies and genetic programming)and swarm intelligence algorithms, being simple iterative algorithms of the class or regular languages/finite automata. In the evolutionary automata approach, there is a room to grow to invent new types of evolutionary and swarm intelligence algorithms of the class of evolutionary pushdown automata, evolutionary Turing machines or evolutionary inductive Turing machines.

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On a New BV_σ I-Convergent Double Sequence Spaces

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Abstract

In this article we study ${}_2({}_0BV^I_\sigma(M))$, ${}_2BV^I_\sigma(M)$, ${}_2({}_\infty BV^I_\sigma(M))$ double sequence spaces with the help of BV_σ space and an Orlicz function M. The BV_σ space was introduced and studied by (Mursaleen, 1983). We study some of its properties and prove some inclusion relations.

Keywords: Bounded variation, invariant mean, σ-Bounded variation, ideal, filter, Orlicz function, I-Convergence, I-null, solid space, sequence algebra, convergence free space.
2010 MSC: 40C05, 46A45, 46E30, 46E40, 46B20.

1. Introduction

Let $\mathbb{N}, \mathbb{R}, \mathbb{C}$ be the sets of all natural, real, and complex numbers respectively. We denote

$$ {}_2\omega = \{x = (x_{ij}) : x_{ij} \in \mathbb{R} \text{ or } \mathbb{C}\}, $$

showing the space of all real or complex sequences.

Definition 1.1. A double sequence of complex numbers is defined as a function $X : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$. We denote a double sequence as (x_{ij}) where the two subscripts run through the sequence of natural numbers independent of each other. A number a$\in \mathbb{C}$ is called double limit of a double sequence (x_{ij}) if for every $\epsilon > 0$ there exists some $N = N(\epsilon) \in \mathbb{N}$ such that,

$$ |(x_{ij}) - a| < \epsilon, \text{ for all } i, j \geqslant N, \tag{1.1} $$

(see (Habil, 2006)). Let l_∞ and c denote the Banach space bounded and convergent sequences, respectively, with norm $\|x\|_\infty = \sup_k |x_k|$. Let v be denote the space of sequences of bounded variation. That is,

$$ v = \{x = (x_k) : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty, x_{-1} = 0\} \tag{1.2} $$

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where v is a Banach space normed by $\|x\| = \sum_{k=0}^{\infty} |x_k - x_{k-1}|$ (see (Mursaleen, 1983)). Let σ be an injective mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or σ-mean if and only if:
1. $\phi(x) \geqslant 0$ where the sequence $x = (x_k)$ has $x_k \geqslant 0$ for all k,
2. $\phi(e) = 1$ where e={1,1,1,1,......},
3. $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

If $x = (x_k)$, write $Tx = (Tx_k) = (x_{\sigma(k)})$. It can be shown that

$$V_\sigma = \{x = (x_k) : \lim_{m\to\infty} t_{m,k}(x) = L \text{ uniformly in } k, L = \sigma - \lim x\} \tag{1.3}$$

where m$\geqslant 0, k > 0$.

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} + ... + x_\sigma^m(k)}{m+1} \text{ and } t_{-1,k} = 0, \tag{1.4}$$

where $\sigma^m(k)$ denote the m^{th}-iterate of $\sigma(k)$ at k . In this case σ is the translation mapping,that is, $\sigma(k) = k + 1, \sigma$-mean is called a Banach limit and V_σ, the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequences. The special case of (1.4) in which $\sigma(k) = k + 1$ was given by (Lorentz, 1948), and that the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on c in the sense that

$$\phi(x) = \lim x, \text{ for all } x \in c. \tag{1.5}$$

Theorem 1.1. *A σ-mean extends the limit functional on* c *in the sense that $\phi(x) = \lim x$ for all $x \in$ c if and only if σ has no finite orbits. That is, if and only if for all $k \geqslant 0, j \geqslant 1, \sigma^j(k) \neq k$, (see (Khan, 2008))*

Put

$$\phi_{m,k}(x) = t_{m,k}(x) - t_{m-1,k}(x), \tag{1.6}$$

assuming that $t_{-1,k}(x) = 0$. A straight forward calculation shows that (Mursaleen, 1983),

$$\phi_{m,k}(x) = \begin{cases} \frac{1}{m(m+1)} \sum_{j=1}^{m} J(x_\sigma^j(k) - x_\sigma^{j-1}(k)), & \text{if } m \geqslant 1 \\ x_k, \quad \text{if } m = 0. \end{cases}$$

For any sequence x,y and scalar λ, we have $\phi_{m,k}(x + y) = \phi_{m,k}(x) + \phi_{m,k}(y)$ and $\phi_{m,k}(\lambda x) = \lambda\phi_{m,k}(x)$.

Definition 1.2. A sequence $x \in l_\infty$ is of σ-bounded variation if and only if:
(i)$\sum |\phi_{m,k}(x)|$ converges uniformly in k,
(ii) $\lim_{m\to\infty} t_{m,k}(x)$, which must exist, should take the same value for all k.

We denote by BV_σ, the space of all sequences of σ-bounded variation (see (Khan, 2008)):

$$BV_\sigma = \{x \in l_\infty : \sum_m |\phi_{m,k}(x)| < \infty, \text{ uniformly in } k\}.$$

Theorem 1.2. *BV_σ is a Banach space normed by*

$$\|x\| = \sup_k \sum_{m=0}^{\infty} |\phi_{m,k}(x)|, \tag{1.7}$$

(see (Khan & Ebadullah, 2012)).
Subsequently invariant mean studied by (Mursaleen, 1983), (Ahmad & Mursaleen, 1988), (Raimi & A., 1963), (Khan & Ebadullah, 2011), (Khan & Ebadullah, 2012), (Schaefer, 1972) and many others.

Definition 1.3. A function $M : [0, \infty) \longrightarrow [0, \infty)$ is said to be an Orlicz function if it satisfies the following conditions;
(i)M is continuous ,convex and non-decreasing,
(ii)$M(0) = 0, M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Remark. (see (Tripathy & Hazarika, 2011)). **(i)** If the convexity of an Orlicz function is replaced by $M(x + y) \leqslant M(x) + M(y)$, then this function is called Modulus function.
(ii) If M is an Orlicz function , then $M(\lambda X) \leqslant \lambda M(x)$ for all λ with $0 < \lambda < 1$.

An Orlicz function M is said to satisfy $\triangle_2$-condition for all values of u if there exists a constant $K > 0$ such that $M(Lu) \leqslant KLM(u)$ for all values of L> 1(see (Tripathy & Hazarika, 2011)). (Lindenstrauss & Tzafriri, 1971) used the idea of an Orlicz function to construct the sequence space $l_M = \{x \in w : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty$ for some $\rho > 0\}$. The space l_∞ becomes a Banach space with the norm

$$\|x\| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leqslant 1\Big\}, \tag{1.8}$$

which is called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(t) = t^p$ for $1 \leqslant p < \infty$. Later on, some Orlicz sequence spaces were investigated by (Hazarika & Esi, 2013), (Maddox, 1970), (Parshar & Choudhary, 1994), (Bhardwaj & Singh, 2000), (Et, 2001), (Tripathy & Hazarika, 2011) and many others. Initially, as a generalization of statistical convergence, the notation of I-convergence was introduced and studied by P. Kostyrko and Wilczynski(Kostyrko *et al.*, 2000). Later on, it was studied by Hazarika and Esi (Hazarika & Esi, 2013) and many others.

Definition 1.4. A double sequence $x = x_{ij} \in {}_2\omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, we have

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geqslant \epsilon\} \in I. \tag{1.9}$$

In this case, we write $I - \lim x_{ij} = L$.

Definition 1.5. Let X be a non empty set. Then, a family of sets $I \subseteq 2^X$ is said to be an Ideal in X if
(i)$\phi \in I$;
(ii)I is additive; that is,$A, B \in I \Rightarrow A \cup B \in I$;
(iii)I is hereditary that is,$A \in I, B \subseteq A \Rightarrow B \in I$.

An Ideal $I \subseteq 2^X$ is called non trivial if $I \neq 2^X$. A non trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$.
A non trivial ideal I is maximal if there cannot exist any non trivial ideal $J \neq I$ containing I as a subset.

Definition 1.6. A non empty family of sets $\mathcal{F} \subseteq 2^X$ is said to be filter on X if and only if
(i) $\emptyset \notin \mathcal{F}$;
(ii) for, A, B $\in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$;
(iii) for each A$\in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$. For each ideal I, there is a filter $\mathcal{F}(I)$ corresponding to I. That is,

$$\mathcal{F}(I) = \{K \subseteq N : K^c \in I\}, \quad \text{where} K^c = N - K. \tag{1.10}$$

Definition 1.7. A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I - null if L=0. In this case, we write

$$I - \lim x_{ij} = 0. \tag{1.11}$$

Definition 1.8. A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists numbers $m = m(\epsilon), n = n(\epsilon)$ such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{mn}| \geqslant \epsilon\} \in I. \tag{1.12}$$

Definition 1.9. A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I-bounded if there exists $M > 0$ such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}| > M\}. \tag{1.13}$$

Definition 1.10. A double sequence space E is said to be solid or normal if $x_{ij} \in E$ implies that $(\alpha_{ij} x_{ij}) \in E$ for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$.

Definition 1.11. A double sequence space E is said to be symmetric if $(x_{\pi(i)\pi(j)}) \in E$ whenever $(x_{ij}) \in E$, where $\pi(i)$ and $\pi(j)$ is a permutation on $\mathbb{N}$.

Definition 1.12. A double sequence space E is said to be sequence algebra if $(x_{ij} y_{ij}) \in E$ whenever $(x_{ij}), (y_{ij}) \in E$.

Definition 1.13. A double sequence space E is said to be convergence free if $(y_{ij}) \in E$ whenever $(x_{ij}) \in E$ and $x_{ij} = 0$ implies $y_{ij} = 0$, for all $(i, j) \in \mathbb{N} \times \mathbb{N}$.

Definition 1.14. Let K=$\{(n_i, k_j) : i, j \in \mathbb{N}; n_1 < n_2 < n_3 < ....and\ \ k_1 < k_2 < k_3 < ....\} \subseteq \mathbb{N} \times \mathbb{N}$ and E be a double sequence space.A K-step space of E is a sequence space

$$\lambda_k^E = \{(\alpha_{ij} x_{ij}) : (x_{ij}) \in E\}.$$

Definition 1.15. A cannonical preimage of a sequence $(x_{n_i k_j}) \in E$ is a sequence $(b_{nk}) \in E$ defined as follows

$$b_{n,k} = \begin{cases} a_{n,k}, & \text{for n , k } \in K \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.16. A sequence space E is said to be monotone if it contains the cannonical preimages of all its stepspaces.

Remark. If $I = I_f$, the class of all finite subsets of $\mathbb{N}$. Then I is an admissible ideal in $\mathbb{N}$ and I_f convergence coincides with the usual convergence.

Definition 1.17. If $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. Then I is an admissible ideal in $\mathbb{N}$ and we call the I_δ -convergence as the logarithmic statistical convergence.

Definition 1.18. If $I = I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Then,I is an admissible ideal in $\mathbb{N}$ and we call the I_d-convergence as asymptotic statistical covergence.

Lemma 1.1. *((Tripathy & Hazarika, 2011)). Every solid space is monotone.*

Lemma 1.2. *Let $\mathcal{F}(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.*

Lemma 1.3. *If $I \subseteq 2^N$ and $M \subseteq N$.If $M \notin I$, then $M \cap N \notin I$.*

2. Main Results

Recently (Khan & Khan, 2013) introduced and studied the following sequence space. For m,n$\geqslant 0$

$$ {}_2BV_\sigma^I = \{x = (x_{ij}) \in {}_2\omega : \{(i,j) \in \mathbb{N} \times \mathbb{N} : |\phi_{mnij}(x) - L| \geqslant \epsilon\} \in I, for\ some\ L \in \mathbb{C}\}. \quad (2.1) $$

In this article we introduce the following double sequence spaces. For m,n$\geqslant 0$

$$ {}_2BV_\sigma^I(M) = \{x = (x_{ij}) \in {}_2\omega : I - \lim M(\frac{|\phi_{mnij}(x) - L|}{\rho}) = 0,\ for\ some\ L \in \mathbb{C}, \rho > 0\} \quad (2.2) $$

$$ {}_2({}_0BV_\sigma^I(M)) = \{x = (x_{ij}) \in {}_2\omega : I - \lim M(\frac{|\phi_{mnij}(x)|}{\rho}) = 0, \rho > 0\}, \quad (2.3) $$

$$ {}_2({}_\infty BV_\sigma^I(M)) = \{x = (x_{ij}) \in {}_2\omega : \{(i,j) \in \mathbb{N} \times \mathbb{N}\ \ \exists\, k > 0\ s.t\ M(\frac{|\phi_{mnij}(x)|}{\rho}) \geqslant k\} \in I, \rho > 0\} \quad (2.4) $$

$$ {}_2({}_\infty BV_\sigma(M)) = \{x = (x_{ij}) \in {}_2\omega : \sup M(\frac{|\phi_{mnij}(x)|}{\rho}) < \infty, \rho > 0\}. \quad (2.5) $$

We also denote

$$ {}_2M_{BV_\sigma}^I(M) = {}_2BV_\sigma^I(M) \cap {}_2({}_\infty BV_\sigma(M)) $$

and

$$ {}_2({}_0M_{BV_\sigma}^I(M)) = {}_2({}_0BV_\sigma^I(M)) \cap {}_2({}_\infty BV_\sigma(M)). $$

Theorem 2.1. *For any Orlicz function M, the classes of double sequence ${}_2({}_0BV_\sigma^I(M))$,${}_2BV_\sigma^I(M)$, ${}_2({}_0M_{BV_\sigma}^I(M))$, and ${}_2M_{BV_\sigma}^I(M)$ are linear spaces.*

Proof. Let $x = (x_{ij}), (y_{ij}) \in \ _2BV_\sigma^I(M)$ be any two arbitrary elements, and let α, β are scalars. Now, since $(x_{ij}), (y_{ij}) \in \ _2BV_\sigma^I(M)$. Then this implies that $\exists$ some positive numbers $L_1, L_2 \in \mathbb{C}$ and $\rho_1, \rho_2 > 0$ such that,

$$I - \lim_{i,j} M\Big(\frac{|\phi_{mnij}(x) - L_1|}{\rho_1}\Big) = 0, \tag{2.6}$$

$$I - \lim_{i,j} M\Big(\frac{|\phi_{mnij}(y) - L_2|}{\rho_2}\Big) = 0. \tag{2.7}$$

$\Rightarrow$ for any given $\epsilon > 0$, the sets

$$\Rightarrow \{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{|\phi_{mnij}(x) - L_1|}{\rho_1}\Big) \geqslant \frac{\epsilon}{2}\} \in I, \tag{2.8}$$

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{|\phi_{mnij}(y) - L_2|}{\rho_2}\Big) \geqslant \frac{\epsilon}{2}\} \in I. \tag{2.9}$$

Now let

$$A_1 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{|\phi_{ij}(x) - L_1|}{\rho_1}\Big) < \frac{\epsilon}{2}\} \in I, \tag{2.10}$$

$$A_2 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{|\phi_{ij}(y) - L_2|}{\rho_2}\Big) < \frac{\epsilon}{2}\} \in I. \tag{2.11}$$

be such that $A_1^c, A_2^c \in I$. Let $\rho_3 = max\{2|\alpha|\rho_1\ ,\ 2|\beta|\rho_2\}$
Since M is non decreasing and convex function,we have

$$M(\frac{|\phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3}) = M(\frac{|(\alpha\phi_{mnij}(x) + \beta\phi_{mnij}(y)) - (\alpha L_1 + \beta L_2)|}{\rho_3})$$

$$\begin{aligned} &= M(\tfrac{|\alpha(\phi_{mnij}(x) - L_1) + \beta(\phi_{mnij}(y) - L_2)|}{\rho_3}) \\ &\leqslant M(\tfrac{|\alpha||\phi_{mnij}(x) - L_1|}{\rho_3}) + M(\tfrac{|\beta||\phi_{mnij}(y) - L_2|}{\rho_3}) \\ &\leqslant M(\tfrac{|\alpha||\phi_{mnij}(x) - L_1|}{\rho_1}) + M(\tfrac{|\beta||\phi_{mnij}(y) - L_2|}{\rho_2}) \\ &\leqslant \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \end{aligned}$$

$\Rightarrow \{(i,j) \in \mathbb{N} \times \mathbb{N} : M(\frac{|\phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3}) > \epsilon\} \in I$
implies that , $I - \lim_{i,j} M(\frac{|\phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3}) = 0$.
Thus $\alpha(x_{ij}) + \beta(y_{ij}) \in \ _2BV_\sigma^I(M)$. As (x_{ij}) and (y_{ij}) are two arbitrary element then $\alpha x_{ij} + \beta y_{ij} \in \ _2BV_\sigma^I(M)$ for all $x_{ij},\ y_{ij} \in \ _2BV_\sigma^I(M)$, for all scalars α, β . Hence $_2BV_\sigma^I(M)$ is linear space. The proof for other spaces will follow similarly. □

Theorem 2.2. *Let M_1, M_2 be two Orlicz functions and statisfying $\triangle_2$ condition ,then*
(a)$X(M_2) \subseteq X(M_1 M_2)$
(b)$X(M_1) \cap X(M_2) \subseteq X(M_1 + M_2)$ for $X = \ _2BV_\sigma^I,\ _2(_0BV_\sigma^I),\ _2M_{BV_\sigma}^I,\ _2(_0M_{BV_\sigma}^I)$.

Proof. (a)Let $x = (x_{ij}) \in {}_2({}_0BV^I_\sigma(M_2))$ be an arbitrary element
$\Rightarrow \rho > 0$ such that

$$I - \lim M_2(\frac{|\phi_{mnij}(x)|}{\rho}) = 0. \tag{2.12}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_1(t) < \epsilon$ for $0 < t \leqslant \delta$.
Write $y_{ij} = M_2(\frac{|\phi_{mnij}(x)|}{\rho})$ and
consider,

$$\lim_{ij} M_1(y_{ij}) = \lim_{y_{ij} \leqslant \delta, i,j \in \mathbb{N}} M_1(y_{ij}) + \lim_{y_{ij} > \delta, i,j \in \mathbb{N}} M_1(y_{ij}). \tag{2.13}$$

Now, since M_1 is an Orlicz function so we have $M_1(\lambda x) \leqslant \lambda M_1(x), 0 < \lambda < 1$.
Therefore we have,

$$\lim_{y_{ij} \leqslant \delta,\ i,j \in \mathbb{N}} M_1(y_{ij}) \leqslant M_1(2) \lim_{y_{ij} \leqslant \delta,\ i,j \in \mathbb{N}} (y_{ij}). \tag{2.14}$$

For $y_{ij} > \delta$, we have $y_{ij} < \frac{y_{ij}}{\delta} < 1 + \frac{y_{ij}}{\delta}$. Now, since M_1 is non-decreasing and convex, it follows that,

$$M_1(y_{ij}) < M_1(1 + \frac{y_{ij}}{\delta}) < \frac{1}{2} M_1(2) + \frac{1}{2} M_1(\frac{2y_{ij}}{\delta}). \tag{2.15}$$

Since M_1 satisfies the $\triangle_2$- condition we have,

$$\begin{aligned} M_1(y_{ij}) &< \frac{1}{2} K \frac{y_{ij}}{\delta} M_1(2) + \frac{1}{2} K M_1(\frac{2y_{ij}}{\delta}) \\ &< \frac{1}{2} K \frac{y_{ij}}{\delta} M_1(2) + \frac{1}{2} K \frac{y_{ij}}{\delta} M_1(2) \\ &= K \frac{y_{ij}}{\delta} M_1(2). \end{aligned} \tag{2.16}$$

This implies that,

$$M_1(y_{ij}) < K \frac{y_{ij}}{\delta} M_1(2). \tag{2.17}$$

Hence,we have

$$\lim_{y_{ij} > \delta,\ i,j \in \mathbb{N}} M_1(y_{ij}) \leqslant max\{1, K\delta^{-1} M_1(2) \lim_{y_{ij} > \delta, i,j \in \mathbb{N}} (y_{ij})\}. \tag{2.18}$$

Therefore from (2.12),and (2.13) we have

$$I - \lim_{ij} M_1(y_{ij}) = 0.$$

$$\Rightarrow I - \lim_{ij} M_1 M_2(\frac{|\phi_{mnij}(x)|}{\rho}) = 0.$$

This implies that $x = (x_{ij}) \in {}_2({}_0BV^I_\sigma(M_1 M_2))$.Hence $X(M_2) \subseteq X(M_1 M_2)$ for $X = {}_2({}_0BV^I_\sigma)$. The other cases can be proved in similar way.

(b) Let $x = (x_{ij}) \in {}_2({}_0BV_\sigma^I(M_1)) \cap {}_2({}_0BV_\sigma^I(M_2))$. Let $\epsilon > 0$ be given.Then $\exists \rho > 0$ such that,

$$I - \lim M_1(\frac{|\phi_{mnij}(x)|}{\rho}) = 0, \tag{2.19}$$

and

$$I - \lim M_2(\frac{|\phi_{mnij}(x)|}{\rho}) = 0. \tag{2.20}$$

Therefore

$$I - \lim_{ij}(M_1 + M_2)(\frac{|\phi_{mnij}(x)|}{\rho}) = I - \lim_{ij} M_1(\frac{|\phi_{mnij}(x)|}{\rho}) + I - \lim_{ij} M_2(\frac{|\phi_{mnij}(x)|}{\rho}),$$

from eqs (2.19) and (2.20)

$$\Rightarrow I - \lim_{ij}(M_1 + M_2)(\frac{|\phi_{mnij}(x)|}{\rho}) = 0.$$

we get

$$x = (x_{ij}) \in {}_2({}_0BV_\sigma^I(M_1 + M_2)).$$

Hence we get ${}_2({}_0BV_\sigma^I(M_1)) \cap {}_2({}_0BV_\sigma^I(M_2)) \subseteq {}_2({}_0BV_\sigma^I(M_1 + M_2))$.
For $X = {}_2BV_\sigma^I, {}_2({}_0M_{BV_\sigma}^I), {}_2(M_{BV_\sigma}^I)$ the inclusion are similar. □

Corollary 2.1. *$X \subseteq X(M)$ for $X = {}_2(BV_\sigma^I), {}_2BV_\sigma^I, {}_2({}_0M_{BV_\sigma}^I)$ and ${}_2M_{BV_\sigma}^I$.*

Proof. For this let $M(x) = x$, for all $x = (x_{ij}) \in X$. Let us suppose that $x = (x_{ij}) \in {}_2({}_0BV_\sigma^I)$. Then for any given $\epsilon > 0$ we have

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |\phi_{mnij}(x)| \geqslant \epsilon\} \in I.$$

Now let

$$A_1 = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |\phi_{mnij}(x)| < \epsilon\} \in I,$$

be such that $A_1^c \in I$. Now consider , for $\rho > 0$,

$$M(\frac{|\phi_{mnij}(x)|}{\rho}) = \frac{|\phi_{mnij}(x)|}{\rho}$$
$$< \frac{\epsilon}{\rho} < \epsilon.$$

$\Rightarrow I - \lim M(\frac{|\phi_{mnij}(x)|}{\rho}) = 0$, which implies that $x = (x_{ij}) \in {}_2({}_0BV_\sigma^I(M))$. Hence we have

$${}_2({}_0BV_\sigma^I) \subseteq {}_2({}_0BV_\sigma^I(M)).$$

$$\Rightarrow X \subseteq X(M)$$

and the other cases will be proved similarly. □

Theorem 2.3. *For any Orlicz function M, the spaces* $_2({}_0BV^I_\sigma(M))$ *and* $_2({}_0M^I_{BV_\sigma})$ *are solid and monotone.*

Proof. Here we consider $_2({}_0BV^I_\sigma)$ and for $_2({}_0BV^I_\sigma(M))$ the proof shall be similar.
Let $x = x_{ij} \in {}_2({}_0BV^I_\sigma(M))$ be an arbitrary element ,$\Rightarrow \exists\rho > 0$ such that

$$I - \lim_{ij} M(\frac{|\phi_{mnij}(x)|}{\rho}) = 0.$$

Let α_{ij} be a sequence of scalars with $|\alpha_{ij}| \leqslant 1$ for $i, j \in \mathbb{N}$.
Now, M is an Orlicz function. Therefore

$$M(\frac{|\alpha_{ij}\phi_{mnij}(x)|}{\rho}) = M(\frac{|\alpha_{ij}||\phi_{mnij}(x)|}{\rho})$$
$$\leqslant |\alpha_{ij}|M(\frac{|\phi_{mnij}(x)|}{\rho})$$

$\Rightarrow M(\frac{|\alpha_{ij}\phi_{mnij}(x)|}{\rho}) \leqslant M(\frac{|\phi_{mnij}(x)|}{\rho})$ for all $i, j \in \mathbb{N}$.

$$\Rightarrow I - \lim_{ij} M(\frac{|\alpha_{ij}\phi_{mnij}(x)|}{\rho}) = 0.$$

Thus we have $(\alpha_{ij}x_{ij}) \in {}_2({}_0BV^I_\sigma(M))$. Hence $_2({}_0BV^I_\sigma(M))$ is solid. Therefore $_2({}_0BV^I_\sigma(M))$ is monotone. Since every solid sequence space is monotone. □

Theorem 2.4. *For any Orlicz function M,the space* $_2BV^I_\sigma(M)$ *and* $_2(M_{BV^I_\sigma}(M))$ *are neither solid nor monotone in general.*

Proof. Here we give counter example for establishment of this result. Let $X = {}_2BV^I_\sigma$ and $_2(M_{BV^I_\sigma})$.
Let us consider $I = I_f$ and $M(x) = x$, for all $x = x_{ij} \in [0, \infty)$. Consider,the K-step space $X_K(M)$ of $X(M)$ defined as follows:
Let $x = (x_{ij}) \in X(M)$ and $y = (y_{ij}) \in X_K(M)$ be such that $(y_{ij}) = (x_{ij})$, if i,j is even and $(y_{ij}) = 0$, otherwise.
Consider the sequence (x_{ij}) defined by $(x_{ij}) = 1$ for all $i, j \in \mathbb{N}$. Then $x = (x_{ij}) \in {}_2BV^I_\sigma(M)$ and $_2M_{BV^I_\sigma}(M)$, but K-step space preimage does not belong to $BV^I_\sigma(M)$ and $_2M^I_{BV_\sigma}(M)$. Thus $_2BV^I_\sigma(M)$ and $_2M^I_{BV_\sigma}(M)$ are not monotone and hence they are not solid. □

Theorem 2.5. *For an Orlicz function M, the spaces* $_2BV^I_\sigma(M)$ *and* $_2BV^I_\sigma(M)$ *are sequence algebra.*

Proof. Let $x = (x_{ij}), y = (y_{ij}) \in {}_2({}_0(BV^I_\sigma(M)))$ be any two arbitrary elements. $\Rightarrow \rho_1,\rho_2 > 0$ such that,

$$I - \lim_{ij} M(\frac{|\phi_{mnij}(x)|}{\rho_1}) = 0,$$

and

$$I - lim_{ij}M(\frac{|\phi_{mnij}(y)|}{\rho_2}) = 0.$$

Let $\rho = \rho_1\rho_2 > 0$. Then

$$M(\frac{|\phi_{mnij}(x)\ \phi_{mnij}(y)|}{\rho}) = M(\frac{|\phi_{mnij}(x)\ \phi_{mnij}(y)|}{\rho_1\rho_2})$$

$$\Rightarrow I - \lim_{ij} M(\frac{|\phi_{mnij}(x)\ \phi_{mnij}(y)|}{\rho}) = 0.$$

Therefore we have $(x_{ij}y_{ij}) \in\ _2({}_0BV_\sigma^I(M))$. Hence $_2({}_0BV_\sigma^I(M))$ is sequence algebra. □

Theorem 2.6. *For any Orlicz function M, the spaces* $_2({}_0BV_\sigma^I(M))$ *and* $_2BV_\sigma^I(M)$ *are not convergence free.*

Proof. To show this let $I = I_f$ and $M(x) = x$, for all $x = [0, \infty)$. Now consider the double sequence $(x_{ij}), (y_{ij})$ which defined as follows:

$$x_{ij} = \frac{1}{i+j} and\ \ y_{ij} = i + j, \forall i, j \in \mathbb{N}.$$

Then we have (x_{ij}) belong to both $_2({}_0BV_\sigma^I(M))$ and $_2BV_\sigma^I(M)$, but (y_{ij}) does not belong to $_2({}_0BV_\sigma^I(M))$ and $_2BV_\sigma^I(M)$. Hence, the spaces $_2({}_0BV_\sigma^I(M))$ and $_2BV_\sigma^I(M)$ are not convergence free. □

Theorem 2.7. *Let M be an Orlicz function. Then*

$$_2({}_0BV_\sigma^I(M)) \subseteq\ _2BV_\sigma^I(M) \subseteq\ _2({}_\infty BV_\sigma^I(M)).$$

Proof. For this let us consider $x = (x_{ij}) \in\ _2({}_0BV_\sigma^I(M))$. It is obvious that it must belong to $_2BV_\sigma^I(M)$. Now consider

$$M\Big(\frac{|\phi_{mnij}(x) - L|}{\rho}\Big) \leqslant M(\frac{|\phi_{mnij}(x)|}{\rho}) + M\Big(\frac{|L|}{\rho}\Big).$$

Now taking the limit on both sides we get

$$I - lim_{ij} M(\frac{|\phi_{mnij}(x) - L|}{\rho}) = 0.$$

Hence $x = (x_{ij}) \in\ _2BV_\sigma^I(M)$.
Now it remains to show that $_2(BV_\sigma^I(M)) \subseteq\ _2({}_\infty BV_\sigma^I(M))$. For this let us consider $x = (x_{ij}) \in\ _2BV_\sigma^I(M) \Rightarrow \exists\rho > 0$ s.t

$$I - \lim_{ij} M\Big(\frac{|\phi_{mnij}(x) - L|}{\rho}\Big) = 0.$$

Now consider

$$M\Big(\frac{|\phi_{mnij}(x)|}{\rho}\Big) \leqslant M\Big(\frac{|\phi_{mnij}(x) - L|}{\rho}\Big) + M\Big(\frac{|L|}{\rho}\Big).$$

Now taking the supremum on both sides we get

$$\sup_{ij} M(\frac{|\phi_{mnij}(x)|}{\rho}) < \infty.$$

Hence $x = (x_{ij}) \in\ _2({}_\infty BV_\sigma^I(M))$. □ □

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