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# Practical Algorithmic Optimizations for Finding Maximal Matchings in Induced Subgraphs of Grids and Minimum Cost Perfect Matchings in Bipartite Graphs

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**Abstract**

In this paper we present practical algorithmic optimizations addressing two problems. The first one is concerned with computing a maximal matching in an induced subgraph of a grid graph. For this problem we present a faster sequential algorithm using bit operations and a way of implementing it in a parallel environment. The second problem is concerned with computing minimum cost perfect matchings in bipartite graphs. For this problem we extend the idea behind the Hopcroft-Karp maximum matching algorithm and then we consider a more general situation in which multiple minimum cost perfect matchings need to be computed in the same graph, under certain cost restrictions. We present experimental results for all the proposed optimizations.

*Keywords:* Minimum cost perfect matching, maximal matching, maximum flow, grid graph.
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## 1. Introduction

The problem of computing maximum or maximal matchings in bipartite graphs has been considered many times in the scientific literature. Many of the proposed algorithms use the fact that computing a maximum matching in a bipartite graph is equivalent to computing a maximum flow in a slightly modified graph. Thus, results from the theory of network flows can be used for computing maximum matchings. If only a maximal matching is needed, then simpler greedy-type algorithms can be employed. In this paper we present several practical algorithmic improvements for some of the algorithms used for computing maximal matchings in grid graphs and minimum cost perfect matchings in bipartite graphs.

The rest of this paper is structured as follows. In Section 2 we define the main terms and techniques used in this paper. In Section 3 we discuss related work. In Section 4 we present

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faster algorithms for computing maximal matchings in induced subgraphs of grid graphs based on algorithms which use bit operations. In Section 5 we extend an idea used for computing maximum matchings in bipartite graphs to the computation of minimum cost perfect matchings. The idea consists of using multiple edge-disjoint augmenting paths per iteration in order to reduce the number of iterations of the algorithm. In Section 6 we consider another perfect matching problem. In this problem we are interested in computing a minimum cost perfect matching in a complete bipartite graph under certain restrictions regarding the cost computation. The cost of the matching is considered to be equal to the sum of the costs of the edges from the matching *except* for the cost of the minimum cost edge from the matching (i.e. the minimum cost edge of the matching is considered to have cost 0 when computing the cost of the matching). In Section 7 we present experimental evaluations of all the algorithms discussed in this paper. Finally, in Section 8 we conclude.

## 2. Terms and Definitions

A bipartite graph is a graph whose vertices can be split into two sets $L$ (left) and $R$ (right). We consider the vertices to be numbered from 1 to $|L|$ in the left set and from 1 to $|R|$ in the right set (it is acceptable to have vertices with the same number in the graph, because they will be differentiated based on the set $L$ or $R$ to which they belong). Every edge $(x, y)$ of the graph is between a node $x \in L$ and a node $y \in R$. A matching in a bipartite graph is a set of edges such that no two edges in the set have a common vertex. A maximum matching is a matching of maximum cardinality. A maximal matching is a matching to which no more edges can be added (i.e. all the edges outside of the matching have a common vertex with at least one edge from the matching). A perfect matching is a matching in which every node of the graph is an endpoint of an edge from the matching (such a matching may exist only when $|L| = |R|$).

In order to reduce the maximum matching problem to a maximum flow problem we need to construct a directed graph as follows. We will have a special node $S$ called the *source* and another special node $T$ called the *sink*. We will also keep all the nodes from the given bipartite graph. Each edge $(x, y)$ of the original bipartite graph will be replaced by a directed arc from $x$ to $y$ having capacity 1. We will also add capacity 1 arcs from $S$ to every node $x \in L$ and from every node $y \in R$ to $T$. In case the edges of the bipartite graph have costs these costs are maintained on the directed arcs from the nodes $x \in L$ to the nodes $y \in R$ (we will denote by $c(x, y)$ the cost of the edge between $x \in L$ and $y \in R$). All the arcs having $S$ or $T$ as an endpoint will have cost 0.

One of the best known maximum flow algorithms is the Edmonds-Karp algorithm (Edmonds & Karp, 1972). This algorithm can be summarized as follows: As long as possible find a shortest path from $S$ to $T$ in the residual graph and augment the flow along that path. When arc costs are involved the algorithm can be adjusted in order to find a minimum cost path from $S$ to $T$ in the residual graph. Note that the residual graph may contain negative costs. This version of the Edmonds-Karp algorithm is known as the *successive shortest path* algorithm (Todinov, 2013). A simple breadth-first search algorithm is used for finding a shortest path in the first case (i.e. when edge costs are not involved), while a minimum cost path algorithm needs to be used in the second case (i.e. when edge costs are involved), for instance, Bellman-Ford-Moore (Papaefthymiou & Rodrigue, 1997) or even Dijkstra's algorithm (Todinov, 2013) after modifying the graph's arc costs in order to remove negative costs. Thus, the algorithm consists of multiple iterations, in each of which the flow is increased along a single path. The most time consuming part in each iteration is the traversal of the graph in order to find an augmenting path. In a graph with $V$ vertices and $E$ arcs finding the shortest augmenting path takes $O(V + E)$ time when no costs are involved and $O(V \cdot E)$ time when costs are involved (or $O(V + E \cdot log(V))$ or $O(E + V \cdot log(V))$ time when Dijkstra's algorithm is used on the modified residual graph costs). Then, augmenting the flow along the found path is easy (it takes only $O(V)$ time). In the case of bipartite graphs it is sufficient to find a path from $S$ to an unmatched vertex in $R$ (because this vertex is directly connected to $T$ through an existing arc in the residual graph).

## 3. Related Work

The best algorithm for computing a maximum matching in sparse bipartite graphs is the Hopcroft-Karp algorithm (Hopcroft & Karp, 1973), which has a time complexity of $O(E \cdot \sqrt{V})$ where $V$ is the number of vertices and $E$ is the number of edges of the graph. For dense bipartite graphs the algorithm proposed in (Alt *et al.*, 1991) has a slightly better time complexity of $O(V^{1.5}\sqrt{\frac{E}{log(V)}})$. Both of these algorithms have a better time complexity than the Edmonds-Karp algorithm for finding a maximum flow presented in the previous section. However, due to its simplicity, the Edmonds-Karp algorithm is used in many practical implementations. Moreover, experimental evaluations showed that for some types of bipartite graphs some modified versions of the Edmonds-Karp algorithm (which use breadth-first search from all the source's neighbors for finding augmenting paths) are faster than the Hopcroft-Karp algorithm, despite having a worse theoretical time complexity (Cherkassky *et al.*, 1998).

Edmonds-Karp is not the only algorithm for computing maximum flows in graphs. In fact, many such algorithms were proposed in the scientific literature. Some of the most popular ones are Dinic's algorithm (Dinic, 1970), Karzanov's algorithm (Karzanov, 1974) and the push-relabel maximum flow algorithm (Goldberg & Tarjan, 1986).

A minimum cost perfect bipartite matching can be computed in $O(V^3)$ time using the Hungarian algoritm (Munkres, 1957). The *successive shortest path* algorithm for minimum cost maximum flows can be implemented in $O(V \cdot (E + V \cdot log(V)))$ time in order to compute a minimum cost maximum matching by using Fibonacci heaps (Fredman & Tarjan, 1987). The algorithm consists of $O(V)$ iterations and each iteration runs in $O(E + V \cdot log(V))$ time. Dynamic versions of the minimum cost perfect bipartite matching problem, in which edge costs can be changed, have also been considered (Mills-Tettey *et al.*, 2007).

Maximum matchings can also be computed in general graphs, not just bipartite graphs (see, for instance, Gabow's algorithm (Gabow, 1976), having an $O(V^3)$ time complexity). Minimum cost perfect matchings have also been considered in some special classes of graphs, e.g. graphs induced by points in the plane (Varadarajan, 1998). Greedy algorithms for maximal matchings, including parallel versions, were presented in (Blelloch *et al.*, 2012). The problem of maintaining maximal matchings in dynamic graphs has been addressed in (Neiman & Solomon, 2013).

## 4. Faster Algorithm for Maximal Matchings in Induced Subgraphs of Grid Graphs using Bit Operations

We consider an $M \cdot N$ grid graph in which every node has a coordinate $(x, y)$ ($0 \le x \le N-1$, $0 \le y \le M-1$) and some nodes are missing. The graph is defined by the implicit adjacency structure of the existing nodes (i.e. two nodes at distance 1 in the grid are neighbors). We are interested in computing a maximal matching in this graph. Note that a maximal matching simply implies that no other edge of the graph can be added to the matching and not that the matching has maximum cardinality. Computing a maximum cardinality matching can be done easily, because the graph is bipartite (we can separate the nodes into two groups based on the parity of their sum of $x$ and $y$ coordinates) and there are many polynomial-time maximum matching algorithms in such graphs (see Section 3).

Computing a maximal matching can be achieved faster, in only $O(M \cdot N)$ time. Let's consider the following Greedy algorithm (Algorithm 1) which traverses the grid graph in increasing order of the $y$-coordinate and for each $y$ in inceasing order of the $x$-coordinate.

**Algorithm 1** Greedy $O(M \cdot N)$ Algorithm for Finding a Maximal Matching in an Induced Subgraph of a Grid Graph

```
C = 0 {At the end of the algorithm C will be the size of the maximal matching.}
for y = 0 to M − 1 do
  for x = 0 to N − 1 do
    if node (x, y) exists in the graph then
      if y > 0 and node (x, y − 1) exists in the graph and node (x, y − 1) is not matched then
        Match the nodes (x, y) and (x, y − 1)
        C = C + 1
      else if x > 0 and node (x − 1, y) exists in the graph and node (x − 1, y) is not matched
      then
        Match the nodes (x, y) and (x − 1, y)
        C = C + 1
      end if
    end if
  end for
end for
```

We can implement a faster version of the Algorithm 1 by using bit operations. Note that the presented algorithm will only compute the size of the maximal matching and not the matching itself. The speed increase is due to using bit operations and handling multiple nodes at the same time. We will split each row of the grid graph (corresponding to a $y$-coordinate) into blocks of $B$ bits. Block $i$ of each row contains bits referring to the coordinates $i \cdot B, \ldots, (i+1) \cdot B - 1$. We will denote by $block(y, i)$ the block $i$ of the row corresponding to the coordinate $y$. We will have bit $j$ of $block(y, i)$ set to 1 if the node $(i \cdot B + j, y)$ exists in the graph, and set to 0 otherwise ($0 \le j \le B-1$). We will traverse the graph from $y = 0$ to $y = M - 1$ as in Algorithm 1. During the traversal we will maintain a row of blocks corresponding to the previous row in which 1 bits will correspond to existing unmatched nodes. When considering a new row $y$, the first step is to perform an **AND** between the current row and the previous row of unmatched nodes. All the 1 bits in the result of this operation will represent nodes from the current row which are matched to nodes from the previous row. After performing this match we will clear the matched 1 bits from the current row. The next step is to match nodes from the current row which are adjacent to each other faster than $O(N)$ time. In order to achieve this we will need to use a preprocessing step. For each sequence $S$ of $B$ bits we will compute $MCnt(S)$ that is the number of pairs of adjacent bits matched in $S$ and $MRes(S)$ the $B$-bit sequence containing the remaining unmatched 1 bits of $S$. We will start with $MCnt(S) = 0$ and $MRes(S) = S$. Then we will traverse all the bits $j$ of $S$ from 1 to $B-1$. If $MRes(S)(j) = 1$ and $Mres(S)(j-1) = 1$ then we increase $MCnt(S)$ by 1 and we clear the bits $j$ and $j-1$ in $MRes(S)$. Thus, we can compute $MCnt(S)$ and $MRes(S)$ in $O(B)$ time, obtaining a preprocessing time of $O(2^B \cdot B)$. Within the same time complexity we will also compute for each $B$-bit sequence $S$ the number of 1 bits in $S$, $BCnt(S)$.

With these values we can perfom the matching on the current row $y$. We will consider each block $i$ from 0 to $(N-1)/B$ and we will maintain the state of the current row as a sequence of blocks *crow*. First we copy $block(y, i)$ to $crow(i)$. Then, if $i > 0$ and bit $B-1$ of $crow(i-1)$ is 1 and bit 0 of $crow(i)$ is 1 we match these two bits and then we set them to zero. Afterwards we replace $crow(i)$ by $MRes(crow(i))$. The detailed algorithm is presented in Algorithm 2.

The time complexity of Algorithm 2 is $O(2^B \cdot B + M \cdot N/B)$. The $O(2^B \cdot B)$ term is the time complexity of the preprocessing stage and the $O(M \cdot N/B)$ is the time complexity of the actual algorithm. The presented algorithm can even be implemented in a parallel manner. First of all the preprocessing stage is obviously parallelizable: each of the $2^B$ values of the tables *MRes*, *MCnt* and *BCnt* ca be computed independently. In order to parallelize the actual algorithm we will need to refactor it first. We will first perform all the horizontal matchings on each of the $M$ rows. We can first perform the matching within each block of each row independently in parallel and store the result in a variable specific to each $(row, block)$ pair (this means that we would have such a variable for each block of each row). Then we can handle the matching between bit 0 of odd-numbered blocks and bit $B-1$ of the preceding even-numbered block in parallel, followed by another stage in which we handle the matching between bit 0 of even-numbered blocks and bit $B-1$ of the preceding odd-numbered block in parallel. Then we can handle matchings between nodes on different rows. In order to parallelize this stage we will first consider all the rows corresponding to odd $y$ coordinates being matched to the adjacent row with a smaller and even $y$ coordinate. Obviously, each block of all of these $M/2$ (we consider integer division) rows can be handled independently in parallel. Then we will consider all the rows corresponding to even $y$ coordinates being matched to the adjacent row with a smaller and odd $y$ coordinate. Each block of these $M - M/2$ rows can also be handled independently, in parallel. The parallel algorithm presented here can use up to $2^B$ processors in the preprocessing stage and up to $M \cdot N/B$ processors in the maximal matching computation stage. Note that the result of the parallel version may differ from the result of the sequential algorithm (Algorithm 2) because a different maximal matching will be computed (due to the different order of performing the vertical and horizontal matches).

**Algorithm 2** Greedy $O(2^B \cdot B + M \cdot N/B)$ Algorithm for Finding a Maximal Matching in an Induced Subgraph of a Grid Graph

```
Compute the tables MRes, MCnt and BCnt.
C = 0
prow(i) = 0 (0 ≤ i ≤ (N − 1)/B)
for y = 0 to M − 1 do
   crow(i) ← block(y, i) (0 ≤ i ≤ (N − 1)/B)
   for i = 0 to (N − 1)/B do
      vmatch(i) = crow(i) AND prow(i)
      C = C + BCnt(vmatch(i)))
      crow(i) = crow(i) XOR vmatch(i)
   end for
   C = C + MCnt(crow(0))
   crow(0) = MRes(crow(0))
   for i = 1 to (N − 1)/B do
      if bit B − 1 of crow(i − 1) is 1 and bit 0 of crow(i) is 1 then
         C = C + 1
         Clear bit B − 1 of crow(i − 1) and bit 0 of crow(i).
      end if
      C = C + MCnt(crow(i))
      crow(i) = MRes(crow(i))
   end for
   prow(i) = crow(i) (0 ≤ i ≤ (N − 1)/B)
end for
```

## 5. Using a Maximal Set of Edge-Disjoint Paths for Reducing the Number of Iterations of Minimum Cost Perfect Matching Algorithms

The best algorithm for computing a maximum matching in a sparse bipartite graph is the Hopcroft-Karp algorithm (Hopcroft & Karp, 1973) which has a time complexity of $O(E \cdot \sqrt{V})$. The main idea behind that algorithm is to enhance a standard augmenting path algorithm as follows. After each BFS traversal of the graph in order to find an augmenting path, the matching will not be increased only along one path, but rather along a maximal set of edge-disjoint shortest paths (note that in this case edge-disjoint paths are also vertex-disjoint paths, because they are paths in a shortest path tree; the only common vertex is the source $S$).

The same idea can be used when computing a minimum cost perfect matching. At each iteration of the *successive shortest path* algorithm (Todinov, 2013) we need to find a minimum cost path in the residual graph. Note that the residual graph may have arcs with negative costs, but does not have negative cycles. Thus, we either need to use a shortest path algorithm which supports negative costs (e.g. Bellman-Ford-Moore (Papaefthymiou & Rodrigue, 1997)) or we need to modify the costs in order to obtain non-negative costs only and, thus, use Dijkstra's algorithm (Todinov, 2013).

No matter what shortest path algorithm we use, at the end of the algorithm we have the minimum cost of a path starting at the source and ending at each node $x \in R$. We can sort all these nodes in ascending order of the cost of the minimum cost path to reach them (ignoring the unreachable nodes, if any). Then, rather than only increasing the matching along the minimum cost augmenting path, we can consider these nodes in sorted order. For each node $x$ we trace back its shortest path to the source. If the matching was already augmented at the current iteration along at least one edge of the path, then we ignore node $x$ and we move on to the next one. If the current path does not intersect with any of the paths along which the matching was augmented at the current iteration then we can augment the matching along this path and mark its edges in order to know that no other shortest path containing (some of) these edges can be used for augmenting the matching at the current iteration.

A direct implementation of this modified matching augmentation algorithm takes $O(V^2)$ time per iteration, because there may be $O(V)$ verified paths and each verification may take $O(V)$ time. On the other hand, we cannot guarantee that the matching will be augmented along more than one path. A scenario in which all the paths have the first edge in common (from the source to a vertex $x \in L$) and the minimum cost path has only this edge in common with the other paths is quite possible. Since $O(V^2)$ may be a higher time complexity than that of computing the minimum cost paths, we may end up increasing the running time of the algorithm instead of decreasing it. Thus, we need to reduce the time complexity of the matching augmentation part. This can be achieved as follows. Let's remember that the minimum cost paths are paths in a shortest path tree (where the length of a path is its cost). We will consider the paths in the same order as before and we will consider all the edges to be initially unmarked. If the last edge of the current path is not marked then we will be able to augment the matching along the current path. After augmenting the matching along the current path, let $x \in L$ be the first vertex on the path (after the source). We will traverse the whole subtree of the shortest path tree rooted at $x$ and we will mark all the edges of this subtree. Augmenting the matching along all the possible paths takes at most $O(V)$ time overall (because the paths are edge-disjoint). Marking the edges of the shortest path tree also takes at most $O(V)$ time overall, because there are $O(V)$ edges in the shortest path tree and each edge is marked at most once. Thus, the matching augmentation algorithm takes only $O(V)$ time plus the time needed for sorting the paths in increasing order of their costs (e.g. $O(V \cdot log(V))$ time).

The first version of the algorithm proposed in this section is described in pseudocode in Algorithm 3. The input to the algorithm consists of two maps: *dist*, containing the cost of the shortest path from $S$ to every vertex $x \in R$ (we consider $dist(x) = +\infty$ if the vertex $x$ is not reachable from $S$), and *parent*, containing the *parent* in the shortest path tree for each vertex of the graph. Left-set vertices $x$ are denoted as $(x, L)$ in the algorithm and right-set vertices $y$ are denoted as $(y, R)$. In order to maintain the pseudocode simpler, we will mark the graph vertices instead of the edges (because, as mentioned earlier, in this case the edge-disjoint paths are also vertex-disjoint). The second version of the algorithm is described in pseudocode in Algorithm 4. The input to the Algorithm 4 also consists of the same two maps *dist* and *parent*, together with an extra map, *children*, which contains the children in the shortest path tree of each vertex of the graph.

This algorithm basically augments the matching along a maximal set of edge-disjoint paths (in fact, because they are paths of a shortest path tree, the paths are also vertex-disjoint except for the source vertex). It is important. however, to consider these paths in increasing order of their costs,

**Algorithm 3** Increasing the Matching Along a Maximal Set of Edge-Disjoint Paths - The $O(V^2)$ Algorithm

**Input:** dist, parent.

Set all the vertices of the graph as unmarked.
Sort the vertices $x \in R$ in increasing order of $dist(x)$.
**for** $x \in R$ in increasing order of $dist(x)$ such that $dist(x) < +\infty$ **do**
  $y = (x, R)$, $OK = true$
  **while** $y \neq S$ **and** $OK = true$ **do**
    **if** vertex $y$ is marked **then**
      $OK = false$
    **else**
      $y = parent(y)$
    **end if**
  **end while**
  **if** $OK = true$ **then**
    Increase the matching along the shortest path from $S$ to $(x, R)$ (the reverse of the path can be found by following the parent pointers starting from $(x, R)$).
    $y = (x, R)$
    **while** $y \neq S$ **do**
      Mark vertex $y$ as marked.
      $y = parent(y)$
    **end while**
  **end if**
**end for**

**Algorithm 4** Increasing the Matching Along a Maximal Set of Edge-Disjoint Paths - The $O(V)$ Algorithm

**Input:** dist, parent, children.

Set all the vertices of the graph as unmarked.
Sort the vertices $x \in R$ in increasing order of $dist(x)$.
**for** $x \in R$ in increasing order of $dist(x)$ such that $dist(x) < +\infty$ **do**
  **if** $(x, R)$ is not marked **then**
    Increase the matching along the shortest path from $S$ to $(x, R)$ (the reverse of the path can be found by following the parent pointers starting from $(x, R)$).
    Let $(y, L)$ be the first node on the path from $S$ to $(x, R)$ after $S$. Recursively mark all the vertices located in vertex $(y, L)$'s subtree of the shortest path tree. The *children* map will be used for retrieving for each vertex $v$ the set $children(v)$ that is the set of the shortest path tree children of the vertex $v$.
  **end if**
**end for**

in order to make sure that the residual graph at the next iteration does not contain negative cycles. The reason for which this optimization works is as follows. In a perfect mtaching every vertex has to be matched. When augmenting the matching along a shortest path to a node $x$, even if $x$ is not the right-side node with the minimum cost path, the point is that any future minimum cost path to node $x$ (in any future residual graph) will not have a lower cost than the current shortest path to $x$. So there is no reason for us not to augment the matching along that path, as long as this does not block paths with lower costs along which the matching could have been augmented.

Note that this optimization is not correct in a maximum matching algorithm. It is not correct to augment the matching along a path which does not have the globally minimum cost, because we are not sure if the right side vertex $x$ needs to be in the optimal matching or not. And since vertices added to the matching are never removed by this algorithm, it is possible to make a mistake in this case.

This optimization does not change the theoretical time complexity of the minimum cost perfect matching algorithm, because we cannot provide any extra guarantees regarding the number of augmenting paths per iteration (and, thus, we cannot provide guarantees regarding the reduction in the number of iterations).

## 6. Minimum Cost Perfect Matching With the Minimum Cost Edge Ignored

In this section we consider the following problem. Given a complete bipartite graph with $n$ nodes on the left side and $n$ nodes on the right side and costs on its edges, we want to find a minimum cost perfect matching in which the cost is defined as the sum of the costs of all the edges in the matching except for the cost of the edge with the smallest cost.

A simple method for solving this problem is to iterate over all the edges $(i, j)$ and fix them as the smallest edge in the matching. Then we would compute a (usual) minimum cost maximum matching in the bipartite graph from which left node $i$, right node $j$ and all the edges $(i', j')$ with $c(i', j') < c(i, j)$ (or $c(i', j') = c(i, j)$ and the edge $(i', j')$ was considered before the edge $(i, j)$) are removed. If the maximum matching $M$ has size $n-1$ then we found a potential solution, as follows. The potential solution consists of the $n-1$ edges of the found matching plus the edge $(i, j)$. The cost of the matching (according to the definition used in this section) is equal to the sum of the costs of the $n-1$ edges of $M$. Note that the fixed edge $(i, j)$ is the edge with the minimum cost in the perfect matching (the one whose cost is not considered towards the cost of the matching). Once the edge $(i, j)$ was fixed we needed to minimize the total cost of the other $n-1$ edges of the perfect matching. Moreover, the other edges of the perfect matching needed to have costs which were larger than or equal to $c(i, j)$. The minimum cost maximum matching $M$ contains the $n-1$ edges we were looking for, in case its cardinality is $n-1$. If its cardinality is less than $n-1$ then we can conclude that there is no perfect matching containing the edge $(i, j)$ as the minimum cost edge.

This solution requires the computation of $O(n^2)$ independent minimum cost maximum matchings. The key to obtain a better solution is to notice that the $O(n^2)$ matchings that we need to compute are not totally independent. We will sort all the $n^2$ edges first in ascending cost order and then we will consider them in this order. For the first edge we will compute the minimum cost maximum matching from scratch. Let's assume that we reached the edge $(i, j)$. This time we will not compute the new matching from scratch. Instead, let's consider the matching $M$ obtained for the previous edge in the sorted order. We will remove from $M$ any edge with an endpoint at the left node $i$ or the right node $j$ (if any). Then we will remove from $M$ all the edges with a cost smaller than $c(i, j)$ (or equal to $c(i, j)$ but for which the corresponding edge was considered before the current edge $(i, j)$ in the sorted order). All the other edges of $M$ will be maintained. We will start the (usual) minimum cost maximum matching for the edge $(i, j)$ with all the remaining edges from $M$ as part of the matching. Note that the algorithm may replace some of these edges by other edges. This can happen if the reverse of an edge $(x, y)$ ($x \in L$ and $y \in R$) from $M$ is, at some point, part of the shortest path from $S$ to $T$ in the (new) residual graph. When considering the maximum matching problem as a maximum flow problem, the fact that an edge $(x, y)$ is removed from the matching means that the flow is pushed back along that edge (in order to be redirected somewhere else).

By applying the optimizations from the previous paragraph we expect that the number of iterations required for computing each new minimum cost maximum matching will be significantly reduced.

This problem can also be viewed as a dynamic minimum cost perfect matching problem, in which the edge costs can be modified (for instance, instead of removing edges from the graph we can consider that their cost increased to $+\infty$).

## 7. Experimental Results

We implemented the three optimizations presented in this paper and compared them against their unoptimized versions. All the tests were run on a machine running Windows 7 with an Intel Atom N450 1.66 GHz CPU and 1 GB RAM. All the algorithms were implemented in the C++ programming language and the code was compiled using the G++ compiler version 3.3.1.

First we tested our new algorithm for computing a maximal matching in an induced subgraph of a grid graph. We chose $M = N = 2048$ and we randomly generated the induced subgraph - each point $(x, y)$ ($0 \le x \le N - 1$, $0 \le y \le M - 1$) had an equal probability of being part of the subgraph or not. Thus, each of the tested subgraphs had approximately 50% of the nodes of the full grid graph. We generated 100 subgraphs and ran Algorithm 1 (the unoptimized version) and Algorithm 2 (the optimized version) on each of them. We computed the total running time for all the graphs. Algorithm 1 took 5.3 seconds. For Algorithm 2 we considered two values for $B$: $B = 16$ and $B = 8$. For $B = 16$ the running time was 2.74 sec and for $B = 8$ it was 1.13 sec. Note that in this case we computed the tables $MCnt$, $Mres$ and $BCnt$ each time (i.e. for each of the 100 tests). However, when running the algorithm on multiple tests with the same value of $B$, these tables only need to be computed once, in the beginning. Thus, we changed the algorithm in order to compute these tables only once in the very beginning and not for each of the 100 tests. The new running times were 0.92 sec for $B = 16$ and 1.13 sec for $B = 8$. Note that the running time is unchanged for $B = 8$ because the sizes of the tables are small and the time needed to compute them is negligible compared to the time needed to compute the matching. However, for $B = 16$, when the sizes of the tables increase significantly, it is much better to compute the tables in the beginning and reuse them for each test.

Then we repeated the tests for induced subgraphs of grid graphs containing 75% and 100% of the nodes of a full grid graph. For graphs with 75% of the nodes of a full grid graph the running times of our optimized algorithm were: 1.21 sec for $B = 8$ and 0.93 sec for $B = 16$ (note that we only considered the case when the tables are computed just once). The running time of the unoptimized version was 4.69 sec. When considering the full grid graph we obtained the following running times: 0.90 sec for $B = 8$ and 0.55 sec for $B = 16$ for the optimized version and 3.40 sec for the unoptimized version.

We did not test other values of $B$ because the implementation would become less feasible. For $B > 16$ the sizes of the precomputed tables would become too large. For $B = 8$ and $B = 16$ we were able to make use of existing C/C++ data types (*unsigned char* and *unsigned short int*) in order to store a block. For $B \neq 8$ and $B \neq 16$ (and $B \leq 16$) we cannot exactly fit a block into an existing C/C++ data type.

Second we tested the improvement brought by the use of multiple edge-disjoint paths for augmenting the matching in a minimum cost bipartite perfect matching algorithm. The expected improvement consisted in a reduction of the number of iterations. The standard algorithm would use $n$ iterations where $|L| = |R| = n$. We chose $n = 256$ and we generated 100 complete bipartite graphs. The cost of each edge was chosen to be a random integer between 1 and 10000 (inclusive). The unoptimized algorithm we used was the standard *successive shortest path* algorithm with the Bellman-Ford-Moore algorithm for computing minimum cost paths at each iteration. The optimized algorithm simply included the $O(n^2)$ matching augmentation along multiple edge-disjoint minimum paths described in Section 5. We measured both the total running time and the total number of iterations. The total running time (for all the 100 graphs) and the total number of iterations of the standard algorithm were: 23.17 sec and 25600 iterations. With our optimization the total running time was 2.3 sec and the total number of iterations was 1022. We notice that, even with the most basic implementation of our optimization, the running time was reduced 10 times and the number of iterations was reduced 25 times. Although the running time improvements may not translate directly when other minimum cost path computation algorithms are used (e.g. Dijkstra's algorithm) or when the $O(V + V \cdot log(V))$ matching augmentation optimization is used (instead of the $O(V^2)$ version), the improvement in the number of iterations does not depend on these algorithms and, thus, it is applicable to any implementation of the *successive shortest path* minimum cost bipartite perfect matching algorithm.

Then we considered the same testing scenario, except that the edge costs were chosen as random integers between 1 and 2 (inclusive). The total running time of our optimized algorithm was 2.98 sec and the total number of iterations was 5100. The number of iterations of the standard algorithm remained the same (as expected), but its running time dropped to 13.83 sec.

We also considered the complete bipartite graph with the following costs $c(x, y) = min(x, y)$ ($1 \leq x, y \leq n$) and $n = 256$. In this case our optimization did not reduce the number of iterations at all (due to the special structure of the bipartite graph it was never able to augment the matching along more than one path per iteration). However, the running time with our optimization enabled was almost identical to the unoptimized version. We conclude that our optimization has a great potential for reducing the number of iterations of the *successive shortest path* minimum cost perfect matching algorithm and even in the pathological cases when it cannot reduce the number of iterations, it doesn't cause any significant overhead.

Although there is a large difference between the number of iterations obtained by our optimization in the cases of random complete bipartite graphs and in the case of the specific bipartite graph from the previous paragraph, we did not consider other types of bipartite graphs for testing. Understanding the correlation between the performance of our optimization and the specific properties of the costs of the edges of the bipartite graph is an interesting topic, but we defer its study to a later date, because we feel that this topic is more appropriate for a separate, more experimentally focused, paper.

For the problem presented in Section 6 we tested our optimization of not recomputing the perfect matching from scratch each time. The first minimum cost perfect matching algorithm that we used was the one which contained our matching augmentation optimization presented in Section 5 and tested earlier. We generated 10 bipartite graphs with $n = 128$ and edge costs randomly selected between 1 and 10000 (inclusive). We computed the total execution time and the total number of iterations of the *successive shortest path* algorithm. When the matching was computed from scratch each time ($O(n^2)$ times) the total running time was 805 sec and the total number of iterations was 1305200. When we applied our optimization from Section 6, the total running time was 221 sec and the total number of iterations was 244886. When using the standard *successive shortest path* algorithm in order to compute a minimum cost perfect matching (and not using our optimization of not recomputing the matching from scratch each time), the total running time was 5798 sec and the total number of iterations was 20610157. When we applied our optimization from Section 6 and also used the standard minimum cost perfect matching algorithm the total running time was 232 sec and the total number of iterations was 251789. We can see that our optimization of not recomputing each minimum cost perfect matching from scratch is very effective. When combined with the optimization presented in Section 5, of augmenting the matching along multiple paths at each iteration, we obtained the best results. However, even when just the standard *successive shortest path* minimum cost perfect matching algorithm is used in conjunction with our optimization from Section 6 the improvements over the naive unoptimized version are significant. Nevertheless, more tests may need to be performed in the future in order to understand sufficiently well how good our proposed optimization really is.

## 8. Conclusions

In this paper we presented three practical algorithmic optimizations addressing problems like computing maximal matchings in induced subgraphs of grid graphs or computing minimum cost perfect matchings in bipartite graphs (under certain restrictions). The proposed optimizations were evaluated experimentally and compared against the unoptimized algorithms. The execution time was significantly reduced in each case, thus proving the validity and effectiveness of our optimizations.

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# Reich Type Contractions on Cone Rectangular Metric Spaces Endowed with a Graph

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## Abstract

In this paper we prove some fixed point theorems for Reich type contractions on cone rectangular metric spaces endowed with a graph without assuming the normality of cone. The results of this paper extends and generalize several known results from metric, rectangular metric, cone metric and cone rectangular metric spaces in cone rectangular metric spaces endowed with a graph. Some examples are given which illustrate the results.

*Keywords:* Graph, cone rectangular metric space, Reich type contraction, fixed point.
*2010 MSC:* 47H10, 54H25.

## 1. Introduction

In 1906, the French mathematician M. Fréchet Fréchet (1906) introduced the concept of metric spaces. After the work of Fréchet several authors generalized the concept of metric space by applying the conditions on metric function. In this sequel, Branciari Branciari (2000) introduced a class of generalized (rectangular) metric spaces by replacing triangular inequality of metric spaces by similar one which involves four or more points instead of three and improved Banach contraction principle Banach (1922) in such spaces. The result of Branciari is generalized and extended by several authors (see, for example, Flora *et al.* (2009); Bari & Vetro (2012); Chen (2012); Işik & Turkoglu (2013); Lakzian & Samet (2012); Arshad *et al.* (2013); Malhotra *et al.* (2013*a*,*b*) and the references therein).

Let $(X, d)$ be a metric space and $T: X \to X$ be a mapping. Then $T$ is called a Banach contraction if there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \lambda d(x, y) \quad \text{for all} \quad x, y \in X. \tag{1.1}$$

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Banach contraction principle ensures the existence of a unique fixed point of a Banach contraction on a complete metric space.

Kannan Kannan (1968) introduced the following contractive condition: there exists $\lambda \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)] \quad \text{for all} \quad x, y \in X. \tag{1.2}$$

Reich Reich (1971) introduced the following contractive condition: there exist nonnegative constants $\lambda, \mu, \delta$ such that $\lambda + \mu + \delta < 1$ and

$$d(Tx, Ty) \leq \lambda d(x, y) + \mu d(x, Tx) + \delta d(y, Ty) \quad \text{for all} \quad x, y \in X. \tag{1.3}$$

Examples show that (see Kannan (1968); Reich (1971)) the conditions of Banach and Kannan are independent of each other while the condition of Reich is a proper generalization of conditions of Banach and Kannan.

On the other hand, the study of abstract spaces and the vector-valued spaces can be seen in Kurepa (1934, 1987); Rzepecki (1980); Lin (1987); Zabreǐko (1997). L.G. Huang and X. Zhang Huang & Zhang (2007) reintroduced such spaces under the name of cone metric spaces and generalized the concept of a metric space, replacing the set of real numbers, by an ordered Banach space. After the work of Huang and Zhang Huang & Zhang (2007), Azam et al. Azam *et al.* (2009) introduced the notion of cone rectangular metric spaces and proved fixed point result for Banach type contraction in cone rectangular space. Malhotra et al. Malhotra *et al.* (2013*b*) generalized the result of Azam et al. Azam *et al.* (2009) in ordered cone rectangular metric spaces and proved some fixed point results for ordered Reich type contractions.

Recently, Jachymski Jachymski (2007) improved the Banach contraction principle for mappings on a metric space endowed with a graph. Jachymski Jachymski (2007) showed that the results of Ran and Reurings Ran & Reurings (2004) and Edelstein Edelstein (1961) can be derived by the results of Jachymski (2007). The results of Jachymski Jachymski (2007) was generalized by several authors (see, for example, Bojor (2012); Chifu & Petrusel (2012); Samreen & Kamran (2013); Asl *et al.* (2013); Abbas & Nazir (2013) and the references therein).

The fixed point results in cone rectangular metric spaces (also in rectangular metric spaces) endowed with a graph are not considered yet. In this paper, we prove some fixed point theorems for Reich type contractions on the cone rectangular metric spaces endowed with a graph. Our results extend the result of Jachymski Jachymski (2007) and the result of Malhotra et al. Malhotra *et al.* (2013*b*) into the cone rectangular metric spaces endowed with a graph. Some examples are provided which illustrate the results.

## 2. Preliminaries

First we recall some definitions about the cone rectangular metric spaces and graphs.

**Definition 2.1.** Huang & Zhang (2007) Let $E$ be a real Banach space and $P$ be a subset of $E$. The set $P$ is called a cone if:
(i) $P$ is closed, nonempty and $P \neq \{\theta\}$, here $\theta$ is the zero vector of $E$;
(ii) $a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.
Given a cone $P \subset E$, we define a partial ordering " $\preceq$ " with respect to $P$ by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$. While $x \ll y$ if and only if $y - x \in P^0$, where $P^0$ denotes the interior of $P$.
Let $P$ be a cone in a real Banach space $E$, then $P$ is called normal, if there exist a constant $K > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number $K$ satisfying the above inequality is called the normal constant of $P$.

**Definition 2.2.** Huang & Zhang (2007) Let $X$ be a nonempty set, $E$ be a real Banach space. Suppose that the mapping $d : X \times X \to E$ satisfies:

(i) $\theta \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$, and $(X, d)$ is called a cone metric space. In the following we always suppose that $E$ is a real Banach space, $P$ is a solid cone in $E$, i.e., $P^0 \neq \phi$ and " $\preceq$ " is partial ordering with respect to $P$.

For examples and basic properties of normal and non-normal cones and cone metric spaces we refer Huang & Zhang (2007) and Rzepecki (1980).
The following remark will be useful in sequel.

*Remark.* Jungck *et al.* (2009) Let $P$ be a cone in a real Banach space $E$, and $a, b, c \in P$, then:

(a) If $a \preceq b$ and $b \ll c$ then $a \ll c$.
(b) If $a \ll b$ and $b \ll c$ then $a \ll c$.
(c) If $\theta \preceq u \ll c$ for each $c \in P^0$ then $u = \theta$.
(d) If $c \in P^0$ and $a_n \to \theta$ then there exist $n_0 \in \mathbb{N}$ such that, for all $n > n_0$ we have $a_n \ll c$.
(e) If $\theta \preceq a_n \preceq b_n$ for each $n$ and $a_n \to a,\ b_n \to b$ then $a \preceq b$.
(f) If $a \preceq \lambda a$ where $0 \leq \lambda < 1$ then $a = \theta$.

**Definition 2.3.** Azam *et al.* (2009) Let $X$ be a nonempty set. Suppose the mapping $d : X \times X \to E$, satisfies:

(i) $\theta \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, w) + d(w, z) + d(z, y)$ for all $x, y \in X$ and for all distinct points $w, z \in X - \{x, y\}$ [rectangular property].

Then $d$ is called a cone rectangular metric on $X$, and $(X, d)$ is called a cone rectangular metric space. Let $\{x_n\}$ be a sequence in $(X, d)$ and $x \in (X, d)$. If for every $c \in E$, with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to $x$ and $x$ is the limit of $\{x_n\}$. We denote this by $\lim_n x_n = x$ or $x_n \to x$, as $n \to \infty$. If for every $c \in E$ with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and $m \in \mathbb{N}$ we have $d(x_n, x_{n+m}) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in $(X, d)$. If every Cauchy sequence is convergent in $(X, d)$, then $(X, d)$ is called a complete cone rectangular metric space. If the underlying cone is normal then $(X, d)$ is called normal cone rectangular metric space.

The concept of cone metric space is more general than that of a metric space, because each metric space is a cone metric space with $E = \mathbb{R}$ and $P = [0, +\infty)$.

**Example 2.1.** Let $X = \mathbb{N}, E = \mathbb{R}^2$, $\alpha, \beta > 0$ and $P = \{(x, y) : x, y \geq 0\}$.
Define $d : X \times X \to E$ as follows:

$$d(x,y) = \begin{cases} (0,0) & \text{if } x = y, \\ 3(\alpha,\beta) & \text{if } x \text{ and } y \text{ are in } \{1,2\}, x \neq y, \\ (\alpha,\beta) & \text{otherwise.} \end{cases}$$

Now $(X, d)$ is a cone rectangular metric space but $(X, d)$ is not a cone metric space because it lacks the triangular property:

$$3(\alpha,\beta) = d(1,2) \succ d(1,3) + d(3,2) = (\alpha,\beta) + (\alpha,\beta) = 2(\alpha,\beta),$$

as $3(\alpha,\beta) - 2(\alpha,\beta) = (\alpha,\beta) \in P$.

Note that in above example $(X, d)$ is a normal cone rectangular metric space. Following is an example of non-normal cone rectangular metric space.

**Example 2.2.** Let $X = \mathbb{N}, E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ for } t \in [0, 1]\}$. Then this cone is not normal (see Rezapour & Hamlbarani (2008)).
Define $d : X \times X \to E$ as follows:

$$d(x,y) = \begin{cases} \theta & \text{if } x = y, \\ 3e^t & \text{if } x, y \in \{1,2\}, x \neq y, \\ e^t & \text{otherwise.} \end{cases}$$

Then $(X, d)$ is non-normal cone rectangular metric space but $(X, d)$ is not a cone metric space because it lacks the triangular property.

Now we recall some basic notions from graph theory which we need subsequently (see also Jachymski (2007)).

Let $X$ be a nonempty set and $\Delta$ denote the diagonal of the cartesian product $X \times X$. Consider a directed graph $G$ such that the set $V(G)$ of its vertices coincides with $X$, and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. We assume $G$ has no parallel edges, so we can identify $G$ with the pair $(V(G), E(G))$. Moreover, we may treat $G$ as a weighted graph by assigning to each edge the rectangular distance between its vertices.

By $G^{-1}$ we denote the conversion of a graph $G$, that is, the graph obtained from $G$ by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

The letter $\widetilde{G}$ denotes the undirected graph obtained from $G$ by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}). \tag{2.1}$$

If $x$ and $y$ are vertices in a graph $G$, then a path in $G$ from $x$ to $y$ of length $l$ is a sequence $(x_i)_{i=0}^{l}$ of $l+1$ vertices such that $x_0 = x, x_l = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, l$. A graph $G$ is called connected if there is a path between any two vertices of $G$. $G$ is weakly connected if $\widetilde{G}$ is connected.

Throughout this paper we assume that $X$ is nonempty set, $G$ is a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$.

Now we define the G-Reich contractions in a cone rectangular metric space.

**Definition 2.4.** Let $(X, d)$ be a cone rectangular metric space endowed with a graph $G$. A mapping $T \colon X \to X$ is said to be a G-Reich contraction if:

(GR1) $T$ is edge preserving, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;

(GR2) there exist nonnegative constants $\lambda, \mu, \delta$ such that $\lambda + \mu + \delta < 1$ and

$$d(Tx, Ty) \preceq \lambda d(x, y) + \mu d(x, Tx) + \delta d(y, Ty) \tag{2.2}$$

for all $x, y \in X$ with $(x, y) \in E(G)$.

An obvious consequence of symmetry of $d(\cdot, \cdot)$ and (2.1) is the following remark.

*Remark.* If $T$ is a $G$-Reich contraction then it is both a $G^{-1}$-Reich contraction and a $\widetilde{G}$-Reich contraction.

**Example 2.3.** Any constant function $T \colon X \to X$ defined by $Tx = c$, where $c \in X$ is fixed, is a $G$-Reich contraction since $E(G)$ contains all the loops.

**Example 2.4.** Any Reich contraction on a $X$ is a $G_0$-Reich contraction, where $E(G_0) = X \times X$.

**Example 2.5.** Let $(X, d)$ be a cone rectangular metric space, $\sqsubseteq$ a partial order on $X$ and $T \colon X \to X$ be an ordered Reich contraction (see Malhotra *et al.* (2013*b*)), that is, there exist nonnegative constants $\lambda, \mu, \delta$ such that $\lambda + \mu + \delta < 1$ and

$$d(Tx, Ty) \preceq \lambda d(x, y) + \mu d(x, Tx) + \delta d(y, Ty)$$

for all $x, y \in X$ with $x \sqsubseteq y$,. Then $T$ is a $G_1$-Reich contraction, where $E(G_1) = \{(x, y) \in X \times X \colon x \sqsubseteq y\}$.

**Definition 2.5.** Let $(X, d)$ be a cone rectangular metric space and $T \colon X \to X$ be a mapping. Then for $x_0 \in X$, a Picard sequence with initial value $x_0$ is defined by $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. The mapping $T$ is called a Picard operator on $X$ if $T$ has a unique fixed point in $X$ and for all $x_0 \in X$ the Picard sequence $\{x_n\}$ with initial value $x_0$ converges to the fixed point of $T$. The mapping $T$ is called weakly Picard operator, if for any $x_0 \in X$, the limit of Picard sequence $\{x_n\}$ with initial value $x_0$, that is, $\lim_{n\to\infty} x_n$ exits (it may depend on $x_0$) and it is a fixed point of $T$.

Now we can state our main results.

## 3. Main results

Let $(X, d)$ be a cone rectangular metric space, and $G$ be a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$. The set of all fixed points of a self mapping $T$ of $X$ is denoted by Fix$T$, that is, Fix$T = \{x \in X \colon Tx = x\}$ and the set of all periodic points of $T$ is denoted by $P(T)$, that is, $P(T) = \{x \in X \colon T^n x = x, \text{ for some } n \in \mathbb{N}\}$. Also we use the notation $X_T = \{x \in X \colon (x, Tx), (x, T^2 x) \in E(G)\}$. $(X, d)$ is said to have the property (P) if:

$$\begin{aligned} &\text{whenever a sequence } \{x_n\} \text{ in } X \text{ converges to } x \text{ with } (x_n, x_{n+1}) \in E(G) \text{ for all } n \in \mathbb{N},\\ &\text{then there is a subsequence } \{x_{k_n}\} \text{ with } (x_{k_n}, x) \in E(G) \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{P}$$

**Proposition 3.1.** *Let $(X, d)$ be a cone rectangular metric space endowed with a graph $G$ and $T \colon X \to X$ be a $G$-Reich contraction. Then, if $x, y \in$ Fix$T$ are such $(x, y) \in E(G)$ then $x = y$.*

*Proof.* Let $x, y \in$ Fix$T$ and $(x, y) \in E(G)$, then by (GR2) we have

$$\begin{aligned} d(x, y) &= d(Tx, Ty)\\ &\preceq \lambda d(x, y) + \mu d(x, Tx) + \delta d(y, Ty)\\ &= \lambda d(x, y) + \mu d(x, x) + \delta d(y, y) = \lambda d(x, y). \end{aligned}$$

As $\lambda < 1$, by (f) of Remark 2, we have $d(x, y) = \theta$, that is, $x = y$. □

**Theorem 3.1.** *Let $(X, d)$ be a cone rectangular metric space endowed with a graph $G$. Let $T \colon X \to X$ be a $G$-Reich contraction. Then for every $x_0 \in X_T$ the Picard sequence $\{x_n\}$, is a Cauchy sequence.*

*Proof.* Let $x_0 \in X_T$ and define the iterative sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. Since $x_0 \in X_T$ we have $(x_0, Tx_0) \in E(G)$ and $T$ is a $G$-Reich contraction, by (GR1) we have $(Tx_0, T^2 x_0) = (x_1, x_2) \in E(G)$. By induction we obtain $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$.

Now since $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$ by (GR2) we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n)\\ &\preceq \lambda d(x_{n-1}, x_n) + \mu d(x_{n-1}, Tx_{n-1}) + \delta d(x_n, Tx_n)\\ &= \lambda d(x_{n-1}, x_n) + \mu d(x_{n-1}, x_n) + \delta d(x_n, x_{n+1}), \end{aligned}$$

that is,

$$d(x_n, x_{n+1}) \preceq \frac{\lambda + \mu}{1 - \delta} d(x_{n-1}, x_n) = \alpha d(x_{n-1}, x_n),$$

where $\alpha = \dfrac{\lambda + \mu}{1 - \delta} < 1$ (as $\lambda + \mu + \delta < 1$). Setting $d_n = d(x_n, x_{n+1})$ for all $n \geq 0$, we obtain by induction that

$$d_n \preceq \alpha^n d_0 \text{ for all } n \in \mathbb{N}. \tag{3.1}$$

Note that, if $x_0 \in P(T)$ then there exists $k \in \mathbb{N}$ such that $T^k x_0 = x_k = x_0$ and by (3.1) we have

$$d_0 = d(x_0, x_1) = d(x_0, Tx_0) = d(x_k, Tx_k) = d(x_k, x_{k+1}) \preceq \alpha^k d(x_0, x_1) = \alpha^k d_0.$$

Since $\lambda \in [0, 1)$ the above inequality yields a contradiction. Thus, we can assume that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$.
As $x_0 \in X_T$ we have $(x_0, T^2x_0) = (x_0, x_2) \in E(G)$ and by (GR1) we obtain $(Tx_0, Tx_2) = (x_1, x_3) \in E(G)$. By induction we obtain $(x_n, x_{n+2}) \in E(G)$ for all $n \geq 0$. Therefore it follows from (GR2) that

$$\begin{aligned} d(x_n, x_{n+2}) &\preceq d(Tx_{n-1}, Tx_{n+1}) \\ &\preceq \lambda d(x_{n-1}, x_{n+1}) + \mu d(x_{n-1}, Tx_{n-1}) + \delta d(x_{n+1}, Tx_{n+1}) \\ &\preceq \lambda[d(x_{n-1}, x_n) + d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+1})] + \mu d(x_{n-1}, x_n) \\ &\quad + \delta d(x_{n+1}, x_{n+2}), \end{aligned}$$

that is,

$$d(x_n, x_{n+2}) \preceq \frac{\lambda + \mu}{1 - \lambda} d_{n-1} + \frac{\lambda + \delta}{1 - \lambda} d_{n+1}$$

which together with (3.1) yields

$$\begin{aligned} d(x_n, x_{n+2}) &\preceq \frac{\lambda + \mu + [\lambda + \delta]\alpha^2}{1 - \lambda} \alpha^{n-1} d_0 \\ &\preceq \frac{2\lambda + \mu + \delta}{1 - \lambda} \alpha^{n-1} d_0, \end{aligned}$$

that is,

$$d(x_n, x_{n+2}) \preceq \beta\alpha^{n-1} d_0, \tag{3.2}$$

where $\beta = \frac{2\lambda+\mu+\delta}{1-\lambda} \succeq 0$. We shall show that the sequence $\{x_n\}$ is a Cauchy sequence.
We consider the value of $d(x_n, x_{n+p})$ in two cases.
If $p$ is odd, say $2m + 1$, then using rectangular inequality and (3.1) we obtain

$$\begin{aligned} d(x_n, x_{n+2m+1}) &\preceq d(x_{n+2m}, x_{n+2m+1}) + d(x_{n+2m-1}, x_{n+2m}) + d(x_n, x_{n+2m-1}) \\ &= d_{n+2m} + d_{n+2m-1} + d(x_n, x_{n+2m-1}) \\ &\preceq d_{n+2m} + d_{n+2m-1} + d_{n+2m-2} + d_{n+2m-3} + \cdots + d_n \\ &\preceq \alpha^{n+2m} d_0 + \alpha^{n+2m-1} d_0 + \alpha^{n+2m-2} d_0 + \cdots + \alpha^n d_0, \end{aligned}$$

that is,

$$d(x_n, x_{n+2m+1}) \preceq \frac{\alpha^n}{1 - \alpha} d_0. \tag{3.3}$$

If $p$ is even, say $2m$, then using rectangular inequality, (3.1) and (3.2) we obtain

$$\begin{aligned} d(x_n, x_{n+2m}) &\preceq d(x_{n+2m-1}, x_{n+2m}) + d(x_{n+2m-1}, x_{n+2m-2}) + d(x_n, x_{n+2m-2}) \\ &= d_{n+2m-1} + d_{n+2m-2} + d(x_n, x_{n+2m-2}) \\ &\preceq d_{n+2m-1} + d_{n+2m-2} + d_{n+2m-3} + \cdots + d_{n+2} + d(x_n, x_{n+2}) \\ &\preceq \alpha^{n+2m-1} d_0 + \alpha^{n+2m-2} d_0 + \alpha^{n+2m-3} d_0 + \cdots + \alpha^{n+2} d_0 + \beta\alpha^{n-1} d_0, \end{aligned}$$

that is,

$$d(x_n, x_{n+2m}) \preceq \frac{\alpha^n}{1 - \alpha} d_0 + \beta\alpha^{n-1} d_0. \tag{3.4}$$

Since $\beta \geq 0$ and $\alpha < 1$, we have $\frac{\alpha^n}{1-\alpha}d_0, \beta\alpha^{n-1}d_0 \to \theta$ as $n \to \infty$ so it follows from (3.3), (3.4) and (a), (d) of Remark 2 that: for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_{n+p}) \ll c \quad \text{for all} \quad p \in \mathbb{N}.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. □

**Theorem 3.2.** *Let $(X, d)$ be a complete cone rectangular metric space endowed with a graph $G$ and has the property $(P)$. Let $T : X \to X$ be a G-Reich contraction such that $X_T \neq \emptyset$, then $T$ is a weakly Picard operator.*

*Proof.* If $X_T \neq \emptyset$ then let $x_0 \in X_T$. By Theorem 3.1, the Picard sequence $\{x_n\}$, where $x_n = T^{n-1}x_0$ for all $n \in \mathbb{N}$, is a Cauchy sequence in $X$. Since $X$ is complete, there exists $u \in X$ such that

$$x_n \to u \quad \text{as} \quad n \to \infty. \tag{3.5}$$

We shall show that $u$ is a fixed point of $T$. By Theorem 3.1 we have $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$, $d_n \preceq d(x_n, x_{n+1}) \preceq \alpha^n d_0$, where $\alpha = \frac{\lambda+\mu}{1-\delta} < 1$ and by the property (P) there exists a subsequence $\{x_{k_n}\}$ such that $(x_{k_n}, u) \in E(G)$ for all $n \in \mathbb{N}$. Also, we can assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. So, using (2.2) we have

$$\begin{aligned}
d(u, Tu) &\preceq d(u, x_{k_n}) + d(x_{k_n}, x_{k_n+1}) + d(x_{k_n+1}, Tu) \\
&= d(u, x_{k_n}) + d_{k_n} + d(Tx_{k_n}, Tu) \\
&\preceq d(u, x_{k_n}) + \alpha^{k_n}d_0 + \lambda d(x_{k_n}, u) + \mu d(x_{k_n}, Tx_{k_n}) + \delta d(u, Tu) \\
&\preceq (1 + \lambda)d(u, x_{k_n}) + (1 + \mu)\alpha^{k_n}d_0 + \delta d(u, Tu),
\end{aligned}$$

that is,

$$d(u, Tu) \preceq \frac{1 + \lambda}{1 - \delta}d(x_{k_n}, u) + \frac{1 + \mu}{1 - \delta}\alpha^{k_n}d_0 \tag{3.6}$$

Since $\alpha^{k_n}d_0 \to \theta$, $x_n \to u$ as $n \to \infty$ we can choose $n_0 \in \mathbb{N}$ such that, for every $c \in E$ with $\theta \ll c$ we have $d(x_{k_n}, u) \ll \dfrac{1 - \delta}{2(1 + \lambda)}c$ and $\alpha^{k_n}d_0 \ll \dfrac{1 - \delta}{2(1 + \mu)}c$ for all $n > n_0$. Therefore, it follows from (3.6) that: for every $c \in E$ with $\theta \ll c$ we have

$$d(u, Tu) \ll c \quad \text{for all} \quad n > n_0.$$

So, by (c) of Remark 2, we have $d(u, Tu) = \theta$, that is, $Tu = u$ theerefore $u \in \text{Fix}T$. Thus $T$ is a weakly Picard operator. □

In the above theorem the mapping $T$ is not necessarily a Picard operator. Indeed, such mapping $T$ may has infinitely many fixed points. Following example verifies this fact.

**Example 3.1.** Let $X = \mathbb{N} = \bigcup\limits_{k\in\mathbb{N}_0} N_k$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $N_k = \left\{2^k(2n - 1) : n \in \mathbb{N}\right\}$ for all $k \in \mathbb{N}_0$. Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ for } t \in [0, 1]\}$. Let $d : X \times X \to E$ be defined by

$$d(x, y) = \begin{cases} \theta & \text{if } x = y, \\ 3e^t & \text{if } x, y \in \{1, 2\}, x \neq y, \\ e^t & \text{otherwise.} \end{cases}$$

Then $(X, d)$ is a cone rectangular metric space endowed with graph $G$, where

$$E(G) = \Delta \bigcup_{k \in \mathbb{N}_0 \setminus \{1\}} (N_k \times N_k) \bigcup \{(1, x) : x \in N_1\}.$$

Note that $(X, d)$ is not a cone metric space. Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} 2^k, & \text{if } x \in N_k, k \in \mathbb{N}_0 \setminus \{1\}; \\ 6, & \text{if } x = 2; \\ 1, & \text{if } x \in N_1 \setminus \{2\}. \end{cases}$$

Then it is easy to see that $T$ is a $G$-Reich contraction with $\lambda \in [1/3, 1), \mu = \delta = 0$. All the conditions of Theorem 3.2 are satisfied and $T$ has infinitely many fixed points, precisely Fix$T = \left\{2^k : k \in \mathbb{N}_0 \setminus \{1\}\right\}$, therefore $T$ is not a Picard operator but weakly Picard operator. Note that, if a Reich contraction on a cone rectangular metric space has a fixed point then it is unique therefore $T$ is not a Reich contraction in $(X, d)$ since Fix$T$ is not singleton.

*Remark.* Unlike from Reich contraction, the above example shows that there may be more than one fixed points of a $G$-Reich contraction in a cone rectangular metric space and therefore a $G$-Reich contraction in a cone rectangular space need not be a Picard operator.

In following theorem we give a necessary and sufficient condition for $T$ to be a Picard operator.

**Theorem 3.3.** *Let $(X, d)$ be a complete cone rectangular metric space endowed with a graph $G$ and has the property $(P)$. Let $T : X \to X$ be a $G$-Reich contraction such that $X_T \neq \emptyset$, then $T$ is a weakly Picard operator. Furthermore, the subgraph $G_{Fix}$ defined by $V(G_{Fix}) = FixT$ is weakly connected if and only if $T$ is a Picard operator.*

*Proof.* The existence of fixed point follows from Theorem 3.2. Let $u, v \in$ Fix$T$, then since $G_{\text{Fix}}$ is weakly connected there exists a path $(x_i)_{i=0}^{l}$ in $G_{\text{Fix}}$ from $u$ to $v$, that is, $x_0 = u, x_l = v$ and $(x_{i-1}, x_i) \in E(G_{\text{Fix}})$ for $i = 1, 2, \ldots, l$. Therefore by Proposition 3.1 and Remark 2 we obtain $u = v$. Thus, fixed point is unique and $T$ is a Picard operator. □

*Remark.* In Jachymski (2007), for $T$ to be a Picard operator Jachymski assumed that $G$ must be weakly connected. From the above theorem it is clear that for $T$ to be a Picard operator it is sufficient to take that Fix$T$ is weakly connected. Next example will illustrate this fact.

**Example 3.2.** Let $X = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$, $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ for } t \in [0, 1]\}$. Let $d : X \times X \to E$ be defined by

$$d\left(\frac{1}{2}, \frac{1}{3}\right) = d\left(\frac{1}{4}, \frac{1}{5}\right) = \frac{3}{10}e^t, \qquad d\left(\frac{1}{2}, \frac{1}{5}\right) = d\left(\frac{1}{3}, \frac{1}{4}\right) = \frac{1}{5}e^t,$$

$$d\left(\frac{1}{2}, \frac{1}{4}\right) = d\left(\frac{1}{5}, \frac{1}{3}\right) = \frac{3}{5}e^t, \qquad d(x, x) = \theta = 0 \quad \text{for all} \quad x \in X,$$

$$d\left(1, \frac{1}{n}\right) = \frac{n-1}{n}e^t \text{ for } n = 2, 3, 4, 5, \quad d(x, y) = d(y, x) \quad \text{for all} \quad x, y \in X,$$

Then $(X, d)$ is a cone rectangular metric space endowed with graph $G$, where

$$E(G) = \Delta \cup \left\{ \left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{1}{3}\right) \right\}.$$

Note that $(X, d)$ is not a cone metric space. Define $T: X \to X$ by

$$Tx = \begin{cases} \frac{1}{2}, & \text{if } x = \frac{1}{2}, \frac{1}{5}; \\ \frac{1}{5}, & \text{if } x = \frac{1}{3}; \\ 1, & \text{if } x = \frac{1}{4}; \\ \frac{1}{4}, & \text{if } x = 1. \end{cases}$$

Then $T$ is a $G$-Reich contraction with $\lambda \in \left[\frac{2}{3}, 1\right), \mu = \delta = 0$. All the conditions of Theorem 3.3 are satisfied and $T$ is a Picard operator and Fix$T = \left\{\frac{1}{2}\right\}$. Note that the graph $G$ is not weakly connected. Indeed, there is no path from 1 to $\frac{1}{n}$ or form $\frac{1}{n}$ to 1 for all $n = 2, 3, 4, 5$. Also, one can see that $T$ is neither a Reich contraction in cone rectangular metric space $(X, d)$ nor a $G$-Reich contraction with respect to the usual metric.

With suitable values of constants $\lambda, \mu$ and $\delta$ we obtain the following corollaries.

**Corollary 3.1.** *Let $(X, d)$ be a complete cone rectangular metric space endowed with a graph $G$ and has the property $(P)$. Let $T: X \to X$ be a $G$-contraction, that is,*

(G1) *$T$ is edge preserving, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;*
(G2) *there exists $\lambda \in [0, 1)$ such that*

$$d(Tx, Ty) \preceq \lambda d(x, y) \text{ for all } x, y \in X \text{ with } (x, y) \in E(G).$$

*Then, if $X_T \neq \emptyset$ then $T$ is a weakly Picard operator. Furthermore, the subgraph $G_{Fix}$ defined by $V(G_{Fix}) = FixT$ is weakly connected if and only if $T$ is a Picard operator.*

**Corollary 3.2.** *Let $(X, d)$ be a complete cone rectangular metric space endowed with a graph $G$ and has the property $(P)$. Let $T: X \to X$ be a $G$-Kannan contraction, that is,*

(GK1) *$T$ is edge preserving, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;*
(GK2) *there exists $\lambda \in [0, 1/2)$ such that*

$$d(Tx, Ty) \preceq \lambda[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X \text{ with } (x, y) \in E(G).$$

*Then, if $X_T \neq \emptyset$ then $T$ is a weakly Picard operator. Furthermore, the subgraph $G_{Fix}$ defined by $V(G_{Fix}) = FixT$ is weakly connected if and only if $T$ is a Picard operator.*

Following corollary is a fixed point result for an ordered Reich contraction (see Malhotra *et al.* (2013*b*)) and a generalization of result of Ran and Reurings Ran & Reurings (2004) in cone rectangular metric spaces.

**Corollary 3.3.** *Let* $(X, d)$ *be a complete cone rectangular metric space endowed with a partial order* $\sqsubseteq$ *and* $T: X \to X$ *be a mapping. Suppose the following conditions hold:*

(A) *$T$ is an ordered Reich contraction;*
(B) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$;*
(C) *$T$ is nondecreasing with respect to $\sqsubseteq$;*
(D) *if $\{x_n\}$ is a nondecreasing sequence in $X$ and converging to some $z$, then $x_n \sqsubseteq z$.*

*Then $T$ is a weakly Picard operator. Furthermore, FixT is well ordered (that is, all the elements of FixT are comparable) if and only if $T$ is a Picard operator.*

*Proof.* Let $G$ be a graph defined by $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x \sqsubseteq y\}$. Then by conditions (A) and (C), $T$ is a $G$-Reich contraction and by condition (B) we have $X_T \neq \emptyset$. Also by condition (D) we see that property (P) is satisfied. Now proof follows from Theorem 3.3. □

**Conclusion.** In the present paper we have proved the existence and uniqueness of fixed point theorems for a $G$-Reich contraction in cone rectangular metric spaces endowed with a graph. We note that the results of this paper generalize the ordered version of theorem of Reich (see Reich (1971) and Malhotra *et al.* (2013*b*)). Note that, in usual metric spaces the fixed point theorem for $G$-contractions generalizes and unifies the ordered version as well as the cyclic version of corresponding fixed point theorems (see Kirk *et al.* (2003) and Kamran *et al.* (2013)). We conclude with an open problem that: is it possible to prove the cyclic version of the result of Reich in cone rectangular metric spaces or rectangular metric spaces?

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# Hadamard Product of Simple Sets of Polynomials in $\mathbb{C}^n$

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**Abstract**

In this paper we give some convergence properties of Hadamard product set of polynomials defined by several simple monic sets of several complex variables in complete Reinhardt domains and in hyperelliptical regions too.

*Keywords:* Basic sets of polynomials, Hadamard product, complete Reinhardt domains, hyperelliptical regions.
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## 1. Introduction

In 1933, Whittaker (Whittaker, 1933), (Whittaker, 1949) introduced the subject of basic sets of polynomials of a single complex variable. This subject is developed by several authors using one and several complex variables. It is of fundamental importance in the theory of basic sets of polynomials of several complex variables to define some kinds of basic sets of polynomials in $\mathbb{C}^n$. This is the main aim of this paper. We will define and study Hadamard products of basic sets of polynomials in complete Reinhardt domains and in hyperelliptical regions.

We start with basic concepts, notations and terminology on this paper.

Let $\mathbb{C}$ represent the field of complex variables. In the space $\mathbb{C}^2$ of the two complex variables $z$ and $w$, the successive monomial $1, z, w, z^2, zw, w^2, ...$ are arranged so that the enumeration number of the monomial $z^j w^k$ in the above sequence is

$$\frac{1}{2}(j+k)(j+k)+k; \quad j,k \geq 0.$$

The enumeration number of the last monomial of a polynomial $P(z,w)$ in two complex variables is called the degree of the polynomial. A sequence $\{P_i(z;w)\}_0^\infty$ of polynomials in two complex variables in which the order of each polynomial is equal to its degree is called a simple set

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( see (Kishka, 1993), (Kumuyi & Nassif, 1986) and (Sayyed & Metwally, 1998)). Such a set is conveniently denoted by $\{P_i(z;w)\}$, where the last monomial in $P_{m,n}(z,w)$ is $z^m w^n$.

If further, the coefficient of this last monomial is 1, the simple set is termed monic. Thus, in the simple monic set $\{P_{m,n}(z;w)\}$ the polynomial $P_{m,n}(z,w)$ is represented as follows.

$$P_{m,n}(z,w) = \sum_{k=0}^{m+n} \sum_{j=0}^{k} P_{k-j,j}^{m,n} z^{k-j} w^j \quad (P_{m,n}^{m,n} = 1; P_{m+n-j,j}^{m,n} = 0, j > n).$$

Let $\mathbf{z} = (z_1, z_2, ..., z_n)$ be an element of $\mathbb{C}^n$; the space of several complex variables. The following definition is introduced in (Mursi & Makar, 1955*a*,*b*).

**Definition 1.1.** A set of polynomials $\{P_{\mathbf{m}}[\mathbf{z}]\} = \{P_0, P_1, P_2, ..., P_n, ...\}$ is said to be basic when every polynomial in the complex variables $z_s$; $s \in I = \{1, 2, 3, ..., n\}$ can be uniquely expressed as a finite linear combination of the elements of the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$.

Thus according to (Mursi & Makar, 1955*b*), the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be basic if and only if there exists a unique row-finite matrix $\overline{P}$ such that $\overline{P}P = P\overline{P} = \mathbf{I}$, where $P = [P_{\mathbf{m},\mathbf{h}}]$ is the matrix of coefficients, $\overline{P}$ is the matrix of operators of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and $\mathbf{I}$ is the infinite unit matrix.

Similar definition for a simple monic set can be extended to the case of several complex variables by replacing $m, n$ by $(\mathbf{m}) = (m_1, m_2, m_3, ..., m_n)$ , $j, k$ by $(\mathbf{h}) = (h_1, h_2, h_3, ..., h_n)$ and $z, w$ by $\mathbf{z}$, where each of $(\mathbf{m})$ and $(\mathbf{h})$ be multi-indices of non-negative integers.

The fact that the simple monic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of several complex variables is necessarily basic follows from the observation that the matrix $[P_{\mathbf{m},\mathbf{h}}]$ of coefficients of the polynomials of the set is a lower triangular matrix with non-zero diagonal elements. (These elements are each equal to 1 for monic sets).

**Definition 1.2.** The basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is said to be algebraic of degree $\ell$ when its matrix of coefficients $P$ satisfies the usual identity

$$\alpha_0 P^{\ell} + \alpha_1 P^{\ell-1} + ... + \alpha_{\ell} I = 0.$$

Thus, we have a relation of the form

$$\overline{P}_{\mathbf{m},\mathbf{h}} = \delta_{\mathbf{m},\mathbf{h}}\gamma_0 + \sum_{s_1=1}^{\ell-1} \gamma_{s_1} P_{\mathbf{m},\mathbf{h}}^{(s_1)},$$

where $P_{\mathbf{m},\mathbf{h}}^{(s_1)}$ are the elements of the power matrix $P^{s_1}$ and $\gamma_{s_1}$, $s_1 = 0, 1, 2, ..., \ell - 1$ are constant numbers. In the space of several complex variables $\mathbb{C}^n$. Let $\mathbf{z} = (z_1, z_2, ..., z_n)$ be an element of $\mathbb{C}^n$; the space of several complex variables, a closed complete Reinhardt domain of radii $\alpha_s r (> 0)$; $s \in I = \{1, 2, 3, ..., n\}$ is here denoted by $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ and is given by

$\bar{\Gamma}_{[\alpha\mathbf{r}]} = \bar{\Gamma}_{[\alpha_1 r, \alpha_2 r, ..., \alpha_n r]} = \{\mathbf{z} \in \mathbb{C}^n : |z_s| \leq \alpha_s r \quad ; s \in I\}$, where $\alpha_s$ are positive numbers. The open complete Reinhardt domain is here denoted by $\Gamma_{[\alpha\mathbf{r}]}$ and is given by

$\Gamma_{[\alpha\mathbf{r}]} = \Gamma_{[\alpha_1 r, \alpha_2 r, ..., \alpha_n r]} = \{\mathbf{z} \in \mathbb{C}^n : |z_s| < \alpha_s r \quad ; s \in I\}$.

Consider unspecified domain containing the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha\mathbf{r}]}$. This domain will be of radii $\alpha_s r_1$; $r_1 > r$, then making a contraction to this domain, we will get the domain $\bar{D}([\alpha\mathbf{r}^+]) = \bar{D}([\alpha_1 r^+, \alpha_2 r^+, ..., \alpha_n r^+])$, where $r^+$ stands for the right-limit of $r_1$ at $r$.

Now let $\mathbf{m} = (m_1, m_2, ..., m_n)$ be multi-indices of non-negative integers. The entire function $f(\mathbf{z})$ of several complex variables has the following representation:

$$f(\mathbf{z}) = \sum_{\mathbf{m}=\mathbf{0}}^{\infty} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}.$$

Suppose now that the function $f(\mathbf{z})$, is given by

$$f(\mathbf{z}) = \sum_{\mathbf{m}=0}^{\infty} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$$

is regular in $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ and

$$M[f; \alpha_s r] = \sup_{\bar{\Gamma}_{[\alpha\mathbf{r}]}} \left| f(\mathbf{z}) \right|.$$

For the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and its inverse $\left\{\overline{P}_{\mathbf{m}}[\mathbf{z}]\right\}$, we have

$$P_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}},$$

$$\overline{P}_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}},$$

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \overline{P}_{\mathbf{h}}[\mathbf{z}].$$

Let $N_{\mathbf{m}} = N_{m_1, m_2, ..., m_n}$ be the number of non-zero coefficients $\overline{P}_{\mathbf{m},\mathbf{h}}$ in the last equality.

A basic set satisfying the condition

$$\lim_{\langle\mathbf{m}\rangle\to\infty} \{N_{\mathbf{m}}\}^{\frac{1}{\langle\mathbf{m}\rangle}} = 1, \tag{1.1}$$

is called, as in (Mursi & Makar, 1955*a*,*b*) and (Kishka & El-Sayed Ahmed, 2003) a Cannon set.

Let $\{P_{\mathbf{m}}[\mathbf{z}]\}$ be a basic set of polynomials of the several complex variables $z_s$; $s \in I$, then the Cannon sum for this set in the complete Reinhardt domains is given as follows:

$$\Omega(P_{\mathbf{m}}, [\alpha\mathbf{r}]) = \prod_{s=1}^{n} (\alpha_s r)^{<\mathbf{m}>-\mathbf{m}_s} \sum_{\mathbf{h}} \left| \bar{P}_{\mathbf{m},\mathbf{h}} \right| M(P_{\mathbf{m}}, [\alpha\mathbf{r}]),$$

where

$$M(P_{\mathbf{m}}, [\alpha\mathbf{r}]) = \max_{\bar{\Gamma}_{[\alpha\mathbf{r}]}} \left| P_{\mathbf{m}}[\mathbf{z}] \right|.$$

The Cannon function is defined by:

$$\Omega(P, [\alpha\mathbf{r}]) = \lim_{\langle\mathbf{m}\rangle\to\infty} \left\{ \Omega(P_{\mathbf{m}}, [\alpha\mathbf{r}]) \right\}^{\frac{1}{\langle\mathbf{m}\rangle}}.$$

When this associated series converges uniformly to $f(\mathbf{z})$ in some domain it is said to represent $f(\mathbf{z})$ in that domain; in other words, as in the classical terminology of Whittaker for a single complex variable (see (Whittaker, 1949)), the basic set $P_{\mathbf{m}}[\mathbf{z}]$ will be effective in that domain. For more information about basic sets of polynomials we refer to ((Abul-Ez, 2000)-(Whittaker, 1949)).

The convergence properties of basic sets of polynomials are classified according to the classes of functions represented by their associated basic series and also to the domain in which are represented.

Concerning the effectiveness of the basic set of polynomials of several complex variables in complete Reinhardt domains, we have the following results from (Mursi & Makar, 1955*a*,*b*).

**Theorem 1.1.** *(Mursi & Makar, 1955a,b) The necessary and sufficient condition for the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables to be effective in the closed complete Reinhardt $\overline{\Gamma}_{[\alpha_s \mathbf{r}]}$ is that*

$$\Omega(P; r_s) = \prod_{s=1}^{n} \alpha_s r. \tag{1.2}$$

In the space of several complex variables $\mathbb{C}^n$, an open elliptical region $\sum_{s=1}^{n} \frac{|z_s|^2}{r_s{}^2} < 1$ is here denoted by $\mathbf{E}_{r_s}$ and its closure $\sum_{s=1}^{n} \frac{|z_s|^2}{r_s{}^2} \le 1$; is denoted by $\overline{\mathbf{E}}_{r_s}$, where $r_s$; $s \in I$ are positive numbers. In terms of the introduced notations these regions satisfy the following inequalities:

$$\mathbf{E}_{r_s} = \{\mathbf{w} : |\mathbf{w}| < 1\}$$

$$\overline{\mathbf{E}}_{r_s} = \{\mathbf{w} : |\mathbf{w}| \le 1\},$$

where $\mathbf{w} = (w_1, w_2, w_3, ..., w_n)$ , $w_s = \frac{z_s}{r_s}$; $s \in I$. Suppose now that the function $f(\mathbf{z})$, is given by

$$f(\mathbf{z}) = \sum_{\mathbf{m}=0}^{\infty} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$$

is regular in $\overline{\mathbf{E}}_{\mathbf{r_s}}$ and

$$M[f; r_s] = \sup_{\overline{\mathbf{E}}_{\mathbf{rs}}} \left| f(\mathbf{z}) \right|.$$

Then it follows that $\{|z_s| \le r_s t_s;\ \ |t_s| = 1\} \subset \overline{\mathbf{E}}_{\mathbf{r_s}}$; hence

$$|a_{\mathbf{m}}| \le \frac{M[f;\rho_s]}{\rho^{\mathbf{m}} t^{\mathbf{m}}} = \frac{M[f;\rho_s]}{\prod_{s=1}^{n} \rho_s^{m_s} t_s^{m_s}} \le \inf_{|t|=1} \frac{M[f;\rho_s]}{\prod_{s=1}^{n} (\rho_s t_s)^{m_s}}$$
$$= \sigma_{\mathbf{m}} \frac{M[f;\rho_s]}{\prod_{s=1}^{n} \rho_s^{m_s}}$$

for all $0 < \rho_s < r_s$; $s \in I$, where

$$\sigma_{\mathbf{m}} = \inf_{|t|=1} \frac{1}{t^{\mathbf{m}}} = \frac{\{\langle \mathbf{m} \rangle\}^{\frac{\langle \mathbf{m} \rangle}{2}}}{\prod_{s=1}^{n} m_s^{\frac{m_s}{2}}}$$

and $1 \leq \sigma_{\mathbf{m}} \leq (\sqrt{n})^{\langle\mathbf{m}\rangle}$ on the assumption that $m_s^{\frac{m_s}{2}} = 1$, whenever $m_s = 0$; $s \in I$. Thus, it follows that

$$\lim_{\langle\mathbf{m}\rangle\to\infty} \sup \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}} \prod_{s=1}^{n} (r_s)^{\langle\mathbf{m}\rangle - m_s}} \right\}^{\frac{1}{\langle\mathbf{m}\rangle}} \leq \frac{1}{\prod_{s=1}^{n} \rho_s} \quad ; \quad \rho_s < r_s; s \in I$$

and since $\rho_s$ can be chosen arbitrary near to $r_s$; $s \in I$, we conclude that

$$\lim_{\langle\mathbf{m}\rangle\to\infty} \sup \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}} \prod_{s=1}^{n} (r_s)^{\langle\mathbf{m}\rangle - m_s}} \right\}^{\frac{1}{\langle\mathbf{m}\rangle}} \leq \frac{1}{\prod_{s=1}^{n} r_s}.$$

Now, write

$$G(P_{\mathbf{m}}; r_s) = \max_{\mu,\nu} \sup_{\overline{\mathbf{E}}_{\mathbf{r_s}}} \left| \sum_{\mathbf{j}=\mu}^{\nu} \overline{P}_{\mathbf{m};\mathbf{j}} P_{\mathbf{j}}[\mathbf{z}] \right|,$$

where, $r_s$; $s \in I$ are positive numbers.

The Cannon sum of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ for $\overline{\mathbf{E}}_{r_s}$ will be

$$\Omega(P_{\mathbf{m}}; r_s) = \sigma_{\mathbf{m}} \prod_{s=1}^{n} \{r_s\}^{\langle\mathbf{m}\rangle - m_s} G(P_{\mathbf{m}}; r_s)$$

and the Cannon function for the same set is

$$\Omega(P; r_s) = \lim_{\langle\mathbf{m}\rangle\to\infty} \{\Omega(P_{\mathbf{m}}; r_s)\}^{\frac{1}{\langle\mathbf{m}\rangle}}.$$

Concerning the effectiveness of the basic set of polynomials of several complex variables in hyperellipse, we have the following results from (El-Sayed Ahmed & Kishka, 2003).

**Theorem 1.2.** *(El-Sayed Ahmed & Kishka, 2003) The necessary and sufficient condition for the basic set* $\{P_{\mathbf{m}}[\mathbf{z}]\}$ *of polynomials of several complex variables to be effective in the closed hyperellipse* $\overline{\mathbf{E}}_{r_s}$ *is that*

$$\Omega(P; r_s) = \prod_{s=1}^{n} r_s.$$

Convergence properties (effectiveness) for Hadamard product set simple monic sets of polynomials of a single complex variable is introduced by Melek and El-Said in (Melek & El-Said, 1985). In (Nassif & Rizk, 1988) Nassif and Rizk introduced an extension of this product in the case of two complex variables using spherical regions. In (El-Sayed Ahmed, 2006), the same author has studied this problem in $\mathbb{C}^n$ using hepespherical regions. It should be mentioned here the study of this problem in Clifford analysis (see (Abul-Ez, 2000)). For more details on basic sets of polynomials in Clifford setting, we refer to (Abul-Ez, 2000; Abul-Ez & De Almeida, 2013; Abul-Ez & Constales, 2003; Aloui *et al.*, 2010; Aloui & Hassan, 2010; Hassan, 2012; Saleem *et al.*, 2012) and others. In the present paper, we aim to investigate the extent of a generalization of this Hadamard product set in $\mathbb{C}^n$ using hyperspherical regions.

In (Nassif & Rizk, 1988), Nassif and Rizk introduced the following definition.

**Definition 1.3.** Let $\{P_{m,n}(z,w)\}$ and $\{q_{m,n}(z,w)\}$ be two simple monic sets of polynomials, where

$$P_{m,n}(z,w) = \sum_{(i,j)=0}^{(m,n)} P_{i,j}^{m,n} z^i w^j,$$

$$q_{m,n}(z,w) = \sum_{(i,j)=0}^{(m,n)} q_{i,j}^{m,n} z^i w^j.$$

Then the Hadamard product of the sets $\{P_{m,n}(z,w)\}$ and $\{q_{m,n}(z,w)\}$ is the simple monic set $\{U_{m,n}(z,w)\}$ given by

$$U_{m,n}(z,w) = \sum_{(i,j)=0}^{(m,n)} U_{i,j}^{m,n} z^i w^j,$$

where

$$U_{i,j}^{m,n} = \frac{\sigma_{m,n}}{\sigma_{i,j}} P_{i,j}^{m,n} q_{i,j}^{m,n}; \quad ((i,j) \le (m,n)),$$

and

$$\sigma_{m,n} = \inf_{|t|=1} \frac{1}{t^{m+n}} = \frac{\{m+n\}^{\frac{m+n}{2}}}{m^{\frac{m}{2}} n^{\frac{n}{2}}}.$$

In this paper, we give an inevitable modification in the definition of Hadamard product of basic sets of polynomials of two complex variables as to yield favorable results in the case of several complex variables in complete Reinhardt domains in $\mathbb{C}^n$, by using $k$ basic sets of polynomials instead of two sets.

Now, we are in a position to extend the above product by using $k$ basic sets of polynomials of several complex variables in complete Reinhardt domains, so we will denote these polynomials by $\{P_{1,\mathbf{m}}[\mathbf{z}]\}, \{P_{2,\mathbf{m}}[\mathbf{z}]\}, ..., \{P_{k,\mathbf{m}}[\mathbf{z}]\}$ and in general write $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$.

**Definition 1.4.** Let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$ be simple monic sets of polynomials of several complex variables, where

$$P_{s_2,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s_2,\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}. \tag{1.3}$$

Then the Hadamard product of the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ is the simple monic set $\{H_{\mathbf{m}}[\mathbf{z}]\}$ given by

$$H_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} H_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}, \tag{1.4}$$

where

$$H_{\mathbf{m},\mathbf{h}} = \left( \prod_{s_2=1}^{k} P_{s_2,\mathbf{m},\mathbf{h}} \right). \tag{1.5}$$

If we substitute by $k = 2$ and consider polynomials of two complex variables instead of several complex variables, then we will obtain Definition 1.3. It should be remarked here that Definition 1.4 is different from that used in (Metwally, 2002).

## 2. Effectiveness in complete Reinhardt domains

In this section, we will study the effectiveness of the extended Hadamard product of simple monic sets of polynomials of several complex variables defined by (1.4) and (1.5) in closed complete Reinhardt domains and at the origin.

Let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ be simple monic sets of polynomials of several complex variables $z_s$; $s \in I$, so that we can write

$$P_{s_2,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s_2,\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}, \tag{2.1}$$

where

$$P^{m_1,m_2,...,m_n}_{s_2,m_1,m_2,...,m_n} = 1; \quad s_2 = 1, 2, ..., k\,.$$

The normalizing functions of the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ are defined by (see (Nassif & Rizk, 1988))

$$\mu(P_{s_2}; \alpha_s r) = \lim_{\langle \mathbf{m}\rangle \to \infty} \sup \Big\{ M[P_{s_2,\mathbf{m}}; \alpha_s r] \Big\}^{\frac{1}{\langle \mathbf{m}\rangle}}, \tag{2.2}$$

where $M[P_{s_2,\mathbf{m}}; \alpha_s r]$ are defined as follows:

$$M[P_{s_2,\mathbf{m}}; \alpha_s r] = \sup_{\bar{\Gamma}_{[\alpha r]}} \Big| P_{s_2,\mathbf{m}}[\mathbf{z}] \Big|.$$

Notice that the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ are monic. By applying Cauchy's inequality in (2.2), we have

$$|P_{s_2,\mathbf{m},\mathbf{h}}| \leq \frac{1}{\prod\limits_{s=1}^{n} (\alpha_s r)^{\langle \mathbf{m}\rangle}} \sup_{\bar{\Gamma}_{[\alpha r]}} \Big| P_{s_2,\mathbf{m}}[\mathbf{z}] \Big|,$$

which implies that

$$M[P_{s_2,\mathbf{m}}; \alpha_s r] \geq \prod_{s=1}^{n} (\alpha_s r)^{\langle \mathbf{m}\rangle}.$$

It follows from (2.2) that

$$\mu(P_{s_2}; \alpha_s r) \geq \prod_{s=1}^{n} \alpha_s r. \tag{2.3}$$

Next, we show if $\rho$ is positive number greater than $r$, then

$$\mu(P_{s_2}; \alpha_s \rho) \leq \frac{\prod\limits_{s=1}^{n} \alpha_s \rho}{\prod\limits_{s=1}^{n} \alpha_s r} \mu(P_{s_2}; \alpha_s r), \quad \alpha_s \rho > \alpha_s r. \tag{2.4}$$

In fact, this relation follows by applying (2.2) to the inequality

$$M[P_{s_2,\mathbf{m}}; \alpha_s r] \leq K \left( \frac{\prod\limits_{s=1}^{n} \alpha_s \rho}{\prod\limits_{s=1}^{n} \alpha_s r} \right)^{\langle \mathbf{m}\rangle} M[P_{s_2,\mathbf{m}}; \alpha_s r],$$

which in its turn, is derivable from (2.2), Cauchy's inequality and the supremum of $\mathbf{z}^{\mathbf{m}}$, where $K = O(\langle \mathbf{m} \rangle + 1)$.

Now, let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$ be simple monic sets of polynomials of several complex variables, and that $\{H^*_{\mathbf{m}}[\mathbf{z}]\}$ is the set defined as follows

$$H^*_{\mathbf{m}}[\mathbf{z}] = \prod_{s_2=1}^{k} P_{s_2,\mathbf{m}}[\mathbf{z}]. \tag{2.5}$$

The following fundamental result is proved.

**Theorem 2.1.** *If, for any $\alpha_s r > 0$*

$$\mu(P_{s_2}; \alpha_s r) = \prod_{s=1}^{n} \alpha_s r, \tag{2.6}$$

*then*

$$\mu(H^*; \alpha_s r) = \prod_{s=1}^{n} \alpha_s r. \tag{2.7}$$

*Proof.* We first observe that, if $\rho$ be any finite number greater than $r$ , then by (2.1), (2.2) and (2.9), we obtain that

$$\mu(P_{s_2}; \alpha_s \rho) = \prod_{s=1}^{n} \alpha_s \rho. \tag{2.8}$$

Now, given $r^* > r$, we choose finite number $r'$ such that

$$\alpha_s r < \alpha_s r' < \alpha_s r^*. \tag{2.9}$$

Then by (2.1) and (2.6), we obtain that

$$M(P_{s_2,\mathbf{h}}; \alpha_s r) < \eta \prod_{s=1}^{n} \alpha_s (r')^{\langle \mathbf{h} \rangle} \quad \text{where} \quad \eta > 1, \tag{2.10}$$

where $\langle \mathbf{h} \rangle = h_1 + h_2 + h_3 + \ldots h_n$. Also from (2.4), we can write

$$H^*_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} \prod_{s_2=1}^{k} P_{s_2,\mathbf{m},\mathbf{h}} P_{s_2,\mathbf{h}}[\mathbf{z}].$$

Hence (2.9) and (2.10) lead to

$$M[H^*_{\mathbf{m}}; \alpha_s r] \leq \eta K \Big( 1 - \Big( \frac{\prod_{s=1}^{n} \alpha_s r'}{\prod_{s=1}^{n} \alpha_s r^*} \Big)^{n} \Big)^{-n} M[P_{s_2,\mathbf{m}}; \alpha_s r^*],$$

Making $\langle \mathbf{m} \rangle \to \infty$ and applying (2.7), we get

$$\mu(H^*; \alpha_s r) = \lim_{\langle \mathbf{m} \rangle \to \infty} \sup \left\{ \sigma_{\mathbf{m}} M[H^*_{\mathbf{m}}; \alpha_s r] \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \leq \mu(P_{s_2}; \alpha_s r^*) = \prod_{s=1}^{n} \alpha_s r^*,$$

which leads to the equality (2.6), by the choice of $r^*$ near to $r$, and our theorem is therefore proved. □

*Remark.* From Theorem 2.1, if we consider the simple monic sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ accord to condition (2.6), then it is not hard to prove by induction for the $j$-power sets $\{P^{(j)}_{s_2,\mathbf{m}}[\mathbf{z}]\}$ that

$$\mu(P^{(j)}_{s_2}; \alpha_s r) = \prod_{s=1}^{n} \alpha_s r. \tag{2.11}$$

Now, we give the following result.

**Theorem 2.2.** *Let* $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$ *be simple monic algebraic sets of polynomials of several complex variables, which accord to condition (10). Then the set will be effective in the closed complete Reinhardt domain* $\bar{\Gamma}_{[\alpha r]}$.

*Proof.* Suppose that the monomial $\mathbf{z}^{\mathbf{m}}$ admit the representation

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}].$$

Since the set $\{P_{1,\mathbf{m}}[\mathbf{z}]\}$ is algebraic, we find there exists a relation of the form

$$\overline{P}_{1,\mathbf{m},\mathbf{h}} = \sum_{j=1}^{k} a_j P^{(j)}_{1,\mathbf{m},\mathbf{h}}; \quad ((\mathbf{h}) \leq (\mathbf{m})), \tag{2.12}$$

where $k$ is a finite positive integer which together with the coefficients $(a_j)_{j=1}^{k}$, is independent of the indices $(\mathbf{m})$, $(\mathbf{h})$. The coefficients $P^{(j)}_{1,\mathbf{m},\mathbf{h}}$ are defined by

$$P^{(j)}_{1,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=1}^{(\mathbf{m})} P^{(j)}_{1,\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}; \quad 1 \leq j \leq k.$$

It follows that

$$\left| P^{(j)}_{1,\mathbf{m},\mathbf{h}} \right| (\alpha_s r)^{\langle \mathbf{m} \rangle} \leq \sigma_{\mathbf{h}} M[P^{(j)}_{1,\mathbf{m}}; \alpha_s r]. \tag{2.13}$$

According to (2.11) for given $r^* > r$ and from the definition corresponding to $\mu(P^{(j)}_1; \alpha_s r)$, we deduce that

$$M[P^{(j)}_{1,\mathbf{h}}; \alpha_s r] < K(\alpha_s r^*)^{\langle \mathbf{h} \rangle}. \tag{2.14}$$

Applying (2.13) and (2.14) in (2.12), we obtain that

$$\left| \overline{P}^{(j)}_{1,\mathbf{m},\mathbf{h}} \right| < \zeta \beta K \frac{\prod_{s=1}^{n} (\alpha_s r^*)^{\langle \mathbf{m} \rangle}}{\prod_{s=1}^{n} (\alpha_s r)^{\langle \mathbf{m} \rangle}}, \tag{2.15}$$

where

$$\beta = \max\{|a_j|; 0 \leq j \leq k\} \quad \text{and} \quad \zeta \quad \text{is a constant.} \tag{2.16}$$

In view of the representation

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}],$$

the Cannon sum of the set $\{P^{(j)}_{1,\mathbf{m}}[\mathbf{z}]\}$ will be

$$\Omega(P^{(j)}_{1,\mathbf{m}}; \alpha_s r) = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} |\overline{P}^{(j)}_{\mathbf{m},\mathbf{h}}| M[P^{(j)}_{1,\mathbf{h}}; \alpha_s r], \tag{2.17}$$

where,

$$M[P^{(j)}_{1,\mathbf{h}}; \alpha_s r] = \sup_{\bar{\Gamma}_{[\alpha \mathbf{r}]}} |P^{(j)}_{1,\mathbf{m}}[\mathbf{z}]|. \tag{2.18}$$

Therefore (2.14), (2.15) and (2.17) (for $r^* > r$) give

$$\Omega(P^{(j)}_{1,\mathbf{m}}; \alpha_s r) < \zeta K \beta \prod_{s=1}^{n} (\alpha_s r^*)^{\langle \mathbf{m} \rangle}. \tag{2.19}$$

Hence the Cannon function of the set $\{P^{(j)}_{1,\mathbf{m}}[\mathbf{z}]\}$ turns out to be

$$\Omega(P^{(j)}_{1}; \alpha_s r) = \lim_{\langle \mathbf{m} \rangle \to \infty} \left\{ \Omega(P^{(j)}_{1,\mathbf{m}}; \alpha_s r) \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} = \prod_{s=1}^{n} \alpha_s r^*,$$

which, by the choice of $r^*$, implies that

$$\Omega(P^{(j)}_{1}; \alpha_s r) = \prod_{s=1}^{n} \alpha_s r.$$

As very similar, we can obtain that the sets $\{P^{(j)}_{\nu,\mathbf{m}}[\mathbf{z}]\}$; $\nu = 2, 3, 4, ..., k$ will be effective in the losed complete Reinhardt domain $\bar{\Gamma}_{[\alpha r]}$. Our theorem is therefore proved. □

## 3. Effectiveness in hyperelliptical regions

Now, we are in a position to extend the above product by using $k$ basic sets of polynomials of several complex variables, so we will denote these polynomials by $\{P_{1,\mathbf{m}}[\mathbf{z}]\}, \{P_{2,\mathbf{m}}[\mathbf{z}]\}, ..., \{P_{k,\mathbf{m}}[\mathbf{z}]\}$ and in general write $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$.

**Definition 3.1.** Let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$ be simple monic sets of polynomials of several complex variables, where

$$P_{s_2,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s_2,\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}. \tag{3.1}$$

Then the Hadamard product of the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ is the simple monic set $\{H_{\mathbf{m}}[\mathbf{z}]\}$ given by

$$H_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} H_{\mathbf{m},\mathbf{h}}\mathbf{z}^{\mathbf{h}}, \tag{3.2}$$

where

$$H_{\mathbf{m},\mathbf{h}} = \left(\frac{\sigma_{\mathbf{m}}}{\sigma_{\mathbf{h}}}\right)^{k-1}\left(\prod_{s_2=1}^{k} P_{s_2,\mathbf{m},\mathbf{h}}\right). \tag{3.3}$$

If we substitute by $k = 2$ and consider polynomials of two complex variables instead of several complex variables, then we will obtain Definition 1.3. It should be remarked here that Definition 1.4 is different from that used in (Metwally, 2002).

Let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ be simple monic sets of polynomials of several complex variables $z_s;\ s \in I$, so that we can write

$$P_{s_2,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s_2,\mathbf{m},\mathbf{h}}\mathbf{z}^{\mathbf{h}}, \tag{3.4}$$

where

$$P^{m_1,m_2,...,m_n}_{s_2,m_1,m_2,...,m_n} = 1; \quad s_2 = 1, 2, ..., k\,.$$

The normalizing functions of the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ are defined by (see (Nassif & Rizk, 1988))

$$\mu(P_{s_2}; r_s) = \lim_{\langle\mathbf{m}\rangle\to\infty} \sup\left\{\sigma_{\mathbf{m}} M[P_{s_2,\mathbf{m}}; r_s]\right\}^{\frac{1}{\langle\mathbf{m}\rangle}}, \tag{3.5}$$

where $M[P_{s_2,\mathbf{m}}; r_s]$ are defined as follows:

$$M[P_{s_2,\mathbf{m}}; r_s] = \sup_{\overline{\mathbf{E}}_{\mathbf{r_s}}}\left|P_{s_2,\mathbf{m}}[\mathbf{z}]\right|.$$

Notice that the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ are monic. By applying Cauchy's inequality, we deduce

$$|P_{s_2,\mathbf{m},\mathbf{h}}| \le \frac{\sigma_{\mathbf{m}}}{[\prod_{s=1}^{n} r_s]^{\langle\mathbf{m}\rangle}} \sup_{\overline{\mathbf{E}}_{\mathbf{r_s}}}\left|P_{s_2,\mathbf{m}}[\mathbf{z}]\right|,$$

which implies that

$$M[P_{s_2,\mathbf{m}}; r_s] \ge \frac{[\prod_{s=1}^{n} r_s]^{\langle\mathbf{m}\rangle}}{\sigma_{\mathbf{m}}}.$$

It follows from (3.4) that

$$\mu(P_{s_2}; r_s) \ge r_s. \tag{3.6}$$

Next, we show if $\rho_s$ are positive numbers greater than $r_s$, then

$$\mu(P_{s_2}; \rho_s) \le \frac{\prod_{s=1}^{n}\rho_s}{\prod_{s=1}^{n} r_s}\,\mu(P_{s_2}; r_s), \quad \rho_s > r_s. \tag{3.7}$$

In fact, this relation follows by applying (3.4) to the inequality

$$M[P_{s_2,\mathbf{m}};\rho_s] \leq K\Big(\frac{\prod_{s=1}^{n}\rho_s}{\prod_{s=1}^{n} r_s}\Big)^{\langle\mathbf{m}\rangle} M[P_{s_2,\mathbf{m}};r_s],$$

which in its turn, is derivable from (3.4), Cauchy's inequality and the supremum of $\mathbf{z}^{\mathbf{m}}$, where $K = O(\langle\mathbf{m}\rangle + 1)$.

Now, let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$ be simple monic sets of polynomials of several complex variables, and that $\{H^*_{\mathbf{m}}[\mathbf{z}]\}$ is the set defined as follows

$$H^*_{\mathbf{m}}[\mathbf{z}] = \prod_{s_2=1}^{k} P_{s_2,\mathbf{m}}[\mathbf{z}]. \tag{3.8}$$

The following fundamental result is proved.

**Theorem 3.1.** *If, for any* $r_s > 0$

$$\mu(P_{s_2};r_s) = \prod_{s=1}^{n} r_s, \tag{3.9}$$

*then*

$$\mu(H^*;r_s) = \prod_{s=1}^{n} r_s. \tag{3.10}$$

*Proof.* We first observe that, if $\rho$ be any finite number greater than $r$ , then by (3.4), (3.5) and (3.7), we obtain that

$$\mu(P_{s_2};\rho_s) = \prod_{s=1}^{n} \rho_s. \tag{3.11}$$

Now, given $r^*_s > r_s$, we choose finite number $r'_s$ such that

$$r_s < r'_s < r^*. \tag{3.12}$$

Then by (3.4) and (3.8), we obtain that

$$M(P_{s_2,\mathbf{h}};r_s) < \frac{\eta}{\sigma_{\mathbf{h}}}[\prod_{s=1}^{n} r'_s]^{\langle\mathbf{h}\rangle} \quad \text{where} \quad \eta > 1, \tag{3.13}$$

where $\langle\mathbf{h}\rangle = h_1 + h_2 + h_3 + \ldots h_n$. Also from (3.7), we can write

$$H^*_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} \prod_{s_2=1}^{k} P_{s_2,\mathbf{m},\mathbf{h}} P_{s_2,\mathbf{h}}[\mathbf{z}].$$

Hence (3.9) and (3.10) lead to

$$\mu(H^*; r_s) = \lim_{\langle\mathbf{m}\rangle\to\infty} \sup\left\{\sigma_{\mathbf{m}} M[H^*_{\mathbf{m}}; r_s]\right\}^{\frac{1}{\langle\mathbf{m}\rangle}} \leq \mu(P_{s_2}; r^*_s) = \prod_{s=1}^{n} r^*_s,$$

which leads to the equality (3.8), by the choice of $r^*_s$ near to $r_s$, and our theorem is therefore proved. □

*Remark.* From Theorem 3.1 if we consider the simple monic sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ accord to condition (3.8), then it is not hard to prove by induction for the $j$-power sets $\{P^{(j)}_{s_2,\mathbf{m}}[\mathbf{z}]\}$ that

$$\mu(P^{(j)}_{s_2}; r_s) = \prod_{s=1}^{n} r_s. \tag{3.14}$$

*Remark.* It should be remarked that the results of this paper improve some results in (El-Sayed Ahmed, 2006, 2013).

## 4. Conclusion

We have obtained some essential and important results for the effectiveness of the Hadamard product set of polynomials in complete Reinhardt domains and in heperelliptical regions. From the established theorems, representations and convergence of power set of the the Hadamard product set are introduced in complete Reinhardt domains and in heperelliptical regions too. Various problems relating to the properties of the Hadamard set of simple basic sets of polynomials are treated with particular emphasis on distinction between the single and several complex variables cases. An important result is established for the relationship between the Cannon functions of simple sets of polynomials in several complex variables and those of the directly Hadamard sets.

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# Fractional Order Differential Equations Involving Caputo Derivative

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## Abstract

In this paper, the Banach contraction principle and Schaefer theorem are applied to establish new results for the existence and uniqueness of solutions for some Caputo fractional differential equations. Some examples are also discussed to illustrate the main results.

*Keywords:* Caputo derivative, Banach fixed point theorem, Fractional differential equations.
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## 1. Introduction

The theory of fractional differential equations has excited in recent years a considerable interest both in mathematics and in applications, (see (Bengrine & Dahmani, 2012; Delbosco & Rodino, 1996; Diethelm & Walz, 1998; El-Sayed, 1998)). In particular, existence and uniqueness of solutions for fractional differential equations have attracted the attention of many mathematicians (Diethelm & Ford, 2002; Houas & Dahmani, 2013; Zhang, 2011; Ntouyas, 2012; Su, 2009; Yang, 2012; Zhang, 2011).
This paper deals with the existence and uniqueness of solutions to the following problem

$$\begin{aligned} D^{\alpha} x(t) + f\left(t, y(t), D^{\delta} y(t)\right) &= 0, t \in J, \\ D^{\beta} y(t) + g\left(t, x(t), D^{\sigma} x(t)\right) &= 0, t \in J, \\ x(0) = y(0) = 0, x(1) - \lambda_1 x(\eta) = 0, y(1) - \lambda_1 y(\eta) &= 0, \\ x''(0) = y''(0) = 0, x''(1) - \lambda_2 x''(\xi) = 0, y''(1) - \lambda_2 y''(\xi) &= 0, \end{aligned} \tag{1.1}$$

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where $D^{\alpha}, D^{\beta}, D^{\delta}$ and $D^{\sigma}$, are the Caputo fractional derivatives, $3 < \alpha, \beta \leq 4, \delta \leq \alpha - 1, \sigma \leq \beta - 1, 0 < \xi, \eta < 1$, $J = [0,1]$, $\lambda_1, \lambda_2$ are real constants satisfying $\lambda_1 \eta \neq 1, \lambda_2 \xi \neq 1$ and $f, g$ are two functions which will be specified later.
This paper is organized as follows: In section 2, we present some preliminaries and lemmas. In section 3, we present our main results for the existence and uniqueness of solutions of (1.1). In section 4, some examples are treated to illustrate our results.

## 2. Preliminaries

To present our main results, we need the the following two definitions:

**Definition 2.1.** The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function $f$ on $[0, \infty[$ is defined as:

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \, d\tau, \alpha > 0, \tag{2.1}$$

$$J^0 f(t) = f(t),$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du.$

**Definition 2.2.** The fractional derivative of $f \in C^n([0, \infty[)$ in the Caputo's sense is defined as:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) \, d\tau, n-1 < \alpha, n \in N^*. \tag{2.2}$$

More details about fractional calculus can be found in (Mainardi, 1997; Podlubny *et al.*, 2002).
We need also to introduce the spaces:
$X = \{x : x \in C([0,1]), D^{\sigma} x \in C([0,1])\}$ and $Y = \{y : y \in C([0,1]), D^{\delta} y \in C([0,1])\}$. For these spaces, we associate respectively the norms $\| x \|_X = \| x \| + \| D^{\sigma} x \|; \| x \| = \sup_{t \in J} |x(t)|, \| D^{\sigma} x \| = \sup_{t \in J} \left| D^{\sigma} x(t) \right|$ and $\| y \|_Y = \| y \| + \| D^{\delta} y \|; \| y \| = \sup_{t \in J} |y(t)|, \| D^{\delta} y \| = \sup_{t \in J} \left| D^{\delta} y(t) \right|$. It is clear that, $(X, \| . \|_X)$ and $(Y, \| . \|_Y)$, are two Banach spaces.
Also, $(X \times Y, \|(x,y)\|_{X \times Y})$ is a Banach space. Its norm is given by $\|(x,y)\|_{X \times Y} = \|x\|_X + \|y\|_Y$.

The following lemmas are crucial for our main results (Kilbas & Marzan, 2005; Lakshmikantham & Vatsala, 2008):

**Lemma 2.1.** *For $\alpha > 0$, the general solution of the equation $D^{\alpha} x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{2.3}$$

*where $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1, n = [\alpha] + 1$.*

**Lemma 2.2.**

$$J^{\alpha} D^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{2.4}$$

*for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, n = [\alpha] + 1$.*

We prove also the following lemma which is needed to present the integral solution for the problem (1.1):

**Lemma 2.3.** *Let $h \in C([0,1])$, $t \in J$, $3 < \alpha \leq 4$. Then the solution of the equation*

$$D^{\alpha}x(t) + h(t) = 0, \tag{2.5}$$

*where,*

$$\begin{aligned} x(0) &= 0, x(1) - \lambda_1 x(\eta) = 0, \\ x''(0) &= 0, x''(1) - \lambda_2 x''(\xi) = 0 \end{aligned} \tag{2.6}$$

is given by the following expression

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h\left(s, y(s), D^{\delta}y(s)\right) ds \\ &+\frac{\lambda_1 t}{(\lambda_1\eta - 1)\Gamma(\alpha)}\int_0^{\eta} (\eta-s)^{\alpha-1} h\left(s, y(s), D^{\delta}y(s)\right) ds \\ &-\frac{1}{(\lambda_1\eta - 1)\Gamma(\alpha)}\int_0^{1} (1-s)^{\alpha-1} h\left(s, y(s), D^{\delta}y(s)\right) ds \\ &+\frac{\left(\lambda_2 - \lambda_2\lambda_1\eta^3\right)t + (\lambda_2\lambda_1\eta - \lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^{\xi} (\xi-s)^{\alpha-3} h\left(s, y(s), D^{\delta}y(s)\right) ds \\ &-\frac{\left(1 - \lambda_1\eta^3\right)t + (\lambda_1\eta - 1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^{1} (1-s)^{\alpha-3} h\left(s, y(s), D^{\delta}y(s)\right) ds. \end{aligned} \tag{2.7}$$

**Proof:** Let $c_i \in \mathbb{R}, i = 0,1,2,3$. Then by lemmas 2.1, 2.2, the general solution of (2.5) can be written as:

$$x(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\, ds - c_0 - c_1 t - c_2 t^2 - c_3 t^3. \tag{2.8}$$

Using (2.6), we immediately get $c_0 = c_2 = 0$. On the other hand, we have

$$\begin{aligned} c_1 &= -\frac{\lambda_1}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}h(s)\,ds + \frac{1}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)\,ds \\ &-\frac{\lambda_2\left(1-\lambda_1\eta^3\right)}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3}h(s)\,ds \\ &+\frac{(1-\lambda_1\eta)}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^1(1-s)^{\alpha-3}h(s)\,ds. \end{aligned} \tag{2.9}$$

To obtain the value of $c_3$, we remarque that

$$c_3 = -\frac{\lambda_2}{6(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3}h(s)\,ds + \frac{1}{6(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^1(1-s)^{\alpha-3}g(s)\,ds. \tag{2.10}$$

Finally, substituting the values of $c_1$ and $c_3$ in (2.8), we obtain (2.7).

## 3. Main results

We begin by introducing the quantities:

$$\begin{aligned}
N_1 &= \frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)},\\
N_2 &= \frac{1}{\Gamma(\alpha-\sigma+1)} + \frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} + \frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\\
&\quad + \frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)},\\
N_3 &= \frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\beta}+1}{|\lambda_1\eta-1|\Gamma(\beta+1)} + \frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\beta-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)},\\
N_4 &= \frac{1}{\Gamma(\beta-\delta+1)} + \frac{|\lambda_1|\eta^{\beta}+1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)} + \frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\\
&\quad + \frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}.
\end{aligned}$$

We impose also the hypotheses:

$(H1)$ : The functions $f, g : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous.

$(H2)$ : There exist non negative functions $a_i, b_i \in C([0,1]), i = 1,2$ such that for all $t \in [0,1]$ and $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$, the inequalities

$$\begin{aligned}
|f(t,x_1,y_1) - f(t,x_2,y_2)| &\le a_1(t)|x_1-x_2| + b_1(t)|y_1-y_2|,\\
|g(t,x_1,y_1) - g(t,x_2,y_2)| &\le a_2(t)|x_1-x_2| + b_2(t)|y_1-y_2|,
\end{aligned} \tag{3.1}$$

are valid, and

$$\omega_1 = \sup_{t\in J} a_1(t), \omega_2 = \sup_{t\in J} b_1(t), \varpi_1 = \sup_{t\in J} a_2(t), \varpi_2 = \sup_{t\in J} b_2(t).$$

$(H3)$ : There exist positive constants $L_1$ and $L_2$ such that

$$|f(t,x,y)| \le L_1, |g(t,x,y)| \le L_2 \text{ for each } t\in J \text{ and all } x,y\in\mathbb{R}.$$

Our first main result is given by the following theorem:

**Theorem 3.1.** *Assume that (H2) holds and suppose that*

$$(N_1+N_2)(\omega_1+\omega_2) + (N_3+N_4)(\varpi_1+\varpi_2) < 1. \tag{3.2}$$

*Then the problem (1.1) has a unique solution on $J$.*

**Proof:** We apply Banach fixed point theorem. So, we consider the operator $\phi : X\times Y \to X\times Y$ defined by:

$$\phi(x,y)(t) := (\phi_1 y(t), \phi_2 x(t)), \tag{3.3}$$

where

$$\begin{aligned}\phi_1 y(t) \ : \ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, y(s), D^{\delta}y(s)\right) ds \\ &+\frac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1} f\left(s, y(s), D^{\delta}y(s)\right) ds \\ &-\frac{1}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f\left(s, y(s), D^{\delta}y(s)\right) ds \\ &+\frac{\left(\lambda_2-\lambda_2\lambda_1\eta^3\right)t+(\lambda_2\lambda_1\eta-\lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3} f\left(s, y(s), D^{\delta}y(s)\right) ds \\ &-\frac{\left(1-\lambda_1\eta^3\right)t+(\lambda_1\eta-1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3} f\left(s, y(s), D^{\delta}y(s)\right) ds,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\phi_2 x(t) \ : \ &= -\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g\left(s, x(s), D^{\sigma}x(s)\right) ds \\ &+\frac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\beta)}\int_0^{\eta}(\eta-s)^{\alpha-1} g\left(s, x(s), D^{\sigma}x(s)\right) ds \\ &-\frac{1}{(\lambda_1\eta-1)\Gamma(\beta)}\int_0^1 (1-s)^{\alpha-1} g\left(s, x(s), D^{\sigma}x(s)\right) ds \\ &+\frac{\left(\lambda_2-\lambda_2\lambda_1\eta^3\right)t+(\lambda_2\lambda_1\eta-\lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)}\int_0^{\xi}(\xi-s)^{\alpha-3} g\left(s, x(s), D^{\sigma}x(s)\right) ds \\ &-\frac{\left(1-\lambda_1\eta^3\right)t+(\lambda_1\eta-1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)}\int_0^1 (1-s)^{\alpha-3} g\left(s, x(s), D^{\sigma}x(s)\right) ds.\end{aligned} \tag{3.5}$$

And we shall prove that $\phi$ is a contraction mapping.

Let $(x, y), (x_1, y_1) \in X \times Y$. Then, for each $t \in J$, we have:

$$\begin{aligned}|\phi_1 y(t)-\phi_1 y_1(t)| &\le \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right)-f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds \\ &+\tfrac{|\lambda_1| t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right)-f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds \\ &+\tfrac{t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right)-f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds \\ &+\tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t+|\lambda_2\lambda_1\eta-\lambda_2|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s, y(s), D^{\delta}y(s)\right)-f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds \\ &+\tfrac{\left|1-\lambda_1\eta^3\right|t+|\lambda_1\eta-1|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}\left|f\left(s, y(s), D^{\delta}y(s)\right)-f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds.\end{aligned} \tag{3.6}$$

Thanks to $(H2)$, we obtain

$$|\phi_1 y(t) - \phi_1 y_1(t)| \leq \frac{(|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha+1)\left(\omega_1\|y-y_1\|+\omega_2\left\|D^\delta y-D^\delta y_1\right\|\right)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \frac{\left[\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|\right]\left(\omega_1\|y-y_1\|+\omega_2\left\|D^\delta y-D^\delta y_1\right\|\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}. \quad (3.7)$$

Consequently,

$$|\phi_1 y(t) - \phi_1 y_1(t)| \leq N_1(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right), \quad (3.8)$$

Hence,

$$\|\phi_1(y) - \phi_1(y_1)\| \leq N_1(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right). \quad (3.9)$$

We have also,

$$\begin{aligned}\left|D^\sigma\phi_1 y(t) - D^\sigma\phi_1 y_1(t)\right| \leq{}& \frac{1}{\Gamma(\alpha-\sigma)}\int_0^t (t-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right|ds \\ &+\frac{|\lambda_1|t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right|ds \\ &+\frac{t^{1-\sigma}}{|(\lambda_1\eta-1)|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^1 (1-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right|ds \\ &+\left(\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} + \frac{|\lambda_2\lambda_1\eta-\lambda_2|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\right)\int_0^\xi (\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right|ds \\ &+\left(\frac{\left|1-\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} + \frac{|\lambda_1\eta-1|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\right)\int_0^1 (1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right|ds.\end{aligned} \quad (3.10)$$

By $(H2)$, yields

$$\begin{aligned}\left|D^\sigma\phi_1 y(t) - D^\sigma\phi_1 y_1(t)\right| \leq{}& \frac{(\omega_1+\omega_2)\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)}{\Gamma(\alpha-\sigma+1)} \\ &+\frac{(\omega_1+\omega_2)[|\lambda_1|\eta^\alpha+1]\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \\ &+\frac{(\omega_1+\omega_2)\left[\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|\right]\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} \\ &+\frac{(\omega_1+\omega_2)\left[|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|\right]\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}.\end{aligned} \quad (3.11)$$

This implies that,

$$\begin{aligned}\left|D^\sigma\phi_1 y(t) - D^\sigma\phi_1 y_1(t)\right| \leq{}& \left[\frac{(\omega_1+\omega_2)}{\Gamma(\alpha-\sigma+1)} + \frac{\%(\omega_1+\omega_2)[|\lambda_1|\eta^\alpha+1]}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right]\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right) \quad (3.12) \\ &+\left[\frac{(\omega_1+\omega_2)\left[\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|\right]}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} + \frac{(\omega_1+\omega_2)\left[|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|\right]}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\right]\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right).\end{aligned}$$

Therefore,

$$\left|D^{\sigma}\phi_1 y(t) - D^{\sigma}\phi_1 y_1(t)\right| \leq N_2(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^{\delta}y - D^{\delta}y_1\right\|\right). \tag{3.13}$$

Consequently,

$$\left\|D^{\sigma}\phi_1(y) - D^{\sigma}\phi_1(y_1)\right\| \leq N_2(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^{\delta}y - D^{\delta}y_1\right\|\right). \tag{3.14}$$

By (3.9) and (3.14), we can write

$$\|\phi_1(y) - \phi_1(y_1)\|_X \leq (N_1+N_2)(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^{\delta}y - D^{\delta}y_1\right\|\right). \tag{3.15}$$

With the same arguments as before, we have

$$\|\phi_2(x) - \phi_2(x_1)\|_Y \leq (N_3+N_4)(\varpi_1+\varpi_2)\left(\|x-x_1\| + \left\|D^{\sigma}x - D^{\sigma}x_1\right\|\right). \tag{3.16}$$

Using (3.15) and (3.16), we can state that

$$\|\phi(x,y) - \phi(x_1,y_1)\|_{X\times Y} \leq \left[\begin{array}{c}(N_1+N_2)(\omega_1+\omega_2)\\ +(N_3+N_4)(\varpi_1+\varpi_2)\end{array}\right](\|(x-x_1, y-y_1)\|_{X\times Y}). \tag{3.17}$$

Thanks to (3.2), we conclude that $\phi$ is contraction. As a consequence of Banach fixed point theorem, we deduce that $\phi$ has a unique fixed point which is a solution of (1.1).

The second main result is based on Schaefer theorem. We have:

**Theorem 3.2.** *Suppose that (H1) and (H3) are satisfied. Then, the problem* (1.1) *has at least one solution on J.*

**Proof: A:** Thanks to (H1), we can state that the operator $\phi$ is continuous on $X \times Y$.

**B:** We will prove that $\phi$ maps bounded sets into bounded sets in $X \times Y$.

So, taking $\rho > 0$, and $(x,y) \in B_{\rho} := \{(x,y) \in X\times Y; \|(x,y)\|_{X\times Y} \leq \rho\}$, then for each $t \in J$, we have:

$$\begin{aligned}
|\phi_1 y(t)| \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\
&+\frac{|\lambda_1|t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\
&+\frac{t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\
&+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t+|\lambda_2\lambda_1\eta-\lambda_2|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\
&+\frac{\left|1-\lambda_1\eta^3\right|t+|\lambda_1\eta-1|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\%\int_0^{1}(1-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds.
\end{aligned} \tag{3.18}$$

The condition $(H3)$ implies that

$$\begin{aligned}|\phi_1 y(t)| &\leq \frac{L_1(|\lambda_1\eta-1||\lambda_1|\eta^\alpha+1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{L_1\left[\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left(\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|\right)\right]}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)} \\ &\leq L_1\left[\begin{array}{c}\frac{|\lambda_1\eta-1||\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\ +\frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\end{array}\right].\end{aligned} \tag{3.19}$$

Then,

$$\|\phi_1(y)\| \leq L_1N_1. \tag{3.20}$$

For $D^\sigma$, we have the following inequalities

$$\begin{aligned}\left|D^\sigma\phi_1 y(t)\right| \leq& \frac{1}{\Gamma(\alpha-\sigma)}\int_0^t (t-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\ &+\frac{|\lambda_1|t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\ &+\frac{t^{1-\sigma}}{|(\lambda_1\eta-1)|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^1 (1-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\ &+\frac{1}{\Gamma(\alpha-2)}\left(\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(2-\sigma)} \\ +\frac{|\lambda_2\lambda_1\eta-\lambda_2|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(4-\sigma)}\end{array}\right)\int_0^\xi (\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\ &+\frac{1}{\Gamma(\alpha-2)}\left(\begin{array}{c}\frac{\left|1-\lambda_1\eta^3\right|t^{1-\sigma}}{\%6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(2-\sigma)} \\ +\frac{|\lambda_1\eta-1|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(4-\sigma)}\end{array}\right)\int_0^1 (1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds.\end{aligned} \tag{3.21}$$

By $(H3)$ again, yields the following formula

$$\begin{aligned}\left|D^\sigma\phi_1 y(t)\right| &\leq L_1\left[\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right] \\ &\quad +L_1\left[\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} \\ +\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\end{array}\right] \\ &\leq L_1\left[\begin{array}{c}\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \\ +\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}+\frac{\%|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\end{array}\right].\end{aligned} \tag{3.22}$$

Hence, we can write

$$\left\|D^\sigma\phi_1(y)\right\| \leq L_1N_2. \tag{3.23}$$

Using (3.20) and (3.23), we obtain

$$\|\phi_1(y)\|_X \leq L_1(N_1+N_2). \tag{3.24}$$

As before, we obtain

$$\|\phi_2(x)\|_Y \leq L_2(N_3 + N_4). \tag{3.25}$$

By (3.24) and (3.25), we get

$$\|\phi(x,y)\|_{X\times Y} \leq L_1(N_1 + N_2) + L_2(N_3 + N_4). \tag{3.26}$$

Therefore,

$$\|\phi(x,y)\|_{X\times Y} < \infty. \tag{3.27}$$

**C:** Now, we prove the equi-continuity of $\phi$.
Let us take $(x,y) \in B_\rho$, and $t_1, t_2 \in J$, with $t_1 < t_2$. We have:

$$\begin{aligned}
|\phi_1 y(t_2) - \phi_1 y(t_1)| \leq\; & \tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}\left((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right)\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\
&+\tfrac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\
&+\tfrac{|\lambda_1|(t_2-t_1)}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\
&+\tfrac{(t_1-t_2)}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{1}(1-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\
&+\tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|(t_2-t_1)+|\lambda_2\lambda_1\eta-\lambda_2|\left(t_2^3-t_1^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\
&+\tfrac{\left|1-\lambda_1\eta^3\right|(t_1-t_2)+|\lambda_1\eta-1|\left(t_1^3-t_2^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{1}(1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\
\leq\; & \tfrac{L_1\left(t_1^\alpha-t_2^\alpha+2\left(t_2^\alpha-t_1^\alpha\right)\right)}{\Gamma(\alpha+1)} + \tfrac{L_1|\lambda_1|\eta^\alpha(t_2-t_1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \tfrac{L_1(t_1-t_2)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\
&+\tfrac{L_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}(t_2-t_1)+L_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}\left(t_2^3-t_1^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)} \\
&+\tfrac{L_1\left|1-\lambda_1\eta^3\right|(t_1-t_2)+L_1|\lambda_1\eta-1|\left(t_1^3-t_2^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}.
\end{aligned} \tag{3.28}$$

Therefore,

$$\begin{aligned}
|\phi_1 y(t_2) - \phi_1 y(t_1)| \leq\; & L_1\left[\tfrac{|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_2-t_1) \\
&+L_1\left[\tfrac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \tfrac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_1-t_2) \\
&+L_1\left[\tfrac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right]\left(t_2^3-t_1^3\right) + \tfrac{L_1|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1^3-t_2^3\right) \\
&+\tfrac{L_1}{\Gamma(\alpha+1)}\left(t_1^\alpha-t_2^\alpha\right) + \tfrac{2L_1}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha.
\end{aligned} \tag{3.29}$$

We have also

$$\begin{aligned}\left|D^{\sigma}\phi_1 y(t_2) - D^{\sigma}\phi_1 y(t_1)\right| \leq & \frac{1}{\Gamma(\alpha-\sigma)}\int_0^{t_1}\left((t_1-s)^{\alpha-\sigma-1}-(t_2-s)^{\alpha-\sigma-1}\right)\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\ & +\frac{1}{\Gamma(\alpha-\sigma)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\ & +\frac{|\lambda_1|\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\ & +\frac{\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^{1}(1-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\ & +\left[\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\frac{|\lambda_2\lambda_1\eta-\lambda_2|\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\ & +\left[\begin{array}{c}\frac{\left|1-\lambda_1\eta^3\right|\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\frac{|\lambda_1\eta-1|\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)}\end{array}\right]\int_0^{1}(1-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds.\end{aligned} \tag{3.30}$$

The condition $(H3)$ implies that

$$\begin{aligned}\left|D^{\sigma}\phi_1 y(t_2) - D^{\sigma}\phi_1 y(t_1)\right| \leq & \frac{L_1}{\Gamma(\alpha-\sigma+1)}\left(t_1^{\alpha-\sigma}-t_2^{\alpha-\sigma}+2(t_2-t_1)^{\alpha-\sigma}\right) \\ +L_1\left[\begin{array}{c}\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \\ +\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\end{array}\right]\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right) & +L_1\left[\begin{array}{c}\frac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \\ +\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\end{array}\right]\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right) \\ +\frac{L_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right) & +\frac{L_1|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right).\end{aligned} \tag{3.31}$$

The inequalities (3.29) and (3.31) imply that:

$$\|\phi_1 y(t_2)-\phi_1 y(t_1)\|_X \le L_1\left[\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_2-t_1)$$
$$+L_1\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_1-t_2)+L_1\left[\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right]\left(t_2^3-t_1^3\right)$$
$$+\frac{L_1|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1^3-t_2^3\right)+\frac{L_1}{\Gamma(\alpha+1)}\left(t_1^\alpha-t_2^\alpha+2(t_2-t_1)^\alpha\right)$$
$$+\frac{L_1}{\Gamma(\alpha-\sigma+1)}\left(t_1^{\alpha-\sigma}-t_2^{\alpha-\sigma}+2(t_2-t_1)^{\alpha-\sigma}\right)+L_1\left[\begin{array}{c}\frac{|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\\+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\end{array}\right]\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)$$
$$+L_1\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}+\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\right]\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)$$
$$+\frac{L_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)+\frac{L_1|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right). \tag{3.32}$$

With the same arguments as before, we can write

$$\|\phi_2 x(t_2)-\phi_2 x(t_1)\|_Y \le L_2\left[\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^\beta}{|\lambda_1\eta-1|\Gamma(\beta+1)}+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\right](t_2-t_1) \tag{3.33}$$
$$+L_2\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\beta+1)}+\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\right](t_1-t_2)+L_2\left[\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\right]\left(t_2^3-t_1^3\right)$$
$$+\frac{L_2|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\left(t_1^3-t_2^3\right)+\frac{L_2}{\Gamma(\beta+1)}\left(t_1^\beta-t_2^\beta+2(t_2-t_1)^\beta\right)$$
$$+\frac{L_2}{\Gamma(\alpha-\beta+1)}\left(t_1^{\beta-\delta}-t_2^{\beta-\delta}+2(t_2-t_1)^{\beta-\delta}\right)+L_2\left[\begin{array}{c}\frac{|\lambda_1|\eta^\beta+1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)}\\+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\end{array}\right]\left(t_2^{1-\delta}-t_1^{1-\delta}\right)$$
$$+L_2\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)}+\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\right]\left(t_1^{1-\delta}-t_2^{1-\delta}\right)$$
$$+\frac{L_2|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}\left(t_2^{3-\delta}-t_1^{3-\delta}\right)+\frac{L_2|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}\left(t_1^{3-\delta}-t_2^{3-\delta}\right).$$

Thanks to (3.32) and (3.33), we can state that $\|\phi(x,y)(t_2)-\phi(x,y)(t_1)\|_{X\times Y}\to 0$ as $t_2\to t_1$. Combining **A, B, C** and using Arzela-Ascoli theorem, we conclude that $\phi$ is completely continuous operator.

**D:** We shall show that

$$\Omega := \{(x,y)\in X\times Y, (x,y)=\mu\phi(x,y), 0<\mu<1\}, \tag{3.34}$$

is a bounded set.

Let $(x,y)\in\Omega$, then $(x,y)=\mu\phi(x,y)$, for some $0<\mu<1$. Thus, for each $t\in J$, we have:

$$y(t)=\mu\phi_1 y(t), x(t)=\mu\phi_2 x(t).$$

Therefore,

$$\begin{aligned}\frac{1}{\mu}|y(t)| \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\frac{|\lambda_1|t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\frac{t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t+|\lambda_2\lambda_1\eta-\lambda_2|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^\xi (\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\frac{\left|1-\lambda_1\eta^3\right|t+|\lambda_1\eta-1|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds.\end{aligned} \tag{3.35}$$

Thanks to $(H3)$, we can write

$$\begin{aligned}\frac{1}{\mu}|y(t)| \leq\ & \frac{L_1(|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha+1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\ &+\frac{L_1\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}.\end{aligned} \tag{3.36}$$

Thus,

$$|y(t)| \leq \mu L_1\left[\begin{array}{c}\frac{(|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha+1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\ +\frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\end{array}\right]. \tag{3.37}$$

Hence,

$$|y(t)| \leq \mu N_1 L_1, t \in J. \tag{3.38}$$

On the other hand,

$$\begin{aligned}\frac{1}{\mu}\left|D^\sigma y(t)\right| \leq\ & \frac{1}{\Gamma(\alpha-\delta)}\int_0^t (t-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\frac{|\lambda_1|t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\frac{t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\left[\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\frac{|\lambda_2\lambda_1\eta-\lambda_2|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^\xi (\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\left[\begin{array}{c}\frac{\left|1-\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\frac{|\lambda_1\eta-1|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^1 (1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds.\end{aligned} \tag{3.39}$$

Thanks to $(H3)$, we have

$$\frac{1}{\mu}\left|D^{\sigma}y(t)\right| \leq L_1\left[\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right] + L_1\left[\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\right]. \tag{3.40}$$

Therefore,

$$\left|D^{\sigma}y(t)\right| \leq \mu L_1\left[\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right] + \mu L_1\left[\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\right]. \tag{3.41}$$

Thus,

$$\left|D^{\beta}y(t)\right| \leq \mu L_1 N_2, t \in J. \tag{3.42}$$

From (3.38) and (3.42) , we get

$$\|y\|_X \leq \mu L_1 (N_1 + N_2). \tag{3.43}$$

Analogously, we can obtain

$$\|x\|_Y \leq \mu L_2 (N_3 + N_4). \tag{3.44}$$

It follows from (3.43) and (3.4) that

$$\|(x, y)\|_{X\times Y} \leq \mu L_1 (N_1 + N_2) + \mu L_2 (N_3 + N_4). \tag{3.45}$$

Hence,

$$\|\phi(x, y)\|_{X\times Y} < \infty. \tag{3.46}$$

This shows that the set $\Omega$ is bounded. Thanks to **A,B,C** and **D**, we conclude that $\phi$ has at least one fixed point. Theorem 3.2 is thus proved.

## 4. Examples

**Example 4.1.** Let us consider the coupled equations:

$$\begin{aligned}
&D^{\frac{7}{2}}x(t)+\frac{|y(t)|}{7\left(\pi t^2+3\right)^2(2+|y(t)|)}+\frac{\sqrt{\pi}e^{-\pi t}|\cos(\pi t)|\left|D^{\frac{7}{3}}y(t)\right|}{7\pi(1+e^t)^2\left(2+\left|D^{\frac{7}{3}}y(t)\right|\right)}=0, t\in[0,1],\\
&D^{\frac{11}{3}}y(t)+\frac{3\pi|x(t)|}{\left(5e^{t^2}+3\sqrt{\pi}\right)(1+|x(t)|)}+\frac{\pi e^{-2\pi t}\left|D^{\frac{5}{2}}x(t)\right|}{5\left(t+3\sqrt{\pi}\right)^2\left(1+\left|D^{\frac{5}{2}}x(t)\right|\right)}=0, t\in[0,1],\\
&x(0)=0, x(1)-\tfrac{3}{4}x\left(\tfrac{1}{3}\right)=0, y(0)=0, y(1)-\tfrac{3}{4}y\left(\tfrac{1}{3}\right)=0,\\
&x''(0)=0, x''(1)-\tfrac{4}{5}x''\left(\tfrac{2}{3}\right)=0, y''(0)=0, y''(1)-\tfrac{4}{5}y''\left(\tfrac{2}{3}\right)=0.
\end{aligned} \tag{4.1}$$

It is clear that

$$f(t,x,y) = \frac{|x|}{7\left(\pi t^2+3\right)^2(2+|x|)} + \frac{\sqrt{\pi}e^{-\pi t}|\cos(\pi t)||y|}{7\pi(1+e^t)^2(2+|y|)}, t\in[0,1], x,y\in\mathbb{R},$$
$$g(t,x,y) = \frac{3\pi|x|}{\left(5e^{t^2}+3\sqrt{\pi}\right)(1+|x|)} + \frac{\pi e^{-2\pi t}|y|}{5\left(t+3\sqrt{\pi}\right)^2(1+|y|)}, t\in[0,1], x,y\in\mathbb{R}.$$

For $x, y, x_1, y_1 \in \mathbb{R}, t \in [0,1]$, we have

$$|f(t,x,y) - f(t,x_1,y_1)| \le \frac{1}{7\left(\pi t^2+3\right)^2}|x-x_1| + \frac{\sqrt{\pi}e^{-\pi t}}{7\pi(1+e^t)^2}|y-y_1|,$$

and

$$|g(t,x,y) - g(t,x_1,y_1)| \le \frac{3\pi}{\left(5e^{t^2}+3\sqrt{\pi}\right)}|x-x_1| + \frac{\pi e^{-2\pi t}}{5\left(t+3\sqrt{\pi}\right)^2}|y-y_1|.$$

Hence,

$$a_1(t) = \frac{1}{7\left(\pi t^2+3\right)^2}, b_1(t) = \frac{\sqrt{\pi}e^{-\pi t}}{7\pi(1+e^t)^2},$$

and

$$a_2(t) = \frac{3\pi}{5e^{t^2}+3\sqrt{\pi}}, b_2(t) = \frac{\pi e^{-2\pi t}}{5\left(t+3\sqrt{\pi}\right)^2}.$$

These imply that

$$\omega_1 = \sup_{t\in[0,1]} a_1(t) = \frac{1}{63}, \omega_2 = \sup_{t\in[0,1]} b_1(t) = \frac{\sqrt{\pi}}{28\pi},$$
$$\varpi_1 = \sup_{t\in[0,1]} a_2(t) = \frac{3\pi}{5+3\sqrt{\pi}}, \varpi_2 = \sup_{t\in[0,1]} b_2(t) = \frac{1}{45},$$
$$N_1 = 1,08935, N_2 = 3,444, N_3 = 0,77571, N_4 = 2,51754,$$

and,

$$(N_1+N_2)(\omega_1+\omega_2) + (N_3+N_4)(\varpi_1+\varpi_2) = 0,16329 + 0,36466 = 0,52795 < 1.$$

So by Theorem 3.1, the problem (4.1) has a unique solution $(x, y)$ on $[0,1]$.

**Example 4.2.** The following example illustrates Theorem 3.2. We take:

$$\begin{aligned}
&D^{\frac{15}{4}}x(t)+\frac{1}{\left(t^2+1\right)\left(2+\left|D^{\frac{7}{3}}y(t)\right|\right)}+\frac{2e^{-t}|\cos(ty(t))|}{7(1+e^t)^2\left(2+\left|D^{\frac{7}{3}}y(t)\right|\right)}=0, t\in[0,1],\\
&D^{\frac{10}{3}}y(t)+\frac{1}{\left(e^{t^2}+1\right)(1+|x(t)|)}+\frac{e^{-t}}{(t+1)^2\left(1+\left|D^{\frac{5}{2}}x(t)\right|\right)}=0, t\in[0,1],\\
&x(0)=0, x(1)-\tfrac{3}{4}x\left(\tfrac{1}{3}\right)=0, y(0)=0, y(1)-\tfrac{3}{4}y\left(\tfrac{1}{3}\right)=0,\\
&x''(0)=0, x''(1)-\tfrac{4}{5}x''\left(\tfrac{2}{3}\right)=0, y''(0)=0, y''(1)-\tfrac{4}{5}y''\left(\tfrac{2}{3}\right)=0.
\end{aligned} \tag{4.2}$$

We have

$$f(t,x,y)=\frac{1}{\left(t^2+1\right)(2+|y|)}+\frac{2e^{-t}|\cos(tx)|}{7(1+e^t)^2(2+|y|)}$$

and

$$g(t,x,y)=\frac{1}{\left(e^{t^2}+1\right)(1+|x|)}+\frac{e^{-t}}{(t+1)^2(1+|y|)}$$

So by Theorem 3.2, the problem (4.2) has at least one solution on $[0,1]$.

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# On the Divisibility of Trinomials by Maximum Weight Polynomials over $\mathbb{F}_2$

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## Abstract

Divisibility of trinomials by given polynomials over finite fields has been studied and used to construct orthogonal arrays in recent literature. Dewar et al. (Dewar *et al.*, 2007) studied the division of trinomials by a given pentanomial over $\mathbb{F}_2$ to obtain the orthogonal arrays of strength at least 3, and finalized their paper with some open questions. One of these questions is concerned with generalizations to the polynomials with more than five terms. In this paper, we consider the divisibility of trinomials by a given maximum weight polynomial over $\mathbb{F}_2$ and apply the result to the construction of the orthogonal arrays of strength at least 3.

*Keywords:* Divisibility of trinomials, Maximum weight polynomials, Orthogonal arrays.
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## 1. Introduction

Sparse irreducible polynomials such as trinomials over $\mathbb{F}_2$ are widely used to perform arithmetic in extension fields of $\mathbb{F}_2$ due to fast modular reduction. In particular, primitive trinomials and maximum-length shift register sequences generated by them play an important role in various applications such as stream ciphers (see (Golomb, 1982), (Jambunathan, 2000)). But even irreducible trinomials do not exist for every degree. When a primitive (respectively irreducible) trinomial of a given degree does not exist, an almost primitive (respectively irreducible) trinomial, which is a reducible trinomial with primitive (respectively irreducible) factor, may be used as an alternative (Brent & Zimmermann, 2004). This encouraged the researchers to study divisibility of trinomials by primitive or irreducible polynomials (Cherif, 2008), (Golomb & Lee, 2007), (Kim & Koepf, 2009). The divisibility of trinomials by primitive polynomials is also related to orthogonal arrays.

Let $f$ be a polynomial of degree $m$ over $\mathbb{F}_2$ and let $a = (a_0, a_1, \cdots)$ be a shift-register sequence with characteristic polynomial $f$. Denote by $C_n^f$ the set of all subintervals of this sequence with

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length $n$, where $m < n \leq 2m$, together with the zero vector of length $n$. Munemasa (Munemasa, 1998) observed that very few trinomials of degree at most $2m$ are divisible by a given primitive trinomial of degree $m$ and proved that if $f$ is a primitive trinomial satisfying certain properties, then $C_n^f$ is an orthogonal array of strength 2 having the property of being very close to an orthogonal array of strength 3. Munemasa's work was extended in (Dewar *et al.*, 2007). The authors considered the divisibility of a trinomial of degree at most $2m$ by a given pentanomial $f$ of degree $m$ and obtained the orthogonal arrays of strength 3. They suggested some open questions in the end of their paper. One of them is to extend the results to finite fields other than $\mathbb{F}_2$. In this regard, Panario et al. (Panario *et al.*, 2012) characterized the divisibility of binomials and trinomials over $\mathbb{F}_3$. Another question in (Dewar *et al.*, 2007) is related to extend the results to the polynomials with more than five terms. In this paper we analyze the division of trinomials by a maximum weight polynomial over $\mathbb{F}_2$.

In the theory of shift register sequences it is well known that the lower the weight, i.e. the number of nonzero coefficients of the characteristic polynomial of shift register sequence, is, the faster is the generation of the sequence. But Ahmadi and Menezes (Ahmadi & Menezes, 2007) point out the advantage of maximum weight polynomials over $\mathbb{F}_2$ in the implementation of fast arithmetic in extension fields.

We show that no trinomial of degree at most $2m$ is divisible by a given maximum weight polynomial $f$ of degree $m$, provided that $m > 7$. Using this result we can also obtain the orthogonal arrays of strength at least 3. The rest of the paper is organized as follows. In Section 2, some basic definitions and results are given and in Section 3, some properties of maximum weight polynomials and shift register sequences generated by them are mentioned. We focus on the divisibility of trinomials by maximum weight polynomials in Section 4, and conclude in Section 5.

## 2. Preliminaries

A *period* of a nonzero polynomial $f(x) \in \mathbb{F}_q[x]$ with $f(0) \neq 0$ is the least positive integer $e$ for which $f(x)$ divides $x^e - 1$. A polynomial $f(x) \in \mathbb{F}_q[x]$ is called *reducible* if it has nontrivial factors; otherwise *irreducible*. A polynomial $f(x)$ of degree $m$ is called *primitive* if it is irreducible and has period $2^m - 1$. The *reciprocal polynomial* of $f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \in \mathbb{F}_q[x]$ with $a_m \neq 0$ is defined by

$$f^*(x) = x^m f(1/x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_{m-1}x + a_m.$$

We refer to (Lidl & Niederreiter, 1994) for more information on the polynomials over finite fields. Throughout this paper we only consider a binary field $\mathbb{F}_2$ and all the polynomials are assumed to be in $\mathbb{F}_2[x]$, unless otherwise specified.

A *shift-register sequence* with characteristic polynomial $f(x) = x^m + \sum_{i=0}^{m-1} c_i x^i$ is the sequence $a = (a_0, a_1, \cdots)$ defined by the recurrence relation

$$a_{n+m} = \sum_{i=0}^{m-1} c_i a_{i+n}$$

for $n \geq 0$.

A subset $C$ of $\mathbb{F}_2^n$ is called an *orthogonal array* of strength $t$ if for any $t-$ subset $T = \{i_1, i_2, \cdots, i_t\}$ of $\{1, 2, \cdots, n\}$ and any $t-$tuple $(b_1, b_2, \cdots, b_t) \in \mathbb{F}_2^t$, there exist exactly $|C|/2^t$ elements $c = (c_1, c_2, \cdots, c_n)$ of $C$ such that $c_{i_j} = b_j$ for all $1 \leq j \leq t$ (Munemasa, 1998). From the definition, if $C$ is an orthogonal array of strength $t$, then it is also an orthogonal array of strength $s$ for all $1 \leq s \leq t$.

The next theorem, due to Delsarte, relates orthogonal arrays to linear codes.

**Theorem 2.1.** (Delsarte, 1973) *Let $C$ be a linear code over $\mathbb{F}_q$. Then $C$ is an orthogonal array of maximum strength $t$ if and only if $C^\perp$, its dual code, has minimum weight $t + 1$.*

Munemasa (Munemasa, 1998) described the dual code of the code generated by a shift-register sequence in terms of multiples of its primitive characteristic polynomial and Panario et al. (Panario *et al.*, 2012) generalized this result as follows by removing the primitiveness condition for the characteristic polynomial.

**Theorem 2.2.** (Panario *et al.*, 2012) *Let $a = (a_0, a_1, \cdots)$ be a shift register sequence with minimal polynomial $f$, and suppose that $f$ has degree $m$ with $m$ distinct roots. Let $\rho$ be the period of $f$ and $2 \leq n \leq \rho$. Let $C_n^f$ be the set of all subintervals of the shift register sequence $a$ with length $n$, together with the zero vector of length $n$. Then the dual code of $C_n^f$ is given by*

$$(C_n^f)^\perp = \{(b_1, \cdots, b_n) : \sum_{i=0}^{n-1} b_{i+1} x^i \text{ is divisible by } f.\}$$

A *maximum weight polynomial* is a degree-$m$ polynomial of weight $m$ (where $m$ is odd) over $\mathbb{F}_2$(Ahmadi & Menezes, 2007), namely,

$$f(x) = x^m + x^{m-1} + \cdots + x^{l+1} + x^{l-1} + \cdots + x + 1 = \frac{x^{m+1} + 1}{x + 1} + x^l$$

If you take

$$g(x) = (x + 1) f(x) = x^{m+1} + x^{l+1} + x^l + 1,$$

then the weight of $g(x)$ is 4, and its middle terms are consecutive, so reduction using $g(x)$ instead of $f(x)$ is possible and can be effective in the arithmetic of an extension field $\mathbb{F}_{2^m}$ as if the reduction polynomial were a trinomial or a pentanomial. This fact motivated us to consider the divisibility of trinomials by maximum weight polynomials.

## 3. Character of shift register sequence generated by a maximum weight polynomial

In this section we state a simple property of maximum weight polynomials and characterize the shift register sequences generated by them.

**Proposition 3.1.** *Let $f(x) = x^m + x^{m-1} + \cdots + x^{l+1} + x^{l-1} + \cdots + 1 \in \mathbb{F}_2[x]$. If $f(x)$ is irreducible, then* $\gcd(m, l) = 1$.

*Proof.* Suppose $\gcd(m, l) = d > 1, m = m_1 d$ and $l = l_1 d$. Then we have

$$\begin{aligned} g(x) &:= (x+1)f(x) = x^{m+1} + x^{l+1} + x^l + 1 \\ &= x^{l+1}(x^{m-l}+1) + (x^l+1) = x^{l+1}(x^{m_1 d - l_1 d}+1) + (x^{l_1 d}+1) \\ &= x^{l+1}(x^{d(m_1-l_1)}+1) + (x^{l_1 d}+1). \end{aligned}$$

So $(x^d+1)/(x+1)$ is a factor of $f(x)$, which means $f(x)$ is reducible. □

**Proposition 3.2.** *Let* $f(x) = x^m + x^{m-1} + \cdots + x^{l+1} + x^{l-1} + \cdots + 1 \in \mathbb{F}_2[x]$ *be a primitive polynomial and*

$$a_{n+m} = \sum_{i=0}^{m-1} a_{n+i} + a_{n+l} (n \geq 0)$$

*be a shift-register sequence with characteristic polynomial* $f$*. Then for all positive integer* $n$*,*

$$a_{n+m} = a_{n-1} + a_{n-1+l} + a_{n+l}.$$

*Proof.* Since $f(x)$ is the characteristic polynomial of $(a_0, a_1, \cdots)$, we get $a_l = a_0 + a_1 + \cdots + a_m$ where $a_0, a_1, \cdots, a_{m-1}$ are initial values not all of which are zero. We use induction on $n$.

If $n = 1$,

$$\begin{aligned} a_{m+1} &= a_1 + \cdots + a_l + a_{l+2} + \cdots + a_m \\ &= a_0 + (a_0 + \cdots + a_l + a_{l+1} + a_{l+2} + \cdots + a_m) + a_{l+1} \\ &= a_0 + a_l + a_{l+1}. \end{aligned}$$

Now assume that the equation $a_{n+m} = a_{n-1} + a_{n-1+l} + a_{n+l}$ holds true for all positive integers less or equal to $n$. Then,

$$\begin{aligned} a_{m+n+1} &= a_{n+1} + \cdots + a_{n+l} + a_{n+l+2} + \cdots + a_{n+m} \\ &= (a_0 + \cdots + a_m) + (a_0 + \cdots + a_n) + a_{n+l+1} \\ &\quad +(a_{m+1} + \cdots + a_{m+n}) \\ &= a_l + (a_0 + \cdots + a_n) + a_{n+l+1} + (a_0 + a_l + a_{l+1}) \\ &\quad +(a_1 + a_{l+1} + a_{l+2}) + \cdots + (a_{n-1} + a_{l+n-1} + a_{l+n}) \\ &= a_n + a_{l+n} + a_{n+l+1} \end{aligned}$$

This completes the proof. □

## 4. Divisibility of trinomials by maximum weight polynomials

In this section we consider the divisibility of trinomials by maximum weight polynomials, provided that the degree of the trinomial does not exceed double the degree of the maximum weight polynomial. Let $f(x) = x^m + x^{m-1} + \cdots + x^{l+1} + x^{l-1} + \cdots + 1 \in \mathbb{F}_2[x]$ and suppose that $f(x)$ divides a trinomial $g(x)$ with

$$g(x) = f(x)h(x) = (x^m + x^{m-1} + \cdots + x^{l+1} + x^{l-1} + \cdots + 1) \cdot \sum_{k=0}^{t} x^{i_k},$$

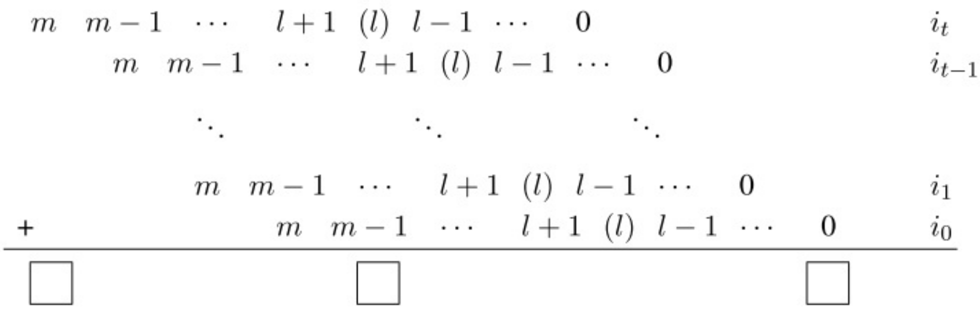

**Figure 1.** An illustration of equation $g(x) = f(x)\sum_{k=0}^{t} x^{i_k}$

where $x^{i_k}$s are the non-zero terms of $h(x)$ and $0 = i_0 < i_1 < \cdots < i_t$. The above equation can be illustrated as in Figure 1.

Here $(l)$ stands for the missing terms. We adopt the same terminology as in (Dewar *et al.*, 2007), (Panario *et al.*, 2012). In particular, if the sum of coefficients in the same column of Figure 1 is 0, then we write that the corresponding terms $x^i$ *cancel* and if the sum is 1 then we say that one of the corresponding terms is *left-over*. The proof of our main results will be done with Figure 1. Since the most top-left term $m + i_t$ and the most bottom-right term $0 + i_0$ are trivial left-over terms, we have only one left-over term undetermined. Below a *left-over term* means the left-over term which is neither $m + i_t$ nor $0 + i_0$. And we always assume that $m + i_0$ is in the same column as $s + i_t, 0 \le s \le m - 1$ and denote the number of terms in $h(x)$ as $N$.

**Lemma 4.1.** *Let* $f(x) = x^m + x^{m-1} + \cdots + x^{l+1} + x^{l-1} + \cdots + 1 \in \mathbb{F}_2[x]$ *and* $g(x)$ *be a trinomial of degree at most* $2m$ *divisible by* $f(x)$ *with* $g(x) = f(x)h(x)$*. Then* $N$ *equals to 3 or 5.*

*Proof.* Since $g(x)$ is a trinomial and $f(x)$ has an odd number of terms, $h(x)$ also has an odd number of terms, that is, $t$ is even. Suppose that $N$ is greater or equal to 7. If $s \ge l$ then for every even number $k$, $m + i_{t-k}$ is a left-over term. Since $t \ge 6$, we have more than 2 left-over terms which contradicts the assumption.

Consider the case of $s < l$. First assume that there exists a unique left-over term to the left of $m + i_0$. It is sufficient to show $l \ge 3$ because if so, $0 + i_2$ is an extra left-term which leads to a contradiction. Observe a position $l + i_t$. If $l + i_t \ge m + i_{t-2}$ then clearly $l \ge i_{t-2} - i_0 \ge 4$, so we have done. Assume that $l + i_t < m + i_{t-2}$. Then $l + i_t \ge m + i_{t-4}$ because if not, then $m + i_{t-2}$ and $m + i_{t-4}$ are left-over terms. Thus we have $l \ge i_{t-4} - i_0$. If $l + i_t > m + i_{t-4}$ then $l > 2$ and if $l + i_t = m + i_{t-4}$ then $i_{t-4} - i_0 > 2$ because if $i_{t-4} - i_0 = 2$ then $m + i_{t-5} = l + i_{t-1}$ and so an extra left-over term appears.

Next assume that there is no left-over term to the left of $m+i_0$. Then it is clear that $m+i_{t-2} = l+i_t$ and $l \ge i_{t-2} - i_0 \ge 5$ hence $0 + i_2$ and $0 + i_4$ are left-over terms; contradiction. □

**Lemma 4.2.** *Under the same condition as in Lemma 1, if* $s < l$ *then* $m + i_0$ *cannot be a left-over term.*

*Proof.* Assume that $m + i_0$ is a left-over term. Then all the remaining terms in other columns must cancel and by Lemma 1 $N = 3$ or $N = 5$. If $N = 3$, then $l + i_1 > m + i_0$ from $s < l$ and thus an extra left-over term occurs in the column of $l + i_1$. Now assume that $N$ is 5. We see easily $l + i_t = m + i_{t-2}$ and $i_t - i_{t-1} = 1$. If there is an extra left-over term to the left of $m + i_0$, then we have done. If there is no any extra left-over term to the left of $m + i_0$, then $i_2 - i_1 = 2$ because if $i_2 - i_1 = 1$ then $m + i_1 = l + i_{t-1}$ and so $m + i_1$ is an extra left-over term and if $i_2 - i_1 > 2$ then $l - 2 + i_t = l - 1 + i_{t-1} = m - 2 + i_2$ and so $l - 2 + i_t$ is an extra left-over term. Then from the condition $i_t \leq m$, it follows $l \geq 3$ and thus $0 + i_2$ is an extra left-over term; contradiction. □

**Theorem 4.1.** *Let* $f(x) = x^m + x^{m-1} + \cdots + x^{l+1} + x^{l-1} + \cdots + 1 \in \mathbb{F}_2[x]$. *If* $g(x)$ *is a trinomial of degree at most* $2m$ *divisible by* $f(x)$ *with* $g(x) = f(x)h(x)$, *then*

1) $f(x)$ *is one of the polynomial exceptions given in Table 1.*

2) $f(x)$ *is the reciprocal of one of the polynomials listed in the previous item.*

**Table 1.** Table of polynomial exceptions

| No | $g(x)$ | $f(x)$ | $h(x)$ |
|---|---|---|---|
| 1 | $x^5 + x^4 + 1$ | $x^3 + x + 1$ | $x^2 + x + 1$ |
| 2 | $x^6 + x^4 + 1$ | $x^3 + x^2 + 1$ | $x^3 + x^2 + 1$ |
| 3 | $x^9 + x^7 + 1$ | $x^5 + x^3 + x^2 + x + 1$ | $x^4 + x + 1$ |
| 4 | $x^7 + x^5 + 1$ | $x^5 + x^4 + x^3 + x + 1$ | $x^2 + x + 1$ |
| 5 | $x^8 + x^5 + 1$ | $x^5 + x^4 + x^3 + x^2 + 1$ | $x^3 + x^2 + 1$ |
| 6 | $x^{14} + x^{13} + 1$ | $x^7 + x^6 + x^5 + x^4 + x^3 + x + 1$ | $x^7 + x^5 + x^2 + x + 1$ |
| 7 | $x^{13} + x^{10} + 1$ | $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + 1$ | $x^6 + x^5 + x^3 + x^2 + 1$ |

*Proof.* We divide into three cases: $s > l$ or $s = l$ or $s < l$.

**Case 1 :** $s > l$.

Since $h(x)$ has an odd number of terms, $s \leq m - 2$ and $m + i_0$ is a left-over term, hence all the remaining terms in other columns must cancel. There is no missing term to the left of $s + i_t$, and therefore $m + i_{t-2}$ is a left-over term. This means $i_0 = i_{t-2}$, namely, $N = 3$. Since $m - 1 + i_0$ must cancel, $s = l + 1$ and $m - 2 + i_0$ cancels up automatically from $i_t - i_{t-1} = 1$. We see easily that $l = 1$ or $m - 3 + i_0$ is a missing term because $m - 3 + i_0$ must cancel up. If $l = 1$, then clearly $m = 5$ and we get the 5th polynomial in Table 1. If $m - 3 + i_0$ is a missing term, then $l = m - 3$. Since $l - 1 + i_0$ must cancel up, $l$ must equal to 2 and so we get the 4th polynomial in Table 1.

**Case 2 :** $s = l$.

In this case, $m + i_0$ cannot be a left-over term because the number of non-zero terms in column of $m + i_0$ is even. If there is a unique left-over term to the left of $m + i_0$, then it must be $m - 1 + i_t$ or $m + i_2$.

**Case 2.1 :** $m - 1 + i_t$ is a unique left-over term to the left of $m + i_0$.

Clearly $i_{t-1} = i_t - 2$. If $N = 3$ then $m - 1 + i_0$ is an extra left-term and if $N = 5$ then $m + i_{t-2}$ is so. This contradicts to the assumption.

**Case 2.2 :** $m + i_2$ is a unique left-over term to the left of $m + i_0$.

This is the case of $N = 5$ and $i_t - i_{t-1} = i_2 - i_1 = 1$. $m - 1 + i_0$ cancels automatically because $m - 1 + i_0 = l + i_{t-1}$. Thus we have only two possible cases: $l = 1$ or $l \neq 1, l + i_2 = m - 2 + i_0$. Assume that $l = 1$ then $m - 3 + i_0$ must be in the column of $l + i_2$ and $m - 5 + i_0$ must cancel with $0 + i_1$ so we get the 7th polynomial in Table 1. And assume that $l \neq 1, l + i_2 = m - 2 + i_0$ then $i_{t-1} - i_2 = 1$ and observing $m - 4 + i_0$ implies that $m - 4 = l, l - 3 \neq 0$ or $m - 4 > l, l = 3$. In these two cases we have an extra left-over term $l - 2 + i_0$; contradiction.

**Case 2.3 :** There is no left-over term to the left of $m + i_0$.

It is obvious that $N = 3$ and $i_t - i_1 = 1$. If $i_1 - i_0 > 3$ then we have two left-over terms among $j + i_0 (1 \le j \le 3)$. Hence $i_1 - i_0$ is less or equals to 3. Examining all cases for $i_1 - i_0$ we get the reciprocals of the 1st, 3rd and 4th polynomials in Table 1.

**Case 3 :** $s < l$.

By lemma 2, $m + i_0$ is not a left-over term. So there exists $z (1 \le z \le t-1)$ such that $m + i_0 = l + i_z$.

**Case 3.1 :** $m + i_0 = l + i_{t-1}$.

Clearly we have $i_{t-1} \ge i_t - 3$. First assume that $i_{t-1} = i_t - 3$. Then $l$ equals to $m - 1$ or $m - 2$. If $l = m - 1$, then $l - 1 + i_t = m - 2 + i_t$ is a left-over term so $l - 3 + i_t = l + i_{t-1} = m + i_0$ and $h(x)$ has three terms. Since the unique left-over term has already been determined, $0 + i_t = l - 1 + i_{t-1} = l + i_0$ and we get the 3rd polynomial in Table 1. If $l = m - 2$, then $m - 1 + i_t$ is a left-over term and $m + i_0$ must cancel with $0 + i_t$ which means $i_1 - i_0 = 2$ and $l = 3$. But then $1 + i_0$ appears as an extra left-over term; contradiction.

Next assume that $i_{t-1} = i_t - 2$. When $l \ne m - 1$, $m - 1 + i_t$ is a left-over term and $l \le m - 3$ because if $l = m - 2$ then $m + i_{t-1}$ is an extra left-over term. $l + i_t$ must cancel with $m + i_{t-2}$ and in fact $N$ is 5. Thus $i_2 - i_1 = 1$. By the condition $m + i_0 = l + i_{t-1}$, we have $i_1 - i_0 = 1$. Since $m - 1 + i_0$ must cancel up, $l - 2 = 0$ or $m - 3 = l$. If $l - 2 = 0$ then we get the 6th polynomial in Table 1 and the equation $m - 3 = l$ leads to a contradiction due to an extra left-over term in column of $l - 3 + i_0$. When $l = m - 1$, clearly $N$ is 3 from the conditin $l + i_{t-1} = m + i_0$. By research of possible values of $l$ we get the reciprocals of the 2nd and 5th polynomials in Table 1.

Next assume that $i_{t-1} = i_t - 1$. If $N = 5$ then $m + i_{t-2}$ is a left-over term and $i_{t-2} - i_1 = 1$, hence an extra left-over term occurs in the comumn of $l + i_t$. Thus $N$ is 3. Since $l - 1 + i_t = l + i_{t-1} = m + i_0$, $l + 1 + i_{t-1}$ is a left-over term. If $m - 1 \ne l$, then $l - 1 = 0$ from consideration of $m - 1 + i_0$ and therefore we get the 2nd polynomial in Table 1. If $m - 1 = l$, then $l - 1$ cannot be zero, so we get the 1st polynomial in Table 1.

**Case 3.2 :** $m + i_0 = l + i_2$.

In this case $N$ is 5 and clearly $2 \le l \le m - 2$. Observe a column of $l + i_t$.

**Case 3.2.1 :** $m + i_2 < l + i_t$.

We have a left-over term in the column of $l + i_t$ and $i_t - i_{t-1} = 1$. Then $m + i_2$ must cancel with $l - 1 + i_t$ and also $i_2 - i_1 - 1$. By the condition $l + i_2 = m + i_0$, $m - 1 + i_0$ must cancel with $l + i_1$. From $i_t \le m$ we have $l \ge 3$ and $i_1 - i_0 = 1$ because if not, then $1 + i_0$ is an extra left-over term. Hence $l$ equals to $m - 2$. Since $m - 1 + i_0$ must cancel up, $l - 4 \ne 0$. Observing the term $l - 1 + i_0$, we see that $l - 5 = 0$ and then $l - 2 + i_0$ appears as an extra left-over term; contradiction.

**Case 3.2.2 :** $m + i_2 = l + i_t$.

Assume that $m - 1 + i_t$ is a left-over term. Then clearly $l < m - 2$ and $i_t - i_{t-1} = 2$. If $i_2 - i_0 = 2$, then $m + i_0$ must concel with $l + i_{t-1}$ which contradicts to the condition $m + i_0 = l + t_{t-2}$. And if $i_2 - i_0 > 2$, then an extra left-over term occurs in the column of $l + 1 + i_t$ or $l + 2 + i_t$ which again leads to a contradiction.

Now assume that $m - 1 + i_t$ is not a left-over term. Then $i_t - i_{t-1} = 1$ and $m + i_1$ cancels with $l + i_{t-1}$ or $m + i_1 < l + i_{t-1}$. If $m + i_1$ cancels with $l + i_{t-1}$ then $m + i_1$ is a left-over term and $i_2 - i_1 = 1$. From $i_t \le m$, we have $0 \le l - 2$. Since if $i_1 - i_0 \ge 2$ then $1 + i_0$ is an extra left-over term, $i_1 - i_0 = 1$ and $l = m - 2 = 4$. Then $l + 2 + i_0$ appears as an extra left-over term; contradiction. If $m + i_1 < l + i_{t-1}$

then $m + i_1$ must cancel with $m - 2 + i_2$ or $m - 3 + i_2$. Briefly considering as above, we arrive at a contradiction in both cases.

**Case 3.2.3 :** $m + i_2 > l + i_t$.

You shall see that $l \leq m - 3, i_t - i_{t-1} = 1$ and $m + i_2$ is a left-over term. Since $m - 1 + i_2$ must cancel, $m - 1 + i_2 = l + i_t$ or $m - 1 + i_2 = m + i_1$. In the first case $i_2 - i_1 = 3$ because $l + i_{t-1} = m - 2 + i_2 = l - 1 + i_t$. Since $m + i_1 < l + i_{t-1}$, $l$ is greater or equals to 3. If $i_1 - i_0 > 1$ then $1 + i_0$ is an extra left-over term and if $i_1 - i_0 = 1$ then $l = 3$ and $m - 2 + i_0$ is an extra left-over term, which leads to a contradiction. In the second case we have $l + i_t = m + i_0$; contradiction.

**Case 3.3 :** $m + i_0 = l + i_1$.

In this case we have $l \geq 3$ from $i_t \leq m$. First assume that $1 + i_0$ is a left-over term. Then clearly $i_1 - i_0 = 2, l + i_0 = 0 + i_2$ and $l + 1 + i_0 = l - 1 + i_1 = 1 + i_2 = 0 + i_{t-1}$. Since $l + 2 + i_0 = l + i_1 = 2 + i_2 = 1 + i_{t-1} = 0 + i_t$, we have $m = l + 2$. Then from $5 + i_2 = 4 + i_{t-1} = 3 + i_t$, we have $l = 5$ which corresponds the reciprocal of the 6th polynomial in Table 1.

Next assume that $1 + i_0$ is not a left-over term. Then $i_1 - i_0 = 1, l = m - 1$ and $0 + i_2$ is a left-over term because if not, then $0 + i_2 = l + i_0$ and thus $N = 3$ which is the case mentioned above. Considering the first and last terms in every rows, we have the following equations:

$$i_{t-1} - i_2 = 1, 0 + i_t = l + i_0, l + i_2 > m + i_1, i_2 - i_1 = 2,$$
$$0 + i_t = l + i_0, i_t - i_{t-1} = 2.$$

This implies the reciprocal of the 7th polynomial in Table 1. □

Note that every polynomial $f(x)$ listed in Table 1 has degree less than 8. From this fact we can immediately get the following corollary.

**Corollary 4.1.** *Let $f(x)$ be a maximum weight polynomial of odd degree $m$ greater than 7 and $g(x)$ be a trinomial of degree at most $2m$. Then $g(x)$ is not divisible by $f(x)$.*

Combining these facts with Theorem 1 and Theorem 2, we get the following corollary on orthogonal arrays of strength 3.

**Corollary 4.2.** *Let $f(x)$ be a primitive maximum weight polynomial of odd degree $m$ greater than 7. If $m \leq n \leq 2m$, then $C_n^f$ is an orthogonal array of strength at least 3.*

## 5. Conclusion

In this paper, we analyzed the divisibility of trinomials by maximum-weight polynomials over $\mathbb{F}_2$ and used the result to obtain the orthogonal arrays of strength 3. More precisely, we showed that if $f(x)$ is a maximum-weight polynomial of degree $m$ greater than 7, then $f(x)$ does not divide any trinomial of degree at most $2m$. Our work gives a partial answer to one of the questions posted in (Dewar *et al.*, 2007). As anticipated in (Dewar *et al.*, 2007), (Panario *et al.*, 2012), one seems to need some new techniques to give a complete answer to the question.

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# Best Approximation in $L^p$-norm and Generalized $(\alpha,\beta)$-growth of Analytic Functions

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## Abstract

Let $0 < p \leq +\infty$ and $\Omega_R = \left\{z \in \mathbf{C}^n; \exp V_E(z) < R\right\}$, for some $R > 1$, where $V_E = \sup\left\{\frac{1}{d}\ln|P_d|, P_d \text{ polynomial of degree} \leq d, \|P_d\|_E \leq 1\right\}$ is the Siciak extremal function of a $L$-regular compact $E$.

The aim of this paper is the characterization of the generalized growth of analytic functions of several complex variables in the open set by means of the best polynomial approximation in $L_p$-norm on a compact $E$ with respect to the set $\Omega_r = \left\{z \in \mathbf{C}^n; \exp V_E(z) \leq r\right\}$, $1 < r < R$.

*Keywords:* Extremal function, L-regular, generalized growth, best approximation of analytic function, $L^p$-norm.
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## 1. Introduction

Let $E$ be a compact L-regular of $\mathbb{C}^n$. For an entire function $f$ in $\mathbb{C}^n$ developed according an extremal polynomial basis $(A_k)_k$ (see Zeriahi (1987)), M. Harfaoui (see Harfaoui (2010) and Harfaoui (2011)) have generalized growth in term of coefficients with respect the sequence $(A_k)_k$. The growth used by M. Harfoui was defined according to the functions $\alpha$ and $\beta$ (see Harfaoui (2010), pp. 5, eq. (2.14)), with respect to the set:

$$\Omega_r = \left\{z \in \mathbb{C}^n,\ exp(V_E)(z) < r\right\},$$

where

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$$V_E = \sup\Big\{\frac{1}{d}\ \log|P_d|, P_d \quad \text{polynomial of degree} \leq d, \|P_d\|_E \leq 1\Big\}$$

is the Siciak's extremal function of $E$ which is continuous in $\mathbb{C}^n$ (Because $E$ is L-regular). The $(\alpha,\beta)$-order and the $(\alpha,\beta)$-type of $f$ an entire function (or generalized order and generalized type) are defined respectively by:

$$\rho(\alpha,\beta) = \limsup_{r\to+\infty}\frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\beta(\log(r))} \quad \text{and} \quad \sigma(\alpha,\beta) = \limsup_{r\to+\infty}\frac{\alpha\Big(\|f\|_{\overline{\Omega}_r}\Big)}{\Big[\beta(r)\Big]^{\rho(\alpha,\beta)}},$$

where

$$\Big\|f\Big\|_{\overline{\Omega}_r} = \sup_{\overline{\Omega}_r}\Big|f(z)\Big|.$$

These results have been used to establish the generalized growth in terms of best approximation in $L_p$-norm for $p \geq 1$.

Let $f$ be a function defined and bounded on $E$. For $k \in \mathbb{N}$ put

$$\pi_k^p(E,f) = \inf\Big\{\Big\|f - P\Big\|_{L^p(E,\mu)},\ P \in \mathcal{P}_k(\mathbb{C}^n)\Big\},$$

where $\mathcal{P}_k(\mathbb{C}^n)$ is the family of all polynomials of degree $\leq k$ and $\mu$ the well-selected measure (The equilibrium measure $\mu = (dd^cV_E)^n$ associated to a $L$-regular compact $E$) (see Zeriahi (1983)) and $L^p(E,\mu)$, $p \geq 1$, is the class of all functions such that:

$$\Big\|f\Big\|_{L^p(E,\mu)} = \Big(\int_E |\ f\ |^p\ d\mu\Big)^{1/p} < \infty.$$

For an entire function $f \in \mathbb{C}^n$ M. Harfaoui established a precise relationship between the general growth with respect to the set (see (Harfaoui (2010)): $\Omega_r = \Big\{z \in \mathbb{C}^n\ :\ \exp(V_E)(z) < r\Big\}$, and the coefficients of the development of $f$ with respect to the sequence $(A_k)_k$, called extremal polynomial (see Zeriahi (1987)). He used these results to give the relationship between the generalized growth of $f$ and the sequence $(\pi_k^p(E,f))_k$. Note that M. Harfaoui did not study the case $0 < p < 1$ because the triangle inequality is not satisfied. A. Janik (see Janik (1991)) characterized the $(\alpha,\beta)$-order of an analytic function $g$ in $\Omega_R$ defned by

$$\Omega_R = \Big\{z \in \mathbb{C}^n,\ exp(V_E(z)) < R\Big\},\ \text{for some } R > 1,$$

by means of polynomial approximation and interpolation to $g$ on on a L-regular compact $E$, with respect to the set

$$\Omega_r = \Big\{z \in \mathbb{C}^n,\ \exp(V_E(z)) < r,\quad 1 < r < R\Big\}.$$

In his work A. Janik used the best approximation defined, for a function defined and bounded on $E$, by:

$$\mathcal{E}_n^{(1)} = \mathcal{E}_n^{(1)}(f,E) = \|\ f - t_n\ \|,$$

$$\mathcal{E}_n^{(2)} = \mathcal{E}_n^{(2)}(f,E) = \|\ f - l_n\ \|,$$

$$\mathcal{E}_{n+1}^{(3)} = \mathcal{E}_{n+1}^{(3)}(f, E) = \| l_{n+1} - l_n \|,$$

where $t_n$ denoted the $n$th Chebychev polynomial of the best approximation to $f$ on $E$ and $l_n$ denoted the $n$th Lagrange interpolation polynomial for $f$ with nodes at extremal points of $E$ (see Siciak (1962)).

The $(\alpha,\beta)$-order of an analytic function was defined as follows:

If $E$ be a compact $L$-regular. If $f$ is an analytic function in

$$\Omega_R = \Big\{z \in \mathbb{C}^n : \ \exp(V_E(z)) < R\Big\}$$

for some $R > 1$. We define the $(\alpha,\beta)$-order of $f$ (or generalized order) by

$$\rho(\alpha,\beta) = \limsup_{r\to R} \frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\beta(R/(R-r))}$$

where $\big\|f\big\|_{\overline{\Omega}_r} = \sup_{\overline{\Omega}_r}\big|f(z)\big| = \sup\Big\{ \mid f(z) \mid : \exp V_E(z) \le r, 1 < r < R\Big\}$.

In this work we study the generalized order and generalized type, which will be defined later, for an analytic function in the open set $\Omega_R$, with respect to the set $\Omega_r$ in terms of coefficients of the analytic function in the development according to the sequence of extremal polynomials. So we obtain a generalization of the results of M. Harfaoui (see Harfaoui (2010) and Harfaoui (2011)) and A. Janik (see Janik (1984), and Janik (1991)) replacing $\mathbb{C}^n$ by $\Omega_R$ and the entire function in $\mathbb{C}^n$ by analytic function in $\Omega_R$ .

After studying the generalized type of an analytic function in $\Omega_R$, for some $R > 1$, we use this results to characterize the generalized type by means of best polynomial approximation on $E$ in $L_p$-norm for $0 < p \le +\infty$.

Recall that the generalized growth used by M. Harfaoui (see Harfaoui (2010) and Harfaoui (2011)) called $(\alpha,\beta)$-growth was defined with respect to functions $\alpha$ and $\beta$ defined as:

Let $\alpha$ and $\beta$ be two positive, strictly increasing to infinity differentiable functions $]0,+\infty[$ to $]0,+\infty[$such that for every $c > 0$:

such that

$$\begin{cases} \lim\limits_{x\to+\infty} \dfrac{\alpha(cx)}{\alpha(x)} = 1, \\ \lim\limits_{x\to+\infty} \dfrac{\beta\Big(1 + x\omega(x)\Big)}{\beta(x)} = 1, \quad \lim\limits_{x\to+\infty} \omega(x) = 0, \\ \lim\limits_{x\to+\infty} \dfrac{d\Big(\beta^{-1}(c\alpha(x))\Big)}{\alpha(\log(x))} \le b. \\ \alpha\Big(x/\beta^{-1}(c\alpha(x))\Big) = (1 + o(x))\alpha(x), \qquad \text{for } x \to +\infty, \end{cases}$$

where $d(u)$ means the differential of $u$.

## 2. Definitions and notations

Before we give some definitions and results which will be frequently used in this paper.

**Definition 2.1.** (Siciak (1977)) Let $E$ be a compact set in $\mathbb{C}^n$ and let $\|.\|_E$ denote the maximum norm on $E$. The function

$$V_E = \sup\left\{\frac{1}{d}\ \log|P_d|, P_d \quad \text{polynomial of degree} \leq d, \|P_d\|_E \leq 1, d \in \mathbb{N}\right\}$$

is called the Siciak's extremal function of the compact $E$.

**Definition 2.2.** Zeriahi (1983) A compact $E$ in $\mathbb{C}^n$ is said to be $L$-regular if the extremal function, $V_E$, associated to $E$ is continuous on $\mathbb{C}^n$.

Regularity is equivalent to the following Bernstein-Markov inequality (see Siciak (1962)): For any $\epsilon > 0$, there exists an open $U \supset E$ such that for any polynomial $P$, $\|P\|_U \leq e^{\epsilon.deg(P)}\|P\|_E$.

In this case we take $U = \left\{z \in \mathbb{C}^n; V_E(z) < \epsilon\right\}$.

Regularity also arises in polynomials approximation. For $f \in C(E)$, we let

$$\epsilon_d(E, f) = \inf\left\{\|f - P\|_E,\ P \in \mathcal{P}_k(\mathbb{C}^n)\right\}$$

where $\mathcal{P}_k(\mathbb{C}^n)$ is the set of polynomials of degree at most $d$. Siciak (see Siciak (1977)) showed:

If $E$ is $L$-regular, then $\limsup_{d\to+\infty}\left(\varepsilon_d(E, f)\right)^{1/d} = \frac{1}{r} < 1$ if and only if $f$ has an analytic continuation to $\left\{z \in \mathbf{C}^n; V_E(z) < \log\left(\frac{1}{r}\right)\right\}$. It is known that if $E$ is an compact $L$-regular of $\mathbb{C}^n$, there exists a measure $\mu$, called extremal measure, having interesting properties (see Siciak (1962) and Siciak (1977)), in particular, we have:

$(P_1)$ Bernstein-Markov inequality: $\forall\epsilon > 0$, there exists $C = C_\varepsilon$ is a constant such that

$$(BM) : \|P_d\|_E = C(1 + \varepsilon)^{s_k}\|P_d\|_{L^2(E,\mu)}, \tag{2.1}$$

for every polynomial of $n$ complex variables of degree at most $d$.

$(P_2)$ Bernstein-Waish (B.W) inequality:

For every set $L$-regular $E$ and every real $r > 1$ we have:

$$\|f\|_E \leq M.r^{deg(f)}\left(\int_E |\ f\ |^p\ .d\mu\right)^{1/p} \tag{2.2}$$

Note that the regularity is equivalent to the Bernstein-Markov inequality.

Let $s : \mathbb{N} \to \mathbb{N}^n, k \to s(k) = (s_1(k), ...., s_n(k))$ be a bijection such that

$$|s(k + 1)| \geq |s(k)| \text{ where } |s(k)| = s_1(k) + .... + s_n(k).$$

A. Zeriahi (see Zeriahi (1987)) has constructed according to the Hilbert Schmidt method a sequence of monic orthogonal polynomials according to a extremal measure (see Siciak (1962)), $(A_k)_k$, called extremal polynomial, defined by

$$A_k(z) = z^{s(k)} + \sum_{j=1}^{k-1} a_j z^{s(j)} \tag{2.3}$$

such that $\left\|A_k\right\|_{L^p(E,\mu)} = \left[\inf\left\{\left\|z^{s(k)} + \sum_{j=1}^{k-1} a_j z^{s(j)}\right\|_{L^2_{(E,\mu)}}, (a_1, a_2, ..., a_n) \in \mathbb{C}^n\right\}\right]^{1/s_k}$.

We need the following notations which will be used in the sequel: $(N_1)$ $\nu_k = \nu_k(E) = \left\|A_k\right\|_{L^2(K,\mu)}$. $(N_2)$ $a_k = a_k(E) = \left\|A_k\right\|_E = \max_{z\in K}\left|A_k(z)\right|$ and $\tau_k = \left(a_k\right)^{1/s_k}$, where $s_k = deg(A_k)$. With that notations and (B.W) inequality we have

$$\left\|A_k\right\|_{\overline{\Omega}_r} \leq a_k.r^{s_k} \tag{2.4}$$

where $s_k = deg(A_k)$. For more details (see Zeriahi (1983)).

**Definition 2.3.** Zeriahi (1983) Let $E$ be a compact $L$-regular. If $f$ is an analytic function in

$$\Omega_R = \left\{z \in \mathbb{C}^n : \exp(V_E(z)) < R\right\}$$

for some $R > 1$. We define the $(\alpha,\beta)$-growth ( $(\alpha,\beta)$-order and $(\alpha,\beta)$-type) of $f$ (or generalized order) by $\rho(\alpha,\beta) = \limsup_{r\to R} \frac{\alpha\left(\log(\|f\|_{\overline{\Omega}_r})\right)}{\beta(R/(R-r))}$, $\sigma(\alpha,\beta) = \limsup_{r\to R} \frac{\alpha\left(\log(\|f\|_{\overline{\Omega}_r})\right)}{\left[\beta(R/(R-r))\right]^{\rho(\alpha,\beta)}}$, where $\left\|f\right\|_{\overline{\Omega}_r} = \sup_{z\in\overline{\Omega}_r}\left|f(z)\right| = \sup\left\{ \mid f(z) \mid : \exp V_E(z) \leq r, 1 < r < R\right\}$.

Note that in the classical case $\alpha(x) = \beta(x) = \log(x)$. We need the following lemma (see Zeriahi (1987)).

**Lemma 2.1.** *( Zeriahi (1987)) If $E$ is a compact $L$-regular subset of $\mathbb{C}^n$, then for every $\theta > 1$, there exists an integer $N_\theta \geq 1$ and a constant $C_\theta > 0$ such that:*

$$\pi_k^p(E, f) \leq C_\theta \frac{(r+1)^{N_\theta}}{(r-1)^{2N-1}} \frac{\|f\|_{\overline{\Omega}_{r\theta}}}{r^k}. \tag{2.5}$$

*for every $k \geq 1$, every $r > 1$ and every $f \in 0\left(\overline{\Omega}_{r\theta}\right)$. If $f = \sum_{k=0}^{+\infty} f_k.A_k$ be an entire function, then for every $\theta > 1$, there exists $N_\theta \in \mathbb{N}^*$ and $C_\theta > 0$ such that*

$$\left|f_k\right|\nu_k \leq C_\theta \frac{(r+1)^{N_\theta}}{(r-1)^{2N-1}} \frac{\|f\|_{\overline{\Omega}_{r\theta}}}{r^{s_k}}, \tag{2.6}$$

*for every $k \geq 0$ and $r > 1$. $C_\theta$ and $N_\theta$ do not depend on $r$ or $k$, or $f$.*

Note that the second assertion of the lemma is a consequence of the first assertion and it replaces Cauchy inequality for complex function defined on the complex plane $\mathbb{C}$.

## 3. Generalized order and coefficient characterizations with respect to extremal polynomial

The purpose of this section is to establish the relationship of the generalized growth of an analytic function in $\Omega_R$ with respect to the set $\Omega_r = \left\{z \in \mathbb{C} : \exp(V_E(z)) < r\right\}$ and coefficients of an entire function $f \in \mathbb{C}^n$ in the development with respect to the sequence of extremal polynomials.

Let $(A_k)_k$ be a basis of extremal polynomial associated to the set $E$ defined the relation (2.3). We recall that $(A_k)_k$ is a basis of $O(\mathbb{C}^n)$ (the set of entire functions on $\mathbb{C}^n$). So if $f$ is an entire function then $f = \sum_{k\geq 1} f_k.A_k$.

Put

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} = \mu(\alpha,\beta). \tag{3.1}$$

To prove the aim result of this section we need the following lemmas:

**Lemma 3.1.** *( Zeriahi (1987)) Let E be a compact L-regular subset of* $\mathbb{C}^n$. *Then*

$$\lim_{k\to+\infty}\left[\frac{|A_k(z)|}{\nu_k}\right]^{1/s_k} = \exp(V_E(z)), \tag{3.2}$$

*for every* $z \in \mathbb{C}^n \setminus \widehat{E}$ *the connected component of* $\mathbb{C}^n \setminus E$,

$$\lim_{k\to+\infty}\left[\frac{\|A_k\|_E}{\nu_k}\right]^{1/s_k} = 1. \tag{3.3}$$

**Lemma 3.2.** *For every* $r > 1$ *and* $\mu > 0$, *the maximum of the function*

$$x \to \omega(x,r) = x.\log(r/R) + \frac{x}{\beta^{-1}\left(\alpha(x)/\mu\right)}$$

*is reached for* $x = x_r$ *solution of the equation*

$$x = \alpha^{-1}\left\{\mu\beta\left[\frac{1 - d\log\left(\beta^{-1}(\alpha(x)/\mu)\right)/d(\log(x))}{\log\left(R/r\right)}\right]\right\}. \tag{3.4}$$

*Proof.* Put $G(x,\mu) = \beta^{-1}\left(\alpha(x)/\mu\right)$, then $\omega(x,r) = x.\log(r/R) + \dfrac{x}{G(x,\mu)}$. The maximum of the function $x \to \omega(x,r)$ is reached for $x = x_r$ solution of the equation of $\dfrac{d\omega(x,r)}{dx} = 0$. We have $\dfrac{\omega(x,r)}{dx} = 0 \Leftrightarrow \log(\dfrac{r}{R}) + \dfrac{G(x,\mu) - x.\dfrac{dG(x,\mu)}{dx}}{\left(G(x,\mu)\right)^2} = 0$, or $G(x,\mu) = \dfrac{1 - \dfrac{x}{G(x,\mu)}.\dfrac{dG(x,\mu)}{dx}}{\log(R/r)}$.

Since $\dfrac{dG(x,\mu)}{dx} = \dfrac{dG(x,\mu)}{d\log(x)}.\dfrac{d\log(x)}{dx} = \dfrac{1}{x}.\dfrac{dG(x,\mu)}{d\log(x)}$, we get

$$G(x,\mu) = \frac{1 - \dfrac{1}{G(x,\mu)}.\dfrac{dG(x,\mu)}{d\log(x)}}{\log(R/r)} = \frac{1 - \dfrac{d\log G(x,\mu)}{d\log(x)}}{\log(R/r)}.$$

We deduce $x = x_r = \alpha^{-1}\left\{\mu\alpha\left[\dfrac{1 - d\Big(\beta^{-1}(\alpha(x)/\mu)\Big)/d(\log(x))}{\log\Big(R/r\Big)}\right]\right\}$. □

**Lemma 3.3.** *Let $f = \sum_{k\geq 0} f_k.A_k$ and $E$ a L-regular compact. For every $r \in ]1, R[$, we put*

$$\begin{cases} \overline{M}(f,r) = \sup_{k\in\mathbb{N}}\Big\{ \| f_k.A_k \|_E .r^k,\ r > 0\Big\} \\ \overline{\rho}(\alpha,\beta) = \limsup_{r\to R} \dfrac{\alpha\Big(\log(\overline{M}(f,r))\Big)}{\beta(R/(R-r))} \end{cases}$$

*then $\overline{\rho}(\alpha,\beta) \leq \mu(\alpha,\beta)$ and $\rho(\alpha,\beta) \leq \overline{\rho}(\alpha,\beta)$.*

*Proof.* By the definition of $\mu$ (3.1) we have, for $r$ sufficiently close to $R$ and $\overline{\mu} = \mu + \epsilon$,

$$\log\Big( | f_k | .\tau_k^{s_k}.R^{s_k}\Big) \leq \frac{\alpha(s_k)}{\beta^{-1}\Big(\dfrac{1}{\overline{\mu}}.\alpha(s_k)\Big)}.$$

Then $\log\Big( | f_k | .\tau_k^{s_k}.r^{s_k}\Big) \leq s_k \log(r/R) + \dfrac{\alpha(s_k)}{\beta^{-1}\Big(\dfrac{1}{\overline{\mu}}.\alpha(s_k)\Big)}$. By the proprieties of $\alpha$ and $\beta$,the function $t \to \log(t)$ and the Lemma 3.3 we get $x_r = (1 + o(1))\alpha^{-1}\Big(\mu.\beta(R/(R-r))\Big)$ as $r \to R$. Indeed this result is a consequence of $\lim_{x\to+\infty}\left|\dfrac{d\Big(\beta^{-1}(c\alpha(x))\Big)}{\alpha(\log(x))}\right| \leq b$, $\log(1+t) = (1+o(t)).t$, $t \to 0$. Therefore $\log\Big( \| f_k.A_k \|_E .r^{s_k}\Big) \leq C_0.\alpha^{-1}\Big(\mu.\beta(R/(R-r))\Big)$, $k \in \mathbb{N}$. Passing to the maximum for the variable $k \in \mathbb{N}$ we obtain, for $r$ sufficiently close to $R$ $\log\Big(\overline{M}(f,r)\Big) \leq C_0.\alpha^{-1}\Big(\mu.\beta(R/(R-r))\Big)$, $k \in \mathbb{N}$. Then, by the proprieties of $\alpha$, we obtain $\dfrac{\alpha\Big(\log(\overline{M}(f,r))\Big)}{\beta(R/(R-r))} \leq \mu$. Passing to upper limit for $r \to R$ we have

$$(*)\quad \overline{\rho}(\alpha,\beta) \leq \mu.$$

Moreover we have for $z \in \Omega_r$ and $k \in \mathbb{N}$, $\|f\|_{\overline{\Omega}_r} \leq \sum_{k\geq 0} | f_k | .\|A_k\|_{\Omega_r} \leq \sum_{k\geq 0} | f_k | .\|A_k\|_E.r^{s_k}$. Write $r = \sqrt{r.R}.\sqrt{r/R}$, then $\|f\|_{\overline{\Omega}_r} \leq \sum_{k\geq 0} | f_k | .\|A_k\|_E.(\sqrt{r.R})^{s_k}.(\sqrt{r/R})^{s_k}$. Because $\sqrt{r/R} < 1$

then $\|f\|_{\overline{\Omega}_r} \leq \sum_{k\geq 0} \sup_{k\in_N}\Big( \mid f_k \mid .\|A_k\|_E.(\sqrt{r.R})^{s_k}\Big).(\sqrt{r/R})^{s_k}$ thus $\|f\|_{\overline{\Omega}_r} \leq \overline{M}(f,r')\sum_{k\geq 0}(\sqrt{r/R})^{s_k} \leq \overline{M}(f,r').\frac{1}{1-\sqrt{r/R}}$. where $r' = \sqrt{r.R}$. Therefore $\log\Big(\|f\|_{\overline{\Omega}_r}\Big) \leq \log\Big(\overline{M}(f,r')\Big) - \log\Big(1-\sqrt{r/R}\Big)$.
We have $\frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\beta(R/(R-r))} \leq \frac{\alpha\Big(\log(\overline{M}(f,\sqrt{r.R})-\log(1-\sqrt{r/R}))\Big)}{\beta(R/(R-\sqrt{r.R}))}.\frac{\beta(R/(R-\sqrt{r.R}))}{\beta(R/(R-r.R))}$. Passing to the upper limit we get

$$(**) \quad \rho(\alpha,\beta) \leq \overline{\rho}(\alpha,\beta).$$

By the relations (*) and (**) we obtain $\rho(\alpha,\beta) \leq \mu(\alpha,\beta)$. □

**Theorem 3.1.** *Let E be a compact L-regular and* $f = \sum_{k\geq 1} f_k.A_k$ *such that*

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\frac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} = \mu(\alpha,\beta) < \infty. \tag{3.5}$$

*Then f is analytic in* $\Omega_R$*, for some* $R > 1$ *and its* $(\alpha,\beta)$*-order* $\rho(\alpha,\beta) = \mu(\alpha,\beta)$*.*

*Proof.* It is known that for every polynomial $P$(see Siciak (1977))

$$| P(z) | \leq \| P \|_E \left(\exp(V_E(z))\right)^{deg(P)}, \quad \text{for every } \ z \in \mathbb{C}^n. \tag{3.6}$$

So for every $r \in ]1,R[$, and for $P = f_k.A_k$ we get

$$| f_k.A_k(z) | \leq | f_k | . \| A_k \|_E \left(\exp(V_E(z))\right)^{s_k}, \text{for every} z \in \mathbb{C}^n. \tag{3.7}$$

Then for every $z \in \Omega_r$, we have $| f_k.A_k(z) | \leq | f_k | . \| A_k \|_E .r^{s_k}$. So, for every $r \in ]1,R[$ the series $\sum f_k.A_k$ is convergent in $\Omega_r$, whence $\sum_{k\geq 1} f_k.A_k$ is analytic in $\Omega_R$.

Now we shall show that $\mu$ is the $(\alpha,\beta)$-order of $f$. By the Lemma 3.3, to complete the proof of the theorem it suffices to show that $\rho(\alpha,\beta) \geq \mu(\alpha,\beta)$. By definition of $\rho$, we have, for every $\epsilon > 0$ there exists $r_\epsilon \in ]1,R[$ such that for every $r \in ]r_\epsilon,R[$ $\log\Big(\|f\|_{\overline{\Omega}_r}\Big) \leq \alpha^{-1}\Big[(\rho(\alpha,\beta)+\epsilon).\beta(R/(R-r))\Big]$. Applying (2.6) and (3.3) we have, for every $k \in \mathbb{N}$ and $r > 1$ sufficiently close to $R$

$$\log\Big( | f_k | \tau_k^{s_k}.R^{s_k}\Big) \leq -s_k \log\Big(r/R\Big) + \log\big(C_0.\frac{(r-1)^{N_\theta}}{(R-r)^{-(2N+1)}}\big) + \log\Big(\|f\|_{\overline{\Omega}_r}\Big), \tag{3.8}$$

then $\log\Big( | f_k | \tau_k^{s_k}.R^{s_k}\Big) \leq \varphi(r,s_k)$, where

$$\varphi(r,s_k) = -s_k \log\Big(r/R\Big) + \log\big(C_0.\frac{(r-1)^{N_\theta}}{(R-r)^{-(2N+1)}} + \beta^{-1}\Big[(\rho(\alpha,\beta)+\epsilon).\beta(R/(R-r))\Big].$$

Put $\rho = \rho(\alpha,\beta)$ and $r_k = R.\left\{1 - \dfrac{1}{\beta^{-1}\left(\dfrac{1}{\rho+\epsilon}.\alpha\Big(\dfrac{s_k}{\beta^{-1}(\alpha(s_k)/(\rho+\epsilon))}\Big)\right)}\right\}$. Replacing in the relation (3.8) $r$ by $r_k$ and applying the proprieties of the functions $\alpha$ and $\beta$:

$$\alpha\Big(x/\beta^{-1}(c\alpha(x))\Big) = (1 + o(x))\alpha(x), \text{ for } c > 0,\ x \to +\infty,$$

and the proprieties of the logarithm, we obtain $\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq C_1.\dfrac{s_k}{\beta^{-1}(\alpha(s_k)/(\rho+\epsilon))}$ where $C_1$ is a constant. Therefore $\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq C_1.\dfrac{s_k}{\beta^{-1}(\alpha(s_k)/(\rho+\epsilon))}$, thus

$$\beta\left(\frac{C_1.s_k}{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}\right) \geq \alpha(s_k)/(\rho+\epsilon).$$

Passing to the upper limit, after a simple calculus, we obtain $\mu(\alpha,\beta) \leq \rho(\alpha,\beta)$. □

## 4. Generalized type and coefficient characterizations with respect to extremal polynomial

The purpose of this section is to establish the relationship of the generalized type of an analytic function in $\Omega_R$ with respect to the set $\Omega_r = \Big\{z \in \mathbb{C} :\ \exp(V_E(z)) < r\Big\}$ and its coefficients in the development according to the sequence of extremal polynomials.

Let $E$ be a compact L-regular and $f = \sum_{k\geq 1} f_k.A_k$ be an analytic function of $(\alpha,\beta)$-order $\rho = \rho(\alpha,\beta)$, and put:

$$\tau_E(\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\left\{\beta\left(\dfrac{s_k}{\log(|f_k|.\tau_k^{s_k}.R^{s_k})}\right)\right\}^{\rho(\alpha,\beta)}}. \tag{4.1}$$

We need the following proposition:

**Proposition 4.1.** *Let $f = \sum_{k\geq 0} f_k.A_k$ and $E$ a L-regular compact. For every $r \in ]1, R[$, we put*

$$\begin{cases} \overline{M}(f,r) = \sup_{k\in\mathbb{N}}\Big\{ \mid f_k \mid . \parallel A_k \parallel_E .r^{s_k}\Big\} \\ \overline{\sigma_1}(\alpha,\beta) = \limsup_{r\to R} \dfrac{\alpha\Big(\log(\overline{M}(f,r))\Big)}{\Big(\beta(R/(R-r))\Big)^{\rho(\alpha,\beta)}} \end{cases}$$

*then $\sigma(\alpha,\beta) \leq \overline{\sigma_1}(\alpha,\beta)$.*

*Proof.* For $z \in \Omega_r$ and $k \in \mathbb{N}$, using the similar arguments and inequalities as in Lemma 2.3

$$\frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\Big[\beta(R/(R-r))\Big]^{\rho(\alpha,\beta)}} \leq \frac{\alpha\Big(\log(\overline{M}(f,\sqrt{r.R}) - \log(1-\sqrt{r/R}))\Big)}{\Big[\alpha(R/(R-\sqrt{r.R}))\Big]^{\rho(\alpha,\beta)}}.\frac{\Big[\alpha(R/(R-\sqrt{r.R}))\Big]^{\rho(\alpha,\beta)}}{\Big[\alpha(R/(R-r))\Big]^{\rho(\alpha,\beta)}}.$$

We have $\limsup_{r\to R} \dfrac{\left[\alpha(R/(R-\sqrt{r.R}))\right]^{\rho(\alpha,\beta)}}{\left[\alpha(R/(R-r))\right]^{\rho(\alpha,\beta)}} = 1.$ □

Proceeding to the upper limit we get

$$(*) \quad \sigma(\alpha,\alpha) \leq \overline{\sigma_1}(\alpha,\beta).$$

**Theorem 4.1.** *Let $E$ be a compact L-regular and $f = \sum_{k\geq 1} f_k.A_k$. If $f$ is of finite generalized $(\alpha,\beta)$-order $\rho(\alpha,\beta)$, and*

$$\tau_E(\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\left\{\beta\left(\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right)\right\}^{\rho(\alpha,\beta)}} < +\infty. \tag{4.2}$$

*Then $f$ is analytic in $\Omega_R$, for some $R > 1$, and its $(\alpha,\beta)$-type $\sigma(\alpha,\beta) = \tau_E(\alpha,\beta)$.*

*Proof.* Put $\tau = \tau_E(\alpha,\beta)$, $\rho = \rho(\alpha,\beta)$, and $\sigma = \sigma(\alpha,\beta)$. The function is analytic by the definition $\tau_E(\alpha,\beta)$ and the arguments used in theorem 3.1.

1. Now we show that $\sigma(\alpha,\beta) \leq \tau_E(\alpha,\beta)$. If $\tau < \infty$, by the definition of $\tau$, for every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_\epsilon$ $\alpha(s_k) \leq (\tau+\epsilon).\left\{\beta\left(\dfrac{s_k}{\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right)}\right)\right\}^{\rho}$. A simple calculus gives for, $\overline{\tau} = \tau + \epsilon$.

$$\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right) \leq \frac{s_k}{\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(s_k)\right)^{1/\rho}\right)}, \tag{4.3}$$

for every $k \geq k_\epsilon$for every $k \geq k_\epsilon$.
Since $\log\left(\mid f_k \mid .\tau_k^{s_k}.r^{s_k}\right) \leq s_k \log(r/R) + \log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right)$. By (4.3), we get

$$\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq s_k \log(r/R) + \frac{s_k}{\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(s_k)\right)^{1/\rho}\right)}. \tag{4.4}$$

For every $r \in ]1, R[$, and $r$ and $r$ sufficiently close to $R$, we put

$$\phi(x,r) = x\log(r/R) + \frac{x}{\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right)}.$$

If we put $F = F(x,\overline{\tau},\frac{1}{\rho}) = \beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right)$ then $\phi(x,r) = x\log(r/R) + \frac{x}{F}$, and the maximum of the function $x \to \phi(x,r)$ is reached for $x = x_r$ solution of the equation of

$$\frac{d\phi(x,r)}{dx} = \frac{\partial\phi}{\partial x}(x,r) = \log(r/R) + \frac{d}{dx}\left\{\frac{x}{F}\right\} = 0.$$

We have $\dfrac{\phi(x,r)}{dx} = 0 \Leftrightarrow \log(\dfrac{r}{R}) + \dfrac{F - x.\dfrac{dF}{dx}}{(F)^2} = 0$, or $F = \dfrac{1 - \dfrac{x}{F}.\dfrac{dF}{dx}}{\log(R/r)}$. Since $\dfrac{dF}{dx} = \dfrac{dF}{d\log(x)}.\dfrac{d\log(x)}{dx} = \dfrac{1}{x}.\dfrac{dF}{d\log(x)}$, we get $F = \dfrac{1 - \dfrac{1}{F}.\dfrac{dF}{d\log(x)}}{\log(R/r)} = \dfrac{1 - \dfrac{d\log F}{d\log(x)}}{\log(R/r)}$, or

$$\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right) = \frac{1 - \dfrac{d\log\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right)}{d\log(x)}}{\log(R/r)}.$$

We deduce $x = x_r = \alpha^{-1}\left\{\left[\overline{\tau}.\beta\left(\dfrac{1 - d\log\left(\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right)\right)/d(\log(x))}{\log\left(R/r\right)}\right)\right]^{\rho}\right\}$. We have $\log\left(\dfrac{r}{R}\right) = \log\left(\dfrac{r-R}{R} + 1\right) \sim \dfrac{r-R}{R}$ $\left(\text{because } \dfrac{r-R}{R} \to 0\right)$ and $\left|\dfrac{d\left[\log\left(\beta^{-1}\left(\left(\alpha(x)\right)^{\rho}\right)\right)\right]}{d\log(x)}\right| \leq b$, where $b$ is a positive constant. Then by the proprieties of $\alpha$ we get

$$x_r = (1 + o(1))\rho.\beta^{-1}\left(\overline{\tau}(\alpha(R/(R-r)))^{\rho}\right).$$

By (4.4), we have $\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq \sup_{r\in\mathbb{N}} \phi(x,r) = \phi(x_r, r)$. Replacing $s_k$ by $x_r$ in this last relation we obtain $\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq \dfrac{(1+o(1))\beta^{-1}\left(\overline{\tau}(\alpha(R/(R-r)))^{\rho}\right)}{R/(R-r)}$. Since $\dfrac{R}{R-r} > 1$ and $\dfrac{\rho-1}{\rho} < 1$, then $\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq C.\beta^{-1}\left(\overline{\tau}.(\alpha(R/(R-r)))^{\rho}\right)$.

Then $\sup_{k\in\mathbb{N}} \log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k} \leq C.\alpha^{-1}\left(\overline{\tau}.(\alpha(R/(R-r)))^{\rho}\right)\right.$ or $\log(\overline{M}(f,r)) \leq C.\beta^{-1}\left(\overline{\tau}.(\alpha(R/(R-r)))^{\rho}\right)$.

Therefore $\dfrac{\alpha(\log(\overline{M}(f,r)))}{(\alpha(R/(R-r)))^{\rho}} \leq \overline{\tau}$.

Proceeding to the upper limit for $r \to R$, get $\overline{\sigma}_1(\alpha,\alpha) = \lim_{r\to R} \dfrac{\alpha(\log(\overline{M}(f,r)))}{(\alpha(R/(R-r)))^{\rho}} \leq \tau$.

By the relations (*) of the proposition 4.1 we obtain $\sigma(\alpha,\alpha) = \lim_{r\to R} \dfrac{\alpha(\log(\overline{M}(f,r)))}{(\alpha(R/(R-r)))^{\rho}} \leq \tau$.

Thus $\sigma(\alpha,\beta) \leq \tau_E(\alpha,\beta)$. The result is obviously holds for $\tau = +\infty$.

2. Now we show that $\sigma(\alpha,\beta) \geq \tau_E(\alpha,\beta)$. Put $\overline{\sigma} = \sigma(\alpha,\beta) + \epsilon$, $\rho = \rho(\alpha,\beta)$. Suppose that $\sigma < \infty$. By definition of $\sigma(\alpha,\beta)$, we have for every $\epsilon > 0$, there exist $r_\epsilon \in ]1, R[$, such that for every

$r > r_\epsilon$ ( $R > r > r_\epsilon > 1$) $\log\Big(\|f\|_{\overline{\Omega}_r}\Big) \leq \alpha^{-1}\Big[\overline{\sigma}.\Big(\beta(R/(R-r))\Big)^{\rho}\Big]$. Applying (3.3) and (2.6) we get, for every $k \in \mathbb{N}$ and $r$ sufficiently close to $R$:

$$\log\Big(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\Big) \leq -s_k \log(r) + \log\big(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\big) + \log\Big(\|f\|_{\overline{\Omega}_r}\Big).$$

As for every $r \in ]1, R[$ $\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) = -s_k \log\Big(r/R\Big) + \log\Big(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\Big)$ then $\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq -s_k \log(r/R) + \log\Big(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\Big) + \log\Big(\|f\|_{\overline{\Omega}_r}\Big)$. or $\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq -s_k \log\Big(r/R\Big) + \log\Big(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\Big) + \alpha^{-1}\Big[\overline{\sigma}.\Big(\beta(R/(R-r))\Big)^{\rho}\Big]$.

Since $s_k \geq 1$, we obtain, for $k$ sufficiently large, $\frac{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}{s_k} \leq \omega(r,k)$ where $\omega(r,k) = -\log(r/R) + \frac{1}{s_k}\log\Big(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\Big) + \frac{1}{s_k}\alpha^{-1}\Big[\overline{\sigma}.\Big(\beta(R/(R-r))\Big)^{\rho}\Big]$.

Since $\lim\limits_{k\to+\infty} \frac{1}{s_k}\log\big(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\big) + \frac{1}{s_k}\alpha^{-1}\Big[\overline{\sigma}.\Big(\beta(R/(R-r))\Big)^{\rho}\Big] = 0$ we get, for $r$ sufficiently close to $R$, $\lim\limits_{k\to+\infty} \omega(r,k) = -\log(r/R) = \log(R/r)$.

Then for $k$ sufficiently large and $r$ sufficiently close to $R$, we have $\omega(r,k) = (1+o(1))\log(R/r)$, $k \to +\infty$, then

$$\frac{1}{s_k}\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq (1+o(1))\log(R/r). \tag{4.5}$$

Choose $r_k = R.\frac{\beta^{-1}\Big(\frac{1}{\overline{\sigma}}\alpha(s_k)\Big)^{1/\rho}}{1+\beta^{-1}\Big(\frac{1}{\overline{\sigma}}\alpha(s_k)\Big)^{1/\rho}}$. Using the relation (4.5) and the proprieties of the function $t \to \log(t)$, we obtain, for $r$ sufficiently close to $R$ $\frac{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}{s_k} \leq (1+o(1))\Big(\frac{R}{r}-1\Big)$.

because $\log\Big(\frac{R}{r}\Big) = \log\big(\frac{R-r+r}{r}\big) = \log\big(1+\frac{R-r}{r}\big) \sim \frac{R-r}{r}$ $(r \to R)$.

Replacing $r$ by the chosen $r_k$ in this last relation we obtain $\frac{R-r_k}{r_k} = \frac{1}{\beta^{-1}\Big(\frac{1}{\overline{\sigma}}\alpha(s_k)\Big)^{1/\rho}}$.

Then, for $r$ sufficiently close to $R$ and $k$ $r$ sufficiently large we get $\frac{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}{s_k} \leq \frac{1}{\beta^{-1}\Big(\frac{1}{\overline{\sigma}}\alpha(s_k)\Big)^{1/\rho}}$, thus $\beta^{-1}\Big(\frac{1}{\overline{\sigma}}\alpha(s_k)\Big)^{1/\rho} \leq \frac{s_k}{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}$ or $\Big(\frac{1}{\overline{\sigma}}\alpha(s_k)\Big)^{1/\rho} \leq \beta\left(\frac{s_k}{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}\right)$.

Therefore $\frac{1}{\overline{\sigma}}\alpha(s_k) \leq \left\{\beta\left(\frac{s_k}{\log\Big(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\Big)}\right)\right\}^{\rho}$ or $\frac{\alpha(s_k)}{\left\{\beta\left(\frac{s_k}{\log\Big(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\Big)}\right)\right\}^{\rho}} \leq \overline{\sigma} = \sigma + \epsilon$.

Proceeding to the upper limit we obtain $\sigma(\alpha,\beta) \geq \tau_E(\alpha,\beta)$. The result is obviously holds for $\sigma(\alpha,\beta) = +\infty$.

□

## 5. Generalized $(\alpha,\beta)$-growth and best polynomial approximation of analytic functions in $L^p$−norm.

Let $E$ a L-regular compact of $\mathbb{C}^n$. The purpose of this paragraph is to give the relationship between the generalized order of an analytic function and speed of convergence to 0 in the best polynomial in $L_p$-norm on $E$. We need the following lemma.

**Lemma 5.1.** . *Let* $f = \sum_{k\geq 0} f_k.A_k$ *an element of* $L^p(E,\mu)$*, for* $p \geq 0$*, and*

$$\pi_k^p(E,f) = \inf\left\{\left\|f - P\right\|_{L^p(E,\mu)},\ P \in \mathcal{P}_k(\mathbb{C}^n)\right\}.$$

*Then*

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} \tag{5.1}$$

*and*

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\left\{\beta\left(\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right)\right\}^{\rho(\alpha,\beta)}} = \limsup_{k\to+\infty} \frac{\alpha(k)}{\left\{\beta\left(\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right)\right\}^{\rho(\alpha,\beta)}}. \tag{5.2}$$

*Proof.* Assume that $p \geq 2$. If $f \in L^p(E,\mu)$ where $p \geq 2$, then $f = \sum_{k=0}^{+\infty} f_k.A_k$ with convergence in $L^2(E,\mu)$, hence for $k \geq 0$, $f_k = \dfrac{1}{{\nu_k}^2}\displaystyle\int_E f.\overline{A}_k d\mu$ and therefore $f_k = \dfrac{1}{{\nu_k}^2}\displaystyle\int_E (f - P_{k-1}).\overline{A}_k d\mu$ (because $deg(A_k) = s_k$). Since the relation, $\left|f_k\right| \leq \dfrac{1}{{\nu_k}^2}\displaystyle\int_E \left|f - P_{k-1}\right|.\left|\overline{A}_k\right|\mu$ is satisfied, is easily verified by using inequalities Bernstein-walsh and Holder that we have for all $\varepsilon > 0$

$$\left|f_k\right|.\nu_k \leq C_\varepsilon.(1+\varepsilon)^{s_k}.\pi^p_{s_k-1}(E,f). \tag{5.3}$$

for all $k \geq 0$.

If $1 \leq p < 2$, let $p'$ such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, we have $p' \geq 2$. According to the inequality of Hölder we have: $\left|f_k\right|.\nu_k^2 \leq \left\|f - P_{k-1}\right\|_{L^p(E,\mu)}.\left\||A_k\right\|_{L^{p'}(E,\mu)}$. But $\left\|A_k\right\|_{L^{p'}(E,\mu)} \leq C.\left\|A_k\right\|_E = C.a_k(E)$. This shows, according to inequality (BM), that: $\left|f_k\right|.\nu_k^2 \leq C.C_\varepsilon.(1+\varepsilon)^{s_k}.\left\|f - P_{s_k-1}\right\|_{L^p(E,\mu)}$.

Hence the result $\left|f_k\right|.\nu_k^2 \leq C'_\varepsilon.(1+\varepsilon)^{s_k}.\pi_k^{s_k-1}(E,f)$. In both cases we have therefore

$$\left|f_k\right|.\nu_k^2 \leq A_\varepsilon.(1+\varepsilon)^{s_k}.\pi_{s_k-1}^p(E,f) \tag{5.4}$$

where $A_\varepsilon$ is a constant which depends only on $\varepsilon$.
After passing to the upper limit in the relation (5.4) and applying the relation (3.3) we get

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} \leq \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}.$$

To prove the other inequality we consider the polynomial of degree $s_k$, $P_k(z) = \sum_{s_j=0}^{k} f_j.A_j$ then $\pi_{s_k-1}^p(E,f) \leq \sum_{s_j=s_k}^{+\infty} |f_j|.\left\|A_j\right\|_{L^p(E,\mu)} \leq C_0 \sum_{s_j=s_k}^{+\infty} |f_j|.\left\|A_j\right\|_E$. By Bernstein-Walsh inequality we have $\pi_k^p(E,f) \leq C_\epsilon \sum_{s_j=s_k}^{+\infty} (1+\epsilon)^{s_j}\left|f_j\right|.\nu_j$ for $k \geq 0$ and $p \geq 1$. If we take as a common factor $(1+\varepsilon)^{s_k}.\left|f_k\right|.\nu_k$ the other factor is convergent thus we have $\pi_k^p(K,f) \leq C(1+\epsilon)^{s_k}.\left|f_k\right|.\nu_k$ and by (3.3) we have, then

$$\pi_k^p(E,f) \leq C(1+\epsilon)^{2s_k}.\left|f_k\right|.\tau_k^{s_k}. \tag{5.5}$$

We deduce $\limsup_{k\to+\infty} \dfrac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} \geq \limsup_{k\to+\infty} \dfrac{\alpha(k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}$. □

Applying this Lemma 5.1 we get the following main result:

**Theorem 5.1.** *Let $f \in L^p(E,\mu)$, then $f$ is $\mu$-almost-surely the restriction to $E$ of an analytic function in $\mathbb{C}^n$ of finite generalized order $\rho(\alpha,\beta)$ if and only if*

$$\rho(\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} + \infty. \tag{5.6}$$

**Theorem 5.2.** *Let $f \in L^p(E,\mu)$, then $f$ is $\mu$-almost-surely the restriction to $E$ of an analytic function in $\mathbb{C}^n$ of finite generalized order $\rho(\alpha,\beta)$ and finite generalized type $\sigma(\alpha,\beta)$ if and only if*

$$\sigma(\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(k)}{\left\{\beta\left(\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right)\right\}^{\rho(\alpha,\beta)}}. \tag{5.7}$$

*Proof.* We prove only the first Theorem 5.1, the second is proved by the same arguments.

Suppose that $f$ is $\mu$-almost-surely the restriction to $E$ of an entire function $g$ of general order $\rho$ $(0 < \rho < +\infty)$ and show that $\rho = \rho(\alpha,\beta)$.

We have $g \in L^p(E,\mu)$, $p \geq 2$ and $g = \sum_{k\geq 0} g_k.A_k$ in $L^2(E,\mu)$ Since $g$ is an element of $L^2(E,\mu)$ then $g = \sum_{k=0}^{+\infty} g_k.A_k$ and according to the Theorem 3.1 $\rho(g,\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\frac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]}$ and with the Lemma 5.1 (relation(5.1)) we have $\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\frac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\frac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}$.

But $g = f$ on $E$ hence $\rho = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\frac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} < +\infty$.

Now suppose that $f$ is a function of $L^p(E,\mu)$ such that the relation (5.6) is verified. The proof is done in three steps $p \geq 2$, $1 \leq p < 2$ and $0 < p < 1$.

**Step.1.** Let $p \geq 2$, then $f = \sum_{k=0}^{+\infty} f_k.A_k$, because $f$ is an element of $L^2(E,\mu)$ $\left((L^p(E,\mu))_{p\geq 1}\right.$ is decreasing sequence$\left.\right)$. Consider in $\mathbb{C}^n$ the series $\sum f_k.A_k$, $k \geq 0$. By the relation (5.6) and the inequality (BW) we have the inequality on coefficients $| A_k |$ (2.4), it can be seen that this series converges normally on all compact of $\mathbb{C}^n$, to an analytic function denoted $f_1$. We have $f_1 = f$, obviously, $\mu$-almost surly on $E$.

We verify easily that this series converges normally on all compact of $\mathbb{C}^n$ to an analytic function denoted $f_1$. We have $f_1 = f$, obviously, $\mu$-almost surly on $E$, and by Theorem 3.1 we have

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\frac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\frac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} < +\infty.$$

According to the Lemma 5.1 we get $\rho(f_1) = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\frac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} < +\infty$.

Let $f_1 = \sum_{k\geq 0} f_k.A_k$, then $f_1(z) = f(z)$ $\mu$-almost surely for every $z$ in $E$. Therefore the $(\alpha,\beta)$-order of $f_1$ is: $\rho(f_1,\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\frac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} < +\infty$ (see Theorem 3.1). By Lemma 5.1 we check $\rho(f_1) = \rho$ so the proof is completed.

**Step.2.** Now let $p \in [1,2[$ and $f \in L^p(E,\mu)$. By (BM) inequality and Hölder inequality we have again the inequality the relation (5.4) and by the previous arguments we obtain the result.

**Step.3.** Let $0 < p < 1$, of course, for $0 < p < 1$ the $L_p$-norm does not satisfy the triangle inequality. But our relations (5.3) and relation (5.4) are also satisfied for $0 < p < 1$ (see Kumar (2011)), because using Holder's inequality we have, for some $M > 0$ and all $r > p$ (p fixed)

$$\|\|\| f \|_{L^p(E,\mu)} \leq M. \| f \|_{L^r(E,\mu)} .$$

Using the inequality $\int_E | f |^p \, d\mu \leq \| f \|_E^{p-r} . \int_E | f |^r \, d\mu$ we get $\| f \|_{L^p(E,\mu)} \leq \| f \|_E^{1-(r/p)} . \| f \|_{L^r(E,\mu)}^{r/p}$ . We deduce that $(E, \mu)$ satisfies the Bernstein.Markov inequality. For $\epsilon > 0$ there is a constant $C = C(\epsilon, p) > 0$ such that, for all (analytic) polynomials $P$ we have

$$\|\|\| P \|_E \leq C(1 + \epsilon)_{deg(P)}. \;\| P \|_{L^p(E,\mu)} .$$

Thus if $(E, \mu)$ satisfies the Bernstein-Markov inequality for one $p > 0$ then (5.4) and (5.5) are satisfied for all$p > 0$.

The rest of proof is easily deduced using the same reasoning as in step 1 and step 2. □

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# Common Fixed Points of Hardy and Rogers Type Fuzzy Mappings on Closed Balls in a Complete Metric Space

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## Abstract

In this paper we obtain some common fixed point theorems for Hardy and Rogers type fuzzy mappings on closed balls in a complete metric space. Our investigation is based on the fact that fuzzy fixed point results can be obtained simply from the fixed point theorem of multi-valued mappings with closed values.In real world problems there are various mathematical models in which the mappings are contractive on the subset of a space under consideration but not on the whole space itself. Our results generalize several results of literature.

*Keywords:* Fuzzy fixed point, Hardy and Rogers mapping, contraction, closed balls, continuous mapping.
*2010 MSC:* 47H10, 54H25, 54A40.

## 1. Introduction

It is a well-known fact that the results of fixed points are very useful for determining the existence and uniqueness of solutions to various mathematical models.In 1922,Banach a Polish mathematician proved a theorem under appropriate of a fixed point this result is called Banach fixed point theorem.This theorem is also applied to prove the existence and uniqueness of the solutions of differential equations. Many authors have made different generalization of Banach fixed point theorem. The study of fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity, and it has a wide range of applications in different areas such as nonlinear and adoptive control systems, parameterize estimation problems, fractal image decoding,computing magneto static fields in a nonlinear medium and convergence of recurrent networks.

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The notion of fixed points for fuzzy mappings was introduced by Weiss (Weiss, 1975) and Butnariu (Butnariu, 1982). Fixed point theorems for fuzzy set valued mappings have been studied by Heilpern (Heilpern, 1981) who introduced the concept of fuzzy contraction mappings and established Banach contraction principle for fuzzy mappings in complete metric linear spaces which is a fuzzy extension of Banach fixed point theorem and Nadlers (Nadler, 1969) theorem for multi-valued mappings. Park and Jeong (Park & Jeong, 1997) proved some common fixed point theorems for fuzzy mappings satisfying in complete metric space which are fuzzy extensions of some theorems in (Azam, 1992; Park & Jeong, 1997). In this paper we obtain some common fixed point theorems of Hardy and Rogers type fuzzy mappings on closed balls.

## 2. Basic concepts

Let $(X, d)$ be a metric space, then we use the following notations: Let

$$2^X = \{A : A \text{ is a subset of } X\},$$

$$CL(2^X) = \{A \in 2^X : A \text{ is nonempty and closed}\},$$

$$C(2^X) = \{A \in 2^X : A \text{ is nonempty and compact}\},$$

$$CB(2^X) = \{A \in 2^X : A \text{ is nonempty, closed and bounded}\},$$

For $A, B \in CB(2^X)$, $d(x, A) = \inf_{y \in A} d(x, y)$, $d(A, B) = \inf_{x \in A, y \in B} d(x, y)$ then the Hausdroff metric $d_H$ on $CB(2^X)$ induced by $d$ is defined as: $d_H(A, B) = \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}$.
A fuzzy set in $X$ is a function with domain $X$ and values in $[0, 1]$ and $I^X$ is the collection of all fuzzy sets in $X$ . If $A$ is a fuzzy set and $x \in X$ then the function values $A(x)$ is called the grade of membership of $x$ in $A$. The $\alpha$-level set of a fuzzy set $A$, is denoted by $[A]_\alpha$, and is defined as:

$$[A]_\alpha = \{x : A(x) \geq \alpha \text{ if } \alpha \in (0, 1]\} \text{ and } [A]_0 = \overline{\{x : A(x) \geq 0\}}.$$

For $x \in X$, we denote the fuzzy set $\chi_{\{x\}}$ by $\{x\}$ unless and until it is stated, where $\chi_A$ is the characteristic function of the crisp set $A$. Now we define a sub-collection of $I^X$ as follows: $\tau(X) = \{A \in I^X : [A]_1 \text{is nonempty and closed}\}$, for A,B $\in I^X$, $A \subset B$ means $A(x) \leq B(x)$ for each $x, y \in X$. For $A, B \in \tau(X)$ then define $D_1\{A, B\} = d_H([A]_1, [B]_1)$.
A point $x^* \in X$ is called a fixed point of a fuzzy mappings $T : X \to I^X$ if $x^* \subset Tx^*$ see (Heilpern, 1981)

**Lemma 2.1.** *(Nadler, 1969) Let A and B be nonempty closed and bounded subsets of a metric space* $(X, d)$ *. If* $a \in A$ *, then* $d(a, B) \leq d_H(A, B)$.

**Lemma 2.2.** *(Nadler, 1969) Let A and B be nonempty closed and bounded subsets of a metric space* $(X, d)$ *and* $0 < \xi \in \mathfrak{R}$ *then for* $a \in A$ *there exists* $b \in B$ *such that* $d(a, B) \leq d_H(A, B) + \xi$.

**Lemma 2.3.** *(Nadler, 1969) The completeness of* $(X, d)$ *implies that* $(CB(2^X), d_H)$ *is complete.*

**Theorem 2.1.** *(Hardy & Rogers, 1973) Let $(X, d)$ be a complete metric space and a mapping $T: X \to X$ suppose there exists non-negative constants $a_1, a_2, a_3, a_4, a_5$ satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ such that for each $x, y \in X$*

$$d(Fx, Fy) \leq a_1 d(x, y) + a_2 d(x, Fx) + a_3 d(y, Fy) + a_4 d(x, Fy) + a_5 d(y, Fx)$$

*holds then F has a unique fixed point in X.*

## 3. Main Results

The mapping satisfies the contractive condition in Theorem (2.1) is called Hardy and Rogers type mapping. It is mentioned that Hardy and Rogers contractive condition does not implies that the mapping T is continuous, which differentiates it from Banach contractive condition for $c \in X$ and $0 < r < R$. Let $S_r(c) = \{x \in X/d(c, x) < r\}$ be the ball of radius r centered at c, the closure of $S_r(c)$ is denoted by $\overline{S_r(c)}$. We present a result regarding the existence of common fixed point for fuzzy mappings satisfying Hardy and Rogers type contractive condition on closed balls. The theorem is as follows:

**Theorem 3.1.** *Let $(X, d)$ be a complete metric space $x_0 \in X$ and mapping $F, T: \overline{S_r(x_0)} \to \tau(X)$. Suppose there exist a constants $a_1, a_2, a_3, a_4, a_5$ satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ with*

$$D_1(Fx, Ty) \leq a_1 d(x, y) + a_2 d(x, [Fx]_1) + a_3 d(y, [Ty]_1) + a_4 d(x, [Ty]_1) + a_5 d(y, [Fx]_1) \tag{3.1}$$

*for all $x, y \in \overline{S_r(x_0)}$ and*

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{(1 - a_3 - a_4)} \tag{3.2}$$

*holds. Then F and T has a common fuzzy fixed point in $\overline{S_r(x_0)}$ that is there exists $x^* \in \overline{S_r(x_0)}$ with $\{x^*\} \subseteq Fx^* \cap Tx^*$.*

*Proof.* Choose $x_1 \in X$ such that $\{x_1\} \subseteq Fx_0$ and

$$d(x_0, x_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{(1 - a_3 - a_4)} \tag{3.3}$$

since $[Fx_0]_1 \neq \phi$ for the sake of simplicity chooses $\lambda = \frac{(a_1+a_2+a_4)}{(1-a_3-a_4)}$ this gives us $d(x_0, x_1) < (1 - \lambda)r$ which implies that $x_1 \in \overline{S_r(x_0)}$. Now choose $\varepsilon > 0$ such that

$$\lambda d(x_0, x_1) + \frac{\varepsilon}{(1 - a_3 - a_4)} < \lambda(1 - \lambda)r. \tag{3.4}$$

Then choose $\varepsilon > 0$ such that $\{x_2\} \subseteq Tx_1$ and by using inequality (3.1) and Lemma 2.1 we have

$$\begin{aligned} d(x_1, x_2) &\leq D_1(Fx_0, Tx_1) + \varepsilon \\ &\leq a_1 d(x_0, x_1) + a_2 d(x_0, [Fx_0]_1) + a_3 d(x_1, [Tx_1]_1) + a_4 d(x_0, [Tx_1]_1) + a_5 d(x_1, [Fx_0]_1 + \varepsilon \\ &\leq a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + a_4 d(x_0, x_2) + a_5 d(x_1, x_1) + \varepsilon \\ &= (a_1 + a_2) d(x_0, x_1) + a_3 d(x_1, x_2) + a_4 d(x_0, x_2) + \varepsilon \end{aligned}$$

i.e. $d(x_1, x_2) \leq \lambda d(x_0, x_1) + \frac{\varepsilon}{(1-a_3-a_4)}$ where $\lambda = \frac{(a_1+a_2+a_4)}{(1-a_3-a_4)}$. Now by inequality (3.4) we get $d(x_1, x_2) < \lambda(1-\lambda)r$. Note that $x_2 \in \overline{S_r(x_0)}$ since

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) < (1-\lambda)r + \lambda(1-\lambda)r = (1-\lambda)r(1+\lambda) \\ &< (1-\lambda)(1+\lambda+\lambda^2+\lambda^3+...)r = r \end{aligned}$$

continue this process and having chosen $\{x_n\}$ in $X$ such that $\{x_{2k+1}\} \subseteq Fx_{2k}$ and $\{x_{2k+2}\} \subseteq Tx_{2k+1}$ with $d(x_{2k+1}, x_{2k+2}) < \lambda^{2k+1}(1-\lambda)r$ where $k = 0, 1, 2, \ldots$

Notice that $\{x_n\}$ is cauchy sequence in $\overline{S_r(x_0)}$ which is complete. Therefore a point $x^* \in \overline{S_r(x_0)}$ exists with $lim_{n\longrightarrow\infty} x_n = x^*$. It remains to show that $\{x^*\} \subseteq Tx^*$ and $\{x^*\} \subseteq Fx^*$. Now by using Lemma 2.1 and inequality (3.1) we get

$$\begin{aligned} d(x^*, [Tx^*]_1) &\leq d(x^*, x_{2n+1}) + d(x_{2n+1}, [Tx^*]_1) \\ &\leq d(x^*, x_{2n+1}) + D_1(Fx_{2n+2}, Tx^*) \\ &\leq d(x^*, x_{2n+1}) + a_1 d(x_{2n+2}, x^*) + a_2 d(x_{2n+2}, [Fx_{2n+2}]_1) + a_3 d(x^*, [Tx^*]_1) \\ &\quad + a_4 d(x_{2n+2}, [Tx^*]_1) + a_5 d(x^*, [Fx_{2n+2}]_1) \\ &\leq d(x^*, x_{2n+1}) + a_1 d(x_{2n+2}, x^*) + a_2 d(x_{2n+2}, x_{2n+1}) + a_3 d(x^*, [Tx^*]_1) \\ &\quad + a_4 d(x_{2n+2}, [Tx^*]_1) + a_5 d(x^*, x_{2n+1}) \\ &\leq d(x^*, x_{2n+1}) + a_1 d(x_{2n+2}, x^*) + a_2 d(x_{2n+2}, x_{2n+1}) + a_4 d(x_{2n+2}, x^*) \\ &\quad + a_4 d(x^*, [Tx^*]_1) + a_5 d(x^*, x_{2n+1}) \\ &\leq d(x^*, x_{2n+1}) + a_1 d(x_{2n+2}, x^*) + a_2 d(x_{2n+2}, x_{2n+1}) \\ &\quad + a_4 d(x_{2n+2}, x^*) + a_5 d(x^*, x_{2n+1}) \\ &\longrightarrow 0 \text{ as } n \longrightarrow \infty \end{aligned}$$

This implies that $d(x^*, [Tx^*]_1) = 0$, which implies that $\{x^*\} \subseteq Tx^*$. Similarly consider that $d(x^*, [Fx^*]_1) \leq d(x^*, x_{2n+2}) + d(x_{2n+2}, [Fx^*]_1)$ to show that $\{x^*\} \subseteq Fx^*$. This implies that the mappings **F** and **T** have a common fixed point $\overline{S_r(x_0)}$, i.e. $\{x^*\} \subseteq Fx^* \cap Tx^*$. □

**Corollary 3.1.** *Let $(X, d)$ be a complete metric space $x_0 \in X$ and mapping $F : \overline{S_r(x_0)} \to \tau(X)$.Suppose there exist a non negative constants $a_1, a_2, a_3, a_4, a_5$ satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ with*

$$D_1(Fx, Fy) \leq a_1 d(x, y) + a_2 d(x, [Fx]_1) + a_3 d(y, [Fy]_1) + a_4 d(x, [Fy]_1) + a_5 d(y, [Fx]_1)$$

*for all $x, y \in \overline{S_r(x_0)}$ and*

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{1 - a_3 - a_4}$$

*holds. Then $F$ has a common fuzzy fixed point in $\overline{S_r(x_0)}$ that is there exists $x^* \in \overline{S_r(x_0)}$ with*

$$\{x^*\} \subseteq Fx^*.$$

*Proof.* Put $T = F$ in Theorem 3.1 we get $x^* \in \overline{S_r(x_0)}$ such that $\{x^*\} \subseteq Fx^*$. □

**Theorem 3.2.** *Let $(X, d)$ be a complete metric space $x_0 \in X$ and mapping $F, T : X : \to \tau(X)$.Suppose there exist a constants $a_1, a_2, a_3, a_4, a_5$ satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ with*

$$D_1(Fx, Ty) \leq a_1 d(x, y) + a_2 d(x, [Fx]_1) + a_3 d(y, [Ty]_1) + a_4 d(x, [Ty]_1) + a_5 d(y, [Fx]_1)$$

*for all $x, y \in X$ and*

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{1 - a_3 - a_4}$$

*holds.Then $F$ and $T$ has a common fuzzy fixed point in $X$ that is there exists $x^* \in X$ with*

$$\{x^*\} \subseteq Fx^* \cap Tx^*.$$

Proof: Fix $x_0 \in X$ and choose $r > 0$ such that

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{1 - a_3 - a_4}$$

Now Theorem 3.1 guarantees that there exists $x^* \in X$ with

$$\{x^*\} \subseteq Fx^* \cap Tx^*.$$

**Corollary 3.2.** *Let $(X, d)$ be a complete metric space $x_0 \in X$ and mapping $F : X \to \tau(X)$.Suppose there exist a constants $a_1, a_2, a_3, a_4, a_5$ satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ with*

$$D_1(Fx, Fy) \leq a_1 d(x, y) + a_2 d(x, [Fx]_1) + a_3 d(y, [Fy]_1) + a_4 d(x, [Fy]_1) + a_5 d(y, [Fx]_1)$$

*for all $x, y \in X$ and*

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{1 - a_3 - a_4}$$

*holds.Then $F$ has a common fuzzy fixed point in $X$ that is there exists $x^* \in X$ with*

$$\{x^*\} \subseteq Fx^*.$$

*Proof: In Theorem 3.2 take T=F we get $x^* \in X$ such that $\{x^*\} \subseteq Fx^*$.*

## 4. The importance and future of this theory:

Fuzzy sets and mappings play important roles in the fuzzification of systems. In particular, in the recent years the fixed point theory for fuzzy mappings and for a family of these mappings obtained via implicit functions named Hardy and Rogers type mappings. In this article can further be used in the process of finding the solution of functional equations in fuzzy systems. As far as the application of contraction mapping is concerned the situation is not fully exploited. It is quite possible that a contraction $T$ is defined on the whole space $X$ but it is contractive on the subset $Y$ of the subset of the space rather on the whole space $X$. Moreover the contraction mapping under consideration may not be continues. If $Y$ is closed, then it is complete,so that a mapping $T$ has a fixed point $x$ in $Y$,and $x_n \to x$ as in the case of whole space X provided we improve a simple restriction on the choice of $x_0$, so that $x_n's$ remains in $Y$. In this paper, we have discussed this concept for fuzzy Hardy and Rogers mappings on a complete metric space $x$ which generalize/improves several classical tendon with (Azam *et al.*, 2013) will become the foundation of fuzzy theory on closed balls.

An example of a fuzzy mapping which is contractive on the subset of a space but not on the whole space is as follows:

**Example 4.1.** Let $X = \mathbb{R}$ and $d : XxX \to \mathbb{R}$ is defined by $d(x, y) = |x - y|$ where $x, y \in X$ consider the mapping $F : X \to \tau(X)$ is defined by

$$F(x) = \begin{cases} \chi_{(1-x)}, & \text{if } x \text{ is irrational;} \\ \chi_{(\frac{1+x}{3})}, & \text{if } x \text{ is rational.} \end{cases}$$

then $F$ is Hardy and Rogers type fuzzy mapping on the closed balls $\overline{S_{(\frac{1}{2})}(\frac{1}{2})} = [0, 1]$ but not on the whole space $X$.

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# Some Fixed Point Theorems for Ordered $F$-generalized Contractions in 0-$f$-orbitally Complete Partial Metric Spaces

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## Abstract

We prove some fixed point theorems for ordered $F$-generalized contractions in ordered 0-$f$-orbitally complete partial metric spaces. Our results generalize some well-known results in the literature, in particular the recent result of Wardowski [Fixed Point Theory Appl. 2012:94 (2012)] from metric spaces to ordered 0-$f$-orbitally complete partial metric spaces. Some examples are given which illustrate the new results.

*Keywords:* Partial metric space, partial order, fixed point, $F$-generalized contraction, 0-$f$-orbitally complete space.
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## 1. Introduction

In 1994, Matthews (Matthews, 1994) introduced the notion of a partial metric space, as a part of the study of denotational semantics of dataflow networks. In a partial metric space, the usual distance was replaced by partial metric, with an interesting property of "nonzero self distance" of points. Also, the convergence of sequences in this space was defined in such a way that the limit of a convergent sequence need not be unique. Matthews showed that the Banach contraction principle is valid in partial metric spaces and can be applied in program verifications. Later on, several authors generalized the results of Matthews (see, for example, (Ahmad *et al.*, 2012; Bari *et al.*, 2012; Kadelburg *et al.*, 2013; Nashine *et al.*, 2012; Vetro & Radenović, 2012)). O'Neill (O'Neill, 1996) generalized the concept of partial metric space a bit further by admitting negative distances. The partial metric defined by O'Neill is called dualistic partial metric. Heckmann (Heckmann,

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1999) generalized it by omitting small self-distance axiom. The partial metric defined by Heckmann is called the weak partial metric. Romaguera (Romaguera, 2010) introduced the notions of 0-Cauchy sequences and 0-complete partial metric spaces and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness.

The existence of fixed points of mappings in partially ordered sets was investigated by Ran and Reurings (Ran & Reurings, 2004) and then by Nieto and Rodríguez-Lopez (Nieto & Lopez, 2005, 2007). In these papers, some results on the existence of a unique fixed point for nondecreasing mappings were applied to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Later on, these results were generalized by several authors in different spaces.

Recently, Wardowski (Wardowski, 2012) has introduced a new concept of $F$-contraction and proved a fixed point theorem which generalizes Banach contraction principle in a different direction than in the known results from the literature in complete metric spaces.

In this paper, we prove some fixed point theorems for ordered $F$-generalized contractions in ordered 0-$f$-orbitally complete partial metric spaces. The results of this paper generalize and extend the results of Wardowski (Wardowski, 2012), Ran and Reurings (Ran & Reurings, 2004), Nieto and Rodríguez-Lopez (Nieto & Lopez, 2005, 2007), Altun et al. (Altun *et al.*, 2010), Ćirić (Ćirić, 1971, 1972) and some other well-known results in the literature. Some examples are given which illustrate our results.

## 2. Preliminaries

First we recall some definitions and properties of partial metric spaces (see, e.g., (Matthews, 1994; Oltra & Valero, 2004; O'Neill, 1996; Romaguera, 2010, 2011)).

**Definition 2.1.** A partial metric on a nonempty set $X$ is a function $p\colon X \times X \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) such that for all $x, y, z \in X$:

(P1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;
(P2) $p(x, x) \leq p(x, y)$;
(P3) $p(x, y) = p(y, x)$;
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair $(X, p)$ such that $X$ is a nonempty set and $p$ is a partial metric on $X$.

It is clear that, if $p(x, y) = 0$, then from (P1) and (P2) $x = y$. But if $x = y$, $p(x, y)$ may not be 0. Also, every metric space is a partial metric space, with zero self distance.

**Example 2.1.** If $p\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $p(x, y) = \max\{x, y\}$, for all $x, y \in \mathbb{R}^+$, then $(\mathbb{R}^+, p)$ is a partial metric space.

For some more examples of partial metric spaces, we refer to (Aydi *et al.*, 2012) and the references therein.

Each partial metric on $X$ generates a $T_0$ topology $\tau_p$ on $X$ which has as a base the family of open $p$-balls $\{B_p(x, \epsilon)\colon x \in X,\ \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X :\ p(x, y) < p(x, x) + \epsilon\}$ for all

$x \in X$ and $\epsilon > 0$. A mapping $f: X \to X$ is continuous if and only if, whenever a sequence $\{x_n\}$ in $X$ converging with respect to $\tau_p$ to a point $x \in X$, the sequence $\{fx_n\}$ converges with respect to $\tau_p$ to $fx \in X$.

**Theorem 2.1.** *(Matthews, 1994) For each partial metric $p: X \times X \to \mathbb{R}^+$ the pair $(X, d)$ where, $d(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ for all $x, y \in X$, is a metric space.*

Here $(X, d)$ is called the induced metric space and $d$ is the induced metric. In further discussion, unless something else is specified, $(X, d)$ will represent the induced metric space.

Let $(X, p)$ be a partial metric space.

(1) A sequence $\{x_n\}$ in $(X, p)$ converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$.
(2) A sequence $\{x_n\}$ in $(X, p)$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$.
(3) $(X, p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges with respect to $\tau_p$ to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.
(4) A sequence $\{x_n\}$ in $(X, p)$ is called a 0-Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. The space $(X, p)$ is said to be 0-complete if every 0-Cauchy sequence in $X$ converges with respect to $\tau_p$ to a point $x \in X$ such that $p(x, x) = 0$.

**Lemma 2.1.** *(Matthews, 1994; Oltra & Valero, 2004; Romaguera, 2010, 2011) Let $(X, p)$ be a partial metric space and $\{x_n\}$ be any sequence in $X$.*

*(i) $\{x_n\}$ is a Cauchy sequence in $(X, p)$ if and only if it is a Cauchy sequence in the metric space $(X, d)$.*
*(ii) $(X, p)$ is complete if and only if the metric space $(X, d)$ is complete. Furthermore, $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.*
*(iii) Every 0-Cauchy sequence in $(X, p)$ is Cauchy in $(X, d)$.*
*(iv) If $(X, p)$ is complete then it is 0-complete.*

The converse assertions of (iii) and (iv) do not hold. Indeed, the partial metric space $(\mathbb{Q} \cap \mathbb{R}^+, p)$, where $\mathbb{Q}$ denotes the set of rational numbers and the partial metric $p$ is given by $p(x, y) = \max\{x, y\}$, provides an easy example of a 0-complete partial metric space which is not complete. Also, it is easy to see that every closed subset of a 0-complete partial metric space is 0-complete.

The proof of the following lemma is easy and for details we refer to (Karapınar, 2012) and the references therein.

**Lemma 2.2.** *Assume $x_n \to z$ as $n \to \infty$ in a partial metric space $(X, p)$ such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for all $y \in X$.*

The notion of orbital completeness of metric spaces was introduced in (Ćirić, 1971) and adapted to partial metric spaces in (Karapınar, 2012) as follows:

Let $(X, p)$ be a partial metric space and $f: X \to X$ be a mapping. For any $x \in X$, the set $O(x) = \{x, fx, f^2x, \ldots\}$ is called the orbit of $f$ at point $x$. $(X, p)$ is called $f$-orbitally complete if every Cauchy sequence in $O(x)$ converges in $(X, p)$.

Now, we define 0-$f$-orbital completeness of a partial metric space.

**Definition 2.2.** Let $(X, p)$ be a partial metric space and $f: X \to X$ be a mapping. $(X, p)$ is said to be 0-$f$-orbitally complete, if every 0-Cauchy sequence in $O(x) = \{x, fx, f^2x, \ldots\}$, $x \in X$, converges with respect to $\tau_p$ to a point $u \in X$ such that $p(u, u) = 0$.

Note that every complete partial metric space is 0-complete, and every 0-complete partial metric space is 0-$f$-orbitally complete for every $f: X \to X$. But, the converse assertions need not hold as shown by the following example.

**Example 2.2.** Let $X = \mathbb{R}^+ \cap (\mathbb{Q} \setminus \{1\})$ and $p: X \times X \to \mathbb{R}^+$ be defined by

$$p(x, y) = \begin{cases} |x - y|, & \text{if } x, y \in [0, 1); \\ \max\{x, y\}, & \text{otherwise.} \end{cases}$$

Define $f: X \to X$ by $fx = \frac{x}{2}$ for all $x \in X$. Then $(X, p)$ is a partial metric space. Note that $(X, p)$ is not complete because the induced metric space $(X, d)$, where

$$d(x, y) = \begin{cases} 2|x - y|, & \text{if } x, y \in [0, 1); \\ |x - y|, & \text{otherwise,} \end{cases}$$

is not complete. Also $(X, p)$ is not 0-complete. Indeed, for $x_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$, we observe that, $p(x_n, x_m) = |\frac{1}{n} - \frac{1}{m}| \to 0$ as $n \to \infty$. But, there is no $u \in X$ such that $\lim_{n\to\infty} p(x_n, u) = p(u, u) = 0$. Now, it is easy to see that $(X, p)$ is 0-$f$-orbitally complete.

Consider, together with Wardowski in (Wardowski, 2012), the following properties for a mapping $F: \mathbb{R}^+ \to \mathbb{R}$:

(F1) $F$ is strictly increasing, that is, for $\alpha, \beta \in \mathbb{R}^+$, $\alpha < \beta$ implies $F(\alpha) < F(\beta)$;
(F2) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We denote the set of all functions satisfying properties (F1)–(F3), by $\mathcal{F}$.

For examples of functions $F \in \mathcal{F}$, we refer to (Wardowski, 2012). Wardowski defined in (Wardowski, 2012) $F$-contractions as follows:

Let $(X, \rho)$ be a metric space. A mapping $T: X \to X$ is said to be an $F$-contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$, $\rho(Tx, Ty) > 0$ we have

$$\tau + F(\rho(Tx, Ty)) \leq F(\rho(x, y)).$$

Similarly, we adopt the following definitions.

**Definition 2.3.** Let $X$ be a nonempty set, $\preceq$ a partial order relation defined on $X$ and $p$ be a partial metric on $X$ (then, $(X, \preceq, p)$ is called an ordered partial metric space). A map $f: X \to X$ is called:

1. an ordered $F$-contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $x \preceq y$ and $p(fx, fy) > 0$ we have

$$\tau + F(p(fx, fy)) \leq F(p(x, y)). \tag{2.1}$$

2. an ordered $F$-weak contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $x \preceq y$ and $p(fx, fy) > 0$ we have

$$\tau + F(p(fx, fy)) \leq F(\max\{p(x, y), p(x, fx), p(y, fy)\}). \tag{2.2}$$

   If inequality (2.2) is satisfied for all $x, y \in X$, then $f$ is called an $F$-weak contraction;

3. an ordered $F$-generalized contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $x \preceq y$ and $p(fx, fy) > 0$ we have

$$\tau + F(p(fx, fy)) \leq F(\max\{p(x, y), p(x, fx), p(y, fy), \frac{p(x, fy) + p(y, fx)}{2}\}). \tag{2.3}$$

   If inequality (2.3) is satisfied for all $x, y \in X$, then $f$ is called an $F$-generalized contraction.

The following example shows that the class of $F$-contractions in partial metric spaces is more general than that in metric spaces.

**Example 2.3.** Let $X = \mathbb{R}^+$ and $p\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Note that the metric induced by $p$ (as well as the usual metric) on $X$ is given by $d(x, y) = |x - y|$ for all $x, y \in X$. Define $f\colon X \to X$ by

$$fx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 1); \\ 0, & \text{if } x = 1. \end{cases}$$

Then for $x = 1, y = \frac{9}{10}$ there is no $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(d(fx, fy)) \leq F(d(x, y)).$$

On the other hand, for $\tau = \log 2$ and $F(\alpha) = \log \alpha + \alpha$, it is easy to see that $f$ is an $F$-contraction in $(X, p)$.

## 3. Main results

The following is our first main result.

**Theorem 3.1.** *Let $(X, \preceq, p)$ be an ordered partial metric space and $f\colon X \to X$ be an ordered $F$-generalized contraction for some $F \in \mathcal{F}$. If $(X, p)$ is 0-$f$-orbitally complete and the following conditions hold:*

*(i) $f$ is nondecreasing with respect to "$\preceq$", that is, if $x \preceq y$ then $fx \preceq fy$;*
*(ii) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;*
*(iii) (a) $f$ is continuous, or*
*(b) $F$ is continuous and for every nondecreasing sequence $\{x_n\}$, $x_n \to u \in X$ as $n \to \infty$ implies $x_n \preceq u$ for all $n \in \mathbb{N}$.*

*Then $f$ has a fixed point $u \in X$. Furthermore, the fixed point of $f$ is unique if and only if the set of all fixed points of $f$ is well-ordered.*

*Proof.* First, we shall show the existence of fixed point of $f$. Let $x_0 \in X$ be the point given by (ii). We define a sequence $\{x_n\}$ in $X$ by $x_{n+1} = fx_n$ for all $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then $x_{n_0}$ is a fixed point of $f$. Therefore, assume that $x_{n+1} \neq x_n$ for all $n \geq 0$. As, $x_0 \preceq fx_0$ we have $x_0 \preceq x_1$, and since $f$ is nondecreasing with respect to $\preceq$, we have $fx_0 \preceq fx_1$ that is $x_1 \preceq x_2$. Similarly, we obtain $x_n \preceq x_{n+1}$ for all $n \geq 0$. Also, $f$ is an ordered $F$-generalized contraction therefore, for any $n \in \mathbb{N}$ it follows from (2.3) and symmetry of $p$ that

$$\begin{aligned}
\tau + F(p(fx_n, fx_{n-1})) &= \tau + F(p(fx_{n-1}, fx_n)) \\
&\leq F(\max\{p(x_n, x_{n-1}), p(x_n, fx_n), p(x_{n-1}, fx_{n-1}), \\
&\quad \frac{p(x_n, fx_{n-1}) + p(x_{n-1}, fx_n)}{2}\}) \\
&= F(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1}), p(x_{n-1}, x_n), \\
&\quad \frac{p(x_n, x_n) + p(x_{n-1}, x_{n+1})}{2}\}) \\
&\leq F(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1}), \\
&\quad \frac{p(x_{n-1}, x_n) + p(x_{n+1}, x_n)}{2}\}).
\end{aligned}$$

Note that, for any $a, b \in \mathbb{R}^+$ we have $\max\{a, b, \frac{a+b}{2}\} = \max\{a, b\}$, therefore it follows from the above inequality that

$$\begin{aligned}
\tau + F(p(x_{n+1}, x_n)) &\leq F(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\}) \\
F(p(x_{n+1}, x_n)) &\leq F(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\}) - \tau. \qquad (3.1)
\end{aligned}$$

If, $\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$ then from (3.1) we have

$$F(p(x_{n+1}, x_n)) \leq F(p(x_n, x_{n+1})) - \tau < F(p(x_n, x_{n+1})),$$

a contradiction. Therefore, $\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\} = p(x_n, x_{n-1})$ and from (3.1) we have

$$F(p(x_{n+1}, x_n)) \leq F(p(x_n, x_{n-1})) - \tau \text{ for all } n \in \mathbb{N}. \qquad (3.2)$$

Setting $p_n = p(x_{n+1}, x_n)$ it follows by successive applications of (3.2) that

$$F(p_n) \leq F(p_{n-1}) - \tau \leq F(p_{n-2}) - 2\tau \leq \cdots \leq F(p_0) - n\tau. \qquad (3.3)$$

From (3.3) we have $\lim_{n\to\infty} F(p_n) = -\infty$, and since $F \in \mathcal{F}$ we must have

$$\lim_{n\to\infty} p_n = 0. \qquad (3.4)$$

Again, as $F \in \mathcal{F}$ there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (p_n)^k F(p_n) = 0. \qquad (3.5)$$

From (3.3) we have

$$(p_n)^k [F(p_n) - F(p_0)] \leq -n\tau(p_n)^k \leq 0.$$

Letting $n \to \infty$ in the above inequality and using (3.4) and (3.5) we obtain

$$\lim_{n\to\infty} n(p_n)^k = 0. \tag{3.6}$$

It follows from (3.6) that there exists $n_1 \in \mathbb{N}$ such that $n(p_n)^k < 1$ for all $n > n_1$, that is,

$$p_n \leq \frac{1}{n^{1/k}} \quad \text{for all} \quad n > n_1. \tag{3.7}$$

Let $m, n \in \mathbb{N}$ with $m > n > n_1$. Then it follows from (3.7) that

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\ &\qquad - [p(x_n, x_n) + p(x_{n+1}, x_{n+1}) + \cdots + p(x_{m-1}, x_{m-1})] \\ &\leq p_n + p_{n+1} + \cdots \\ &\leq \frac{1}{n^{1/k}} + \frac{1}{(n+1)^{1/k}} + \cdots \\ &= \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

As $k \in (0, 1)$, the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/k}}$ converges, so it follows from the above inequality that $\lim_{n\to\infty} p(x_n, x_m) = 0$, that is, the sequence $\{x_n\}$ is a 0-Cauchy sequence in $O(x_0) = \{x_0, fx_0, f^2x_0, \ldots\}$. Therefore, by 0-$f$-orbital completeness of $(X, p)$, there exists $u \in X$ such that

$$\lim_{n\to\infty} p(x_n, u) = \lim_{n,m\to\infty} p(x_n, x_m) = p(u, u) = 0. \tag{3.8}$$

We shall show that $u$ is a fixed point of $f$. For this, we consider two cases.

**Case I:** Suppose (a) is satisfied, that is, $f$ is continuous. Then using (3.8) and Lemma 2.2, we obtain

$$p(u, fu) = \lim_{n\to\infty} p(x_n, fu) = \lim_{n\to\infty} p(fx_{n-1}, fu) = p(fu, fu).$$

Suppose that $p(fu, fu) > 0$. Then as $u \preceq u$, using (2.3) we obtain

$$\begin{aligned} \tau + F(p(fu, fu)) &\leq F(\max\{p(u, u), p(u, fu), p(u, fu), \frac{p(u, fu) + p(u, fu)}{2}\}) \\ &= F(\max\{p(u, u), p(u, fu)\}) \\ &= F(p(u, fu)), \end{aligned}$$

that is, $F(p(fu, fu)) < F(p(u, fu))$ and from $F \in \mathcal{F}$ we have $p(fu, fu) < p(u, fu) = p(fu, fu)$, a contradiction. Therefore, $p(fu, fu) = p(u, fu) = 0$, that is, $fu = u$, so $u$ is a fixed point of $f$.

**Case II:** Suppose that (b) is satisfied. Then we consider two subcases.

**(i):** For each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $p(x_{k_n+1}, fu) = 0$ and $k_n > k_{n-1}$, where $k_0 = 1$. Then, using Lemma 2.2, we have

$$p(u, fu) = \lim_{n\to\infty} p(x_{k_n+1}, fu) = 0.$$

Therefore, $fu = u$, that is, $u$ is a fixed point of $f$.

**(ii):** There exists $n_2 \in \mathbb{N}$ such that $p(x_n, fu) \neq 0$ for all $n > n_2$. In this case, since $\{x_n\}$ is a nondecreasing sequence and $x_n \to u$ as $n \to \infty$, we have $x_n \preceq u$ for all $n \in \mathbb{N}$. Therefore, using (2.3) we obtain

$$\begin{aligned}\tau + F(p(x_{n+1}, fu)) &= \tau + F(p(fx_n, fu)) \\ &\leq F(\max\{p(x_n, u), p(x_n, fx_n), p(u, fu), \frac{p(x_n, fu) + p(u, fx_n)}{2}\}) \\ &\leq F(\max\{p(x_n, u), p(x_n, x_{n+1}), p(u, fu), \\ &\quad \frac{p(x_n, u) + p(u, fu) + p(u, x_{n+1})}{2}\}).\end{aligned}$$

From (3.8), there exists $n_3 \in \mathbb{N}$ such that, for all $n > n_3$ we have

$$\max\{p(x_n, u), p(x_n, x_{n+1}), p(u, fu), \frac{p(x_n, u) + p(u, fu) + p(u, x_{n+1})}{2}\} = p(u, fu),$$

so, for $n > \max\{n_2, n_3\}$ we obtain

$$\tau + F(p(x_{n+1}, fu)) \leq F(p(u, fu)).$$

As $F$ is continuous, letting $n \to \infty$ in the above inequality and using (3.8) and Lemma 2.2 we obtain

$$\tau + F(p(u, fu)) \leq F(p(u, fu)),$$

a contradiction. Therefore, we must have $p(u, fu) = 0$ that is $fu = u$. Thus $u$ is a fixed point of $f$.

Suppose that the set of fixed points of $f$ is well-ordered and $u, v \in F_f$ with $p(u, v) > 0$, where $F_f$ denotes the set of all fixed points of $f$. As $F_f$ is well-ordered, let $u \preceq v$. Then from (2.3) we obtain

$$\begin{aligned}\tau + F(p(u, v)) &= \tau + F(p(fu, fv)) \\ &\leq F(\max\{p(u, v), p(u, fu), p(v, fv), \frac{p(u, fv) + p(v, fu)}{2}\}) \\ &\leq F(\max\{p(u, v), p(u, u), p(v, v), p(v, u)\}) \\ &\leq F(p(u, v)),\end{aligned}$$

a contradiction. Similarly, for $v \preceq u$ we get a contradiction. Therefore, the fixed point of $f$ is unique. For the converse, if the fixed point of $f$ is unique then $F_f$, being a singleton, is well-ordered. □

The following corollaries are immediate consequences of the above theorem.

**Corollary 3.1.** *Let $(X, \preceq, p)$ be an ordered partial metric space and $f\colon X \to X$ be an ordered $F$-contraction. Let $(X, p)$ is 0-$f$-orbitally complete and the following conditions hold:*

*(i) $f$ is nondecreasing with respect to "$\preceq$", that is, if $x \preceq y$ then $fx \preceq fy$;*

*(ii) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;*

*(iii) (a) $f$ is continuous, or*

*(b) $F$ is continuous and for every nondecreasing sequence $\{x_n\}$ such that $x_n \to u \in X$ as $n \to \infty$ it follows that $x_n \preceq u$ for all $n \in \mathbb{N}$.*

*Then $f$ has a fixed point $u \in X$. Furthermore, the fixed point of $f$ is unique if and only if the set of all fixed points of $f$ is well-ordered.*

**Corollary 3.2.** *Let $(X, \preceq, p)$ be an ordered partial metric space and $f : X \to X$ be an ordered $F$-weak contraction. If $(X, p)$ is 0-$f$-orbitally complete and the following conditions hold:*

*(i) $f$ is nondecreasing with respect to "$\preceq$", that is, if $x \preceq y$ then $fx \preceq fy$;*

*(ii) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;*

*(iii) (a) $f$ is continuous, or*

*(b) $F$ is continuous and for every nondecreasing sequence $\{x_n\}$ such that $x_n \to u \in X$ as $n \to \infty$ it follows that $x_n \preceq u$ for all $n \in \mathbb{N}$.*

*Then $f$ has a fixed point $u \in X$. Furthermore, the fixed point of $f$ is unique if and only if the set of all fixed points of $f$ is well-ordered.*

*Remark.* We note that every metric space is a partial metric space with zero self distance. Therefore we can replace the partial metric $p$ by a metric $\rho$ in Theorem 3.1. Also, after this replacement, the 0-$f$-orbital completeness reduces to orbital completeness of the metric space. Therefore, by this replacement in Theorem 3.1, we obtain the fixed point result for ordered $F$-generalized contraction in orbitally complete metric spaces.

In the above theorems the fixed point of the self map $f$ is the limit of a 0-Cauchy sequence and due to 0-$f$-orbital completeness of the space this limit has zero self distance. The next theorem shows that, if an ordered $F$-generalized contraction has a fixed point then its self distance must be zero, that is, it does not depend on the properties of space such as completeness etc.

**Theorem 3.2.** *Let $(X, \preceq, p)$ be an ordered partial metric space and $f : X \to X$ be an ordered $F$-generalized contraction. If $f$ has a fixed point $u$ then $p(u, u) = 0$.*

*Proof.* Suppose that $u \in F_f$ and $p(u, u) > 0$. Then, it follows from (2.3) that

$$\begin{aligned}\tau + F(p(u, u)) &= \tau + F(p(fu, fu))\\ &\leq F(\max\{p(u, u), p(u, fu), p(u, fu), \frac{p(u, fu) + p(u, fu)}{2}\})\\ &= F(p(u, u)).\end{aligned}$$

As $\tau > 0$, the above inequality yields a contradiction. Therefore, we have $p(u, u) = 0$ for all $u \in F_f$. □

The following example illustrates our results.

**Example 3.1.** Let $X = [0,2] \cap (\mathbb{Q} \setminus \{1\})$ and define $p\colon X \times X \to \mathbb{R}^+$ by

$$p(x,y) = \begin{cases} |x-y|, & \text{if } x,y \in [0,1); \\ 0, & \text{if } x = y = 2; \\ \max\{x,y\}, & \text{otherwise.} \end{cases}$$

Then $(X,p)$ is a partial metric space. Define a partial order relation "$\preceq$" on $X$ by

$$\preceq\, = \{(x,y)\colon x,y \in [0,1), y \le x\} \cup \{(x,y)\colon x,y \in (1,2), y \le x\} \cup \{(2,2)\}.$$

Define $f\colon X \to X$ by

$$fx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0,1); \\ \frac{1}{4}, & \text{if } x \in (1,2); \\ 2, & \text{if } x = 2. \end{cases}$$

Then it is easy to see that $(X,p)$ is a 0-$f$-orbitally complete partial metric space. Let $F(\alpha) = \log \alpha$ for all $\alpha \in \mathbb{R}^+$. Then $f$ satisfies all the conditions of Corollary 3.1 (except that the set of fixed points of $f$ is well-ordered) with $\tau \le \log 2$. Note that, $F_f = \{0,2\}$ with $p(0,0) = p(2,2) = 0$ and $(2,0),(0,2) \notin\, \preceq$. Now, the metric $d$ induced by $p$ is given by

$$d(x,y) = \begin{cases} 2|x-y|, & \text{if } x,y \in [0,1); \\ |x-y|, & \text{otherwise,} \end{cases}$$

and $(X,d)$ is not complete. Similarly, if $\rho$ is the usual metric on $X$ then $(X,\rho)$ is not complete, therefore the results from metric cases are not applicable here. This example shows also that an ordered $F$-contraction may not be an $F$-contraction (not even an $F$-generalized contraction). Indeed, for $x \in [0,1)$, $y = 2$ there exists no $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\tau + F(p(fx,fy)) \le F(\max\{p(x,y), p(x,fx), p(y,fy), \frac{p(x,fy)+p(y,fx)}{2}\}).$$

Therefore, $f$ is not an $F$-generalized contraction in $(X,p)$. Similarly, for $x = 0$, $y = 2$ one can see that $f$ is not an $F$-generalized contraction in $(X,d)$ and $(X,\rho)$.

In the following theorem the conditions on self map $f$, "nondecreasing", continuous and 0-$f$-orbital completeness of space, are replaced by another condition on self map $f$.

**Theorem 3.3.** *Let $(X,\preceq,p)$ be an ordered partial metric space and $f\colon X \to X$ be an ordered $F$-generalized contraction. Let there exists $u \in X$ such that $u \preceq fu$ and $p(u,fu) \le p(x,fx)$ for all $x \in X$. Then $f$ has a fixed point $u \in X$. Furthermore, the fixed point of $f$ is unique if and only if the set of all fixed points of $f$ is well-ordered.*

*Proof.* Let $G(x) = p(x,fx)$ for all $x \in X$. Then by assumption we have

$$G(u) \le G(x) \quad \text{for all} \quad x \in X. \tag{3.9}$$

We shall show that $fu = u$. Suppose that $G(u) = p(u, fu) > 0$. Then since $u \preceq fu$, it follows from (2.3) that

$$\begin{aligned} F(G(fu)) &= F(p(fu, ffu)) \\ &\leq F(\max\{p(u, fu), p(u, fu), p(fu, ffu), \frac{p(u, ffu) + p(fu, fu)}{2}\}) - \tau \\ &\leq F(\max\{p(u, fu), p(fu, ffu), \frac{p(u, fu) + p(fu, ffu)}{2}\}) - \tau \\ &= F(\max\{G(u), G(fu), \frac{G(u) + G(fu)}{2}\}) - \tau \\ &= F(\max\{G(u), G(fu)\}) - \tau. \end{aligned}$$

If $\max\{G(u), G(fu)\} = G(fu)$, then it follows from the above inequality that $F(G(fu)) < F(G(fu))$, a contradiction. If $\max\{G(u), G(fu)\} = G(u)$, then again we obtain $F(G(fu)) < F(G(u))$ and $F \in \mathcal{F}$ so $G(fu) < G(u)$, a contradiction. Thus, we must have $G(u) = p(u, fu) = 0$, that is $fu = u$ and so $u$ is a fixed point of $f$.

The necessary and sufficient condition for the uniqueness of fixed point follows from a similar process as used in Theorem 3.1. □

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# On Some Generalized $I$-Convergent Sequence Spaces Defined by a Sequence of Moduli

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**Abstract**

In this article we introduce the sequence spaces $c_0^I(F,p)$, $c^I(F,p)$ and $l_\infty^I(F,p)$ for $F = (f_k)$ a sequence of moduli and $p = (p_k)$ sequence of positive reals and study some of the properties and inclusion relation on these spaces.

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## 1. Introduction

Throughout the article $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ and $\omega$ denotes the set of natural,real,complex numbers and the class of all sequences respectively.

The notion of the statistical convergence was introduced by H. Fast (Fast, 1951). Later on it was studied by J. A. Fridy (Fridy, 1985, 1993) from the sequence space point of view and linked it with the summability theory. The notion of $I$-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Šalát and Wilczyński (Kostyrko & Šalát and W. Wilczyński, 2000). Later on it was studied by Šalát, Tripathy and Ziman (Šalát *et al.*, 1963) and Demirci (Demirci, 2001). Recently it was studied by V. A. Khan and K. Ebadullah (Khan & Ebadullah, 2011; Khan *et al.*, 2011; Khan & Ebadullah, 2012; Khan *et al.*, 2012) and Tripathy and Hazarika (Tripathy & Hazarika, 2009, 2011).

Here we give some preliminaries about the notion of $I$-convergence.

Let $N$ be a non empty set. Then a family of sets $I \subseteq 2^N$ ($2^N$ denoting the power set of $N$) is said to be an ideal if $I$ is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e $A \in I, B \subseteq A \Rightarrow B \in I$.

A non-empty family of sets $\pounds(I) \subseteq 2^N$ is said to be filter on $N$ if and only if $\phi \notin \pounds(I)$,for $A, B \in \pounds(I)$ we have $A \cap B \in \pounds(I)$ and for each $A \in \pounds(I)$ and $A \subseteq B$ implies $B \in \pounds(I)$.

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An Ideal $I \subseteq 2^N$ is called non-trivial if $I \neq 2^N$. A non-trivial ideal $I \subseteq 2^N$ is called admissible if $\{\{x\} : x \in N\} \subseteq I$.

A non-trivial ideal $I$ is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing $I$ as a subset.

For each ideal $I$, there is a filter £($I$) corresponding to $I$. i.e £($I$) $= \{K \subseteq N : K^c \in I\}$,where $K^c = N - K$.

**Definition 1.1.** A sequence $(x_k)$ is said to be $I$-convergent to a number L if for every $\epsilon > 0$. $\{k \in N : |x_k - L| \geq \epsilon\} \in$ I. In this case we write $I$-lim $x_k = L$. The space $c^I$ of all $I$-convergent sequences to $L$ is given by

$$c^I = \{(x_k) : \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I, \text{for some L} \in \mathbb{C}\}.$$

**Definition 1.2.** A sequence $(x_k)$ is said to be $I$-null if L = 0. In this case we write $I$-lim $x_k = 0$.

**Definition 1.3.** A sequence $(x_k)$ is said to be $I$-cauchy if for every $\epsilon > 0$ there exists a number m = m($\epsilon$) such that $\{k \in N : |x_k - x_m| \geq \epsilon\} \in I$.

**Definition 1.4.** A sequence $(x_k)$ is said to be $I$-bounded if there exists M >0 such that $\{k \in N : |x_k| > M\} \in I$.

**Definition 1.5.** Let $(x_k), (y_k)$ be two sequences. We say that $(x_k) = (y_k)$ for *almost all k relative to I (a.a.k.r.I)*, if $\{k \in \mathbb{N} : x_k \neq y_k\} \in I$.

**Definition 1.6.** For any set E of sequences the space of multipliers of E, denoted by $M(E)$ is given by

$$M(E) = \{a \in \omega : ax \in E \ \text{ for all } \ x \in E\}).$$

**Definition 1.7.** The concept of paranorm(See (Maddox, 1969)) is closely related to linear metric spaces. It is a generalization of that of absolute value.

Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if for all $x, y, z \in X$,

(PI) $g(x) = 0 \; if \; x = \theta$,

(P2) $g(-x) = g(x)$,

(P3) $g(x + y) \leq g(x) + g(y)$,

(P4) If $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda \; (n \to \infty)$ and $x_n, a \in X$ with $x_n \to a \;\; (n \to \infty)$ , in the sense that $g(x_n - a) \to 0 \;\; (n \to \infty)$ , in the sense that $g(\lambda_n x_n - \lambda a) \to 0 \;\; (n \to \infty)$.

A paranorm $g$ for which $g(x) = 0$ implies $x = \theta$ is called a total paranorm on $X$, and the pair $(X, g)$ is called a totally paranormed space.

The idea of modulus was structured in 1953 by Nakano. (See (Nakano, 1953)).

A function $f : [0,\infty) \longrightarrow [0,\infty)$ is called a modulus if

(1)$f$(t) = 0 if and only if t = 0,

(2) $f$(t+u)$\leq$ $f$(t)+ $f$(u) for all t,u$\geq$0,

(3) $f$ is increasing, and

(4) $f$ is continuous from the right at zero.

Ruckle (Ruckle, 1968, 1967, 1973) used the idea of a modulus function $f$ to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space,and Ruckle (Ruckle, 1968, 1967, 1973) proved that that the intersecton of all such $X(f)$ spaces is $\phi$, the space of all finite sequences.
The space $X(f)$ is closely related to the space $l_1$ which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$.Thus Ruckle (Ruckle, 1968, 1967, 1973) proved that, for any modulus $f$.

$$X(f) \subset l_1 \text{ and } X(f)^{\alpha} = l_{\infty}.$$

The space $X(f)$ is a Banach space with respect to the norm $\|x\| = \sum_{k=1}^{\infty} f(|x_k|) < \infty$.

Spaces of the type $X(f)$ are a special case of the spaces structured by B. Gramsch in (Gramsch, 1967). From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Symmetric sequence spaces, which are locally convex have been frequently studied by D. J. H Garling (Garling, 1966, 1968)and W. H. Ruckle (Ruckle, 1968, 1967, 1973).

After then E. Kolk (Kolk, 1993, 1994) gave an extension of $X(f)$ by considering a sequence of modulii $F = (f_k)$ and defined the sequence space

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.\text{(See (Kolk, 1993, 1994)).}$$

The following subspaces of $\omega$ were first introduced and discussed by Maddox (Maddox, 1986, 1970, 1969). $l(p) = \{x \in \omega : \sum_k |x_k|^{p_k} < \infty\}$, $l_{\infty}(p) = \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\}$, $c(p) = \{x \in \omega : \lim_k |x_k - l|^{p_k} = 0, \text{ for some } l \in \mathbb{C}\}$, $c_0(p) = \{x \in \omega : \lim_k |x_k|^{p_k} = 0, \}$, where $p = (p_k)$ is a sequence of strictly positive real numbers.

After then Lascarides (Lascarides, 1971, 1983) defined the following sequence spaces:
$l_{\infty}\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sup_k |x_k r|^{p_k} t_k < \infty\}$,
$c_0\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \lim_k |x_k r|^{p_k} t_k = 0, \}$,
$l\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sum_{k=1}^{\infty} |x_k r|^{p_k} t_k < \infty\}$,
Where $t_k = p_k^{-1}$, for all $k \in \mathbb{N}$.

We need the following lemmas in order to establish some results of this article.

**Lemma 1.1.** *Let $h = \inf_k p_k$ and $H = \sup_k p_k$. Then the following conditions are equivalent.(See[28]).*
*(a) $H < \infty$ and $h > 0$.*
*(b) $c_0(p) = c_0$ or $l_{\infty}(p) = l_{\infty}$.*
*(c) $l_{\infty}\{p\} = l_{\infty}(p)$.*
*(d) $c_0\{p\} = c_0(p)$.*
*(e) $l\{p\} = l(p)$.*

**Lemma 1.2.** *Let K∈ £(I) and M⊆N. If M∉I, then M∩K ∉ I.(See (Tripathy & Hazarika, 2009, 2011)). (c.f (Dems, 2005; Gurdal, 2004; Khan & Ebadullah, 2011, 2012; Kolk, 1993; Lascarides, 1971; Tripathy & Hazarika, 2011)).*

## 2. Main Results

Throughout the article $l_\infty, c^I, c_0^I, m^I$ and $m_0^I$ represent the bounded , $I$-convergent, $I$-null, bounded $I$-convergent and bounded $I$-null sequence spaces respectively.

**In this article we introduce the following classes of sequence spaces.**

$$c^I(F,p) = \{(x_k) \in \omega : f_k(|x_k - L|^{p_k}) \geq \epsilon \text{ for some L}\} \in I$$

$$c_0^I(F,p) = \{(x_k) \in \omega : f_k(|x_k|^{p_k}) \geq \epsilon\} \in I.$$

$$l_\infty^I(F,p) = \{(x_k) \in \omega : \sup_k f_k(|x_k|^{p_k}) < \infty\} \in I.$$

Also we denote by $m^I(F,p) = c^I(F,p) \cap l_\infty(F,p)$ and $m_0^I(F,p) = c_0^I(F,p) \cap l_\infty(F,p)$.

**Theorem 2.1.** *Let $(p_k) \in l_\infty$. Then $c^I(F,p), c_0^I(F,p), m^I(F,p)$ and $m_0^I(F,p)$ are linear spaces.*

*Proof.* Let $(x_k), (y_k) \in c^I(F,p)$ and $\alpha, \beta$ be two scalars. Then for a given $\epsilon > 0$ we have

$$\{k \in \mathbb{N} : f_k(|x_k - L_1|^{p_k}) \geq \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \mathrm{C}\} \in I$$

$$\{k \in \mathbb{N} : f_k(|y_k - L_2|^{p_k}) \geq \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \mathrm{C}\} \in I$$

where $M_1 = D.max\{1, \sup_k |\alpha|^{p_k}\}$, $M_2 = D.max\{1, \sup_k |\beta|^{p_k}\}$ and $D = max\{1, 2^{H-1}\}$ where $H = \sup_k p_k \geq 0$. Let $A_1 = \{k \in \mathbb{N} : f_k(|x_k - L_1|^{p_k}) < \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \mathrm{C}\} \in £(I)$, $A_2 = \{k \in \mathbb{N} : f_k(|y_k - L_2|^{p_k}) < \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \mathrm{C}\} \in £(I)$ be such that $A_1^c, A_2^c \in I$. Then

$$A_3 = \{k \in \mathbb{N} : f_k(|(\alpha x_k + \beta y_k) - f_k(\alpha L_1 + \beta L_2)|^{p_k}) < \epsilon\} \supseteq \{k \in \mathbb{N} : |\alpha|^{p_k} f_k(|x_k - L_1|^{p_k}) < \frac{\epsilon}{2M_1}|\alpha|^{p_k}.D\}$$

$$\cap \{k \in \mathbb{N} : |\beta|^{p_k} f_k(|y_k - L_2|^{p_k}) < \frac{\epsilon}{2M_2}|\beta|^{p_k}.D\}.$$

Thus $A_3^c = A_1^c \cap A_2^c \in I$. Hence $(\alpha x_k + \beta y_k) \in c^I(F,p)$. Therefore $c^I(F,p)$ is a linear space. The rest of the result follows similarly. □

**Theorem 2.2.** *Let $(p_k) \in l_\infty$. Then $m^I(F,p)$ and $m_0^I(F,p)$ are paranormed spaces, paranormed by $g(x_k) = \sup_k f_k(|x_k|^{\frac{p_k}{M}})$ where $M = max\{1, \sup_k p_k\}$.*

*Proof.* Let $x = (x_k), y = (y_k) \in m^I(F,p)$. (1) Clearly, $g(x) = 0$ if and only if $x = 0$. (2) $g(x) = g(-x)$ is obvious. (3) Since $\frac{p_k}{M} \leq 1$ and $M > 1$, using Minkowski's inequality and the definition of $f$ we have $\sup_k f_k(|x_k + y_k|^{\frac{p_k}{M}}) \leq \sup_k f_k(|x_k|^{\frac{p_k}{M}}) + \sup_k f(|y_k|^{\frac{p_k}{M}})$ (4) Now for any complex $\lambda$ we have $(\lambda_k)$ such that $\lambda_k \to \lambda$, $(k \to \infty)$. Let $x_k \in m^I(f,p)$ such that $f_k(|x_k - L|^{p_k}) \geq \epsilon$. Therefore, $g(x_k - L) = \sup_k f_k(|x_k - L|^{\frac{p_k}{M}}) \leq \sup_k f_k(|x_k|^{\frac{p_k}{M}}) + \sup_k f_k(|L|^{\frac{p_k}{M}})$. Hence $g(\lambda_n x_k - \lambda L) \leq g(\lambda_n x_k) + g(\lambda L) = \lambda_n g(x_k) + \lambda g(L)$ as $(k \to \infty)$. Hence $m^I(F,p)$ is a paranormed space. The rest of the result follows similarly. □

**Theorem 2.3.** *A sequence $x = (x_k) \in m^I(F, p)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(F, p)$.*

*Proof.* Suppose that $L = I - \lim x$. Then $B_\epsilon = \{k \in \mathbb{N} : |x_k - L|^{p_k} < \frac{\epsilon}{2}\} \in m^I(F, p)$. For all $\epsilon > 0$. Fix an $N_\epsilon \in B_\epsilon$. Then we have $|x_{N_\epsilon} - x_k|^{p_k} \leq |x_{N_\epsilon} - L|^{p_k} + |L - x_k|^{p_k} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ which holds for all $k \in B_\epsilon$. Hence $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(F, p)$.

Conversely, suppose that $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(F, p)$. That is $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(F, p)$ for all $\epsilon > 0$. Then the set $C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m^I(F, p)$ for all $\epsilon > 0$. Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I(F, p)$ as well as $C_{\frac{\epsilon}{2}} \in m^I(f, p)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I(F, p)$. This implies that $J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$ that is $\{k \in \mathbb{N} : x_k \in J\} \in m^I(F, p)$ that is $diamJ \leq diamJ_\epsilon$ where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals $J_\epsilon = I_0 \supseteq I_1 \supseteq ... \supseteq I_k \supseteq ...$ with the property that $diamI_k \leq \frac{1}{2}diamI_{k-1}$ for $(k = 2, 3, 4, ...)$ and $\{k \in \mathbb{N} : x_k \in I_k\} \in m^I(F, p)$ for $(k = 1, 2, 3, ...)$. Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f_k(\xi) = I - \lim f_k(x)$, that is $L = I - \lim f_k(x)$. □

**Theorem 2.4.** *Let $H = \sup_k p_k < \infty$ and I an admissible ideal. Then the following are equivalent.*
*(a) $(x_k) \in c^I(F, p)$;*
*(b) there exists$(y_k) \in c(F, p)$ such that $x_k = y_k$, for a.a.k.r.I; (c) there exists$(y_k) \in c(F, p)$ and $(x_k) \in c_0^I(F, p)$ such that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : f_k(|y_k - L|^{p_k}) \geq \epsilon\} \in I$ ; (d) there exists a subset $K = \{k_1 < k_2 ....\}$ of $\mathbb{N}$ such that $K \in £(I)$ and $\lim_{n\to\infty} f_k(|x_{k_n} - L|^{p_{k_n}}) = 0$.*

*Proof.* (a) implies (b). Let $(x_k) \in c^I(F, p)$. Then there exists $L \in \mathbb{C}$ such that $\{k \in \mathbb{N} : f_k(|x_k - L|^{p_k}) \geq \epsilon\} \in I$. Let $(m_t)$ be an increasing sequence with $m_t \in \mathbb{N}$ such that $\{k \leq m_t : f_k(|x_k - L|^{p_k}) \geq \epsilon\} \in I$. Define a sequence $(y_k)$ as $y_k = x_k$, for all $k \leq m_1$. For $m_t < k \leq m_{t+1}, t \in \mathbb{N}$.
$y_k = \begin{cases} x_k, & \text{if } |x_k - L|^{p_k} < t^{-1}, \\ L, otherwise. \end{cases}$ Then $(y_k) \in c(F, p)$ and form the following inclusion $\{k \leq m_t : x_k \neq y_k\} \subseteq \{k \leq m_t : f_k(|x_k - L|^{p_k}) \geq \epsilon\} \in I$. We get $x_k = y_k$, for a.a.k.r.I.

(b) implies (c).For $(x_k) \in c^I(F, p)$. Then there exists $(y_k) \in c(F, p)$ such that $x_k = y_k$, for a.a.k.r.I. Let $K = \{k \in \mathbb{N} : x_k \neq y_k\}$, then $k \in I$. Define a sequence $(z_k)$ as $z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, otherwise. \end{cases}$ Then $z_k \in c_0^I(F, p)$ and $y_k \in c(F, p)$.

(c) implies (d).Let $P_1 = \{k \in \mathbb{N} : f_k(|x_k|^{p_k}) \geq \epsilon\} \in I$ and $K = P_1^c = \{k_1 < k_2 < k_3 < ...\} \in £(I)$. Then we have $\lim_{n\to\infty} f_k(|x_{k_n} - L|^{p_{k_n}}) = 0$.

(d) implies (a). Let $K = \{k_1 < k_2 < k_3 < ...\} \in £(I)$ and $\lim_{n\to\infty} f_k(|x_{k_n} - L|^{p_{k_n}}) = 0$. Then for any $\epsilon > 0$, and Lemma 1.9, we have $\{k \in \mathbb{N} : f_k(|x_k - L|^{p_k}) \geq \epsilon\} \subseteq K^c \cup \{k \in K : f_k(|x_k - L|^{p_k}) \geq \epsilon\}$. Thus $(x_k) \in c^I(F, p)$. □

**Theorem 2.5.** *Let $(p_k)$ and $(q_k)$ be two sequences of positive real numbers. Then $m_0^I(F, p) \supseteq m_0^I(F, q)$ if and only if $\lim_{k\in K} \inf \frac{p_k}{q_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.*

*Proof.* Let $\liminf_{k\in K} \frac{p_k}{q_k} > 0$. and $(x_k) \in m_0^I(F,q)$. Then there exists $\beta > 0$ such that $p_k > \beta q_k$, for all sufficiently large $k \in K$. Since $(x_k) \in m_0^I(F,q)$, for a given $\epsilon > 0$, we have $B_0 = \{k \in \mathbb{N} : f_k(|x_k|^{q_k}) \geq \epsilon\} \in I$ Let $G_0 = K^c \cup B_0$. Then $G_0 \in I$. Then for all sufficiently large $k \in G_0$, $\{k \in \mathbb{N} : f_k(|x_k|^{p_k}) \geq \epsilon\} \subseteq \{k \in \mathbb{N} : f_k(|x_k|^{\beta q_k}) \geq \epsilon\} \in I$. Therefore $(x_k) \in m_0^I(F,p)$. □

**Theorem 2.6.** *Let $(p_k)$ and $(q_k)$ be two sequences of positive real numbers. Then $m_0^I(F,q) \supseteq m_0^I(F,p)$ if and only if $\liminf_{k\in K} \frac{q_k}{p_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.*

*Proof.* The proof follows similarly as the proof of Theorem 2.5. □

**Theorem 2.7.** *Let $(p_k)$ and $(q_k)$ be two sequences of positive real numbers. Then $m_0^I(F,q) = m_0^I(F,p)$ if and only if $\liminf_{k\in K} \frac{p_k}{q_k} > 0$, and $\liminf_{k\in K} \frac{q_k}{p_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K^c \in I$.*

*Proof.* On combining Theorem 2.5 and 2.6 we get the required result. □

**Theorem 2.8.** *Let $h = \inf_k p_k$ and $H = \sup_k p_k$. Then the following results are equivalent.*
*(a) $H < \infty$ and $h > 0$. (b) $c_0^I(F,p) = c_0^I$.*

*Proof.* Suppose that $H < \infty$ and $h > 0$,then the inequalities $min\{1, s^h\} \leq s^{p_k} \leq max\{1, s^H\}$ hold for any $s > 0$ and for all $k \in \mathbb{N}$. Therefore the equivalent of (a) and (b) is obvious. □

**Theorem 2.9.** *Let $F = (f_k)$ be a sequence of modulii. Then $c_0^I(F,p) \subset c^I(F,p) \subset l_\infty^I(F,p)$ and the inclusions are proper.*

*Proof.* Let $(x_k) \in c^I(F,p)$. Then there exists $L \in \mathbb{C}$ such that $I - \lim f_k(|x_k - L|^{p_k}) = 0$. We have $f_k(|x_k|^{p_k}) \leq \frac{1}{2} f_k(|x_k - L|^{p_k}) + \frac{1}{2} f_k(|L|^{p_k})$. Taking supremum over $k$ both sides we get $(x_k) \in l_\infty^I(F,p)$. The inclusion $c_0^I(F,p) \subset c^I(F,p)$ is obvious. Hence $c_0^I(F,p) \subset c^I(F,p) \subset l_\infty^I(F,p)$. □

**Theorem 2.10.** *If $H = \sup_k p_k < \infty$, then for a sequence of moduli $F$, we have $l_\infty^I \subset M(m^I(F,p))$, where the inclusion may be proper.*

*Proof.* Let $a \in l_\infty^I$. This implies that $\sup_k |a_k| < 1 + K$. for some $K > 0$ and all $k$. Therefore $x \in m^I(F,p)$ implies $\sup_k f_k(|a_k x_k|^{p_k}) \leq (1+K)^H \sup_k f_k(|x_k|^{p_k}) < \infty$. which gives $l_\infty^I \subset M(m^I(F,p))$. To show that the inclusion may be proper, consider the case when $p_k = \frac{1}{k}$ for all $k$. Take $a_k = k$ for all $k$. Therefore $x \in m^I(F,p)$ implies $\sup_k f_k(|a_k x_k|^{p_k}) \leq \sup_k f_k(|k|^{\frac{1}{k}}) \sup_k f_k(|x_k|^{p_k}) < \infty$. Thus in this case $a = (a_k) \in M(m^I(F,p))$ while $a \notin l_\infty^I$. □

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# Some Inequalities Involving Fuzzy Complex Numbers

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## Abstract

In this paper we wish to establish a few inequalities related to fuzzy complex numbers which extend some standard results.

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## 1. Introduction, Definitions and Notations

The idea of fuzzy subset $\mu$ of a set $X$ was primarily introduced by L.A. Zadeh (Zadeh, 1965) as a function $\mu : X \to [0, 1]$ . Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. Among the various types of fuzzy sets, those which are defined on the universal set of complex numbers are of particular importance. They may, under certain conditions, be viewed as fuzzy complex numbers.

A fuzzy set $z_f$ is defined by its membership function $\mu\left(z \mid z_f\right)$ which is a mapping from the complex numbers $\mathbb{C}$ into $[0, 1]$ where $z$ is a regular complex number as $z = x + iy$, is called a fuzzy complex number if it satisfies the following conditions :

1. $\mu\left(z \mid z_f\right)$ is continuous;

2. An $\alpha$-cut of $z_f$ which is defined as $z_f^\alpha = \left\{z \mid \mu\left(z \mid z_f\right) > \alpha\right\}$, where $0 \le \alpha < 1$, is open, bounded, connected and simply connected; and

3. $z_f^1 = \left\{z \mid \mu\left(z \mid z_f\right) = 1\right\}$ is non-empty, compact, arcwise connected and simply connected.

(For detail on the set $z_f$ as mentioned above, one may see (Buckley, 1989)).

Using this concept of fuzzy complex numbers, J. J. Buckley (Buckley, 1989) shown that fuzzy complex numbers is closed under the basic arithmetic operations. In paper (Buckley, 1989) we

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also see the development of fuzzy complex numbers by defining addition and multiplication from the extension principle which has been shown in terms of $\alpha$-cuts.

We now review some definitions used in this paper.

**Definition 1.1.** (Buckley, 1989)The complex conjugate $\bar{z}_f$ of $z_f$ is defined as

$$\mu\left(z \mid \bar{z}_f\right) = \mu\left(\bar{z} \mid z_f\right),$$

where $\bar{z} = x - iy$ is the complex conjugate of $z = x + iy$. The complex conjugate $\bar{z}_f$ of a fuzzy complex number $z_f$ is also a fuzzy complex number because the mapping $z = x + iy \to \bar{z} = x - iy$ is continuous.

**Definition 1.2.** (Buckley, 1989)The modulus $\left|z_f\right|$ of a fuzzy complex number $z_f$ is defined as

$$\mu\left(r \mid \left|z_f\right|\right) = \sup\left\{\mu\left(z \mid z_f\right) \mid |z| = r\right\},$$

where $r$ is the modulus of $z$.

Similarly we may define the modulus of a real fuzzy number $R_f$ as follows:

$$\mu\left(|a| \mid \left|R_f\right|\right) = \sup\{\mu\left(a \mid R_f\right) \mid |a| = a \text{ if } a > 0, |a| = 0 \text{ if } a = 0 \text{ and } |a| = -a \text{ if } a < 0\}.$$

Now in the following, we define two special types of fuzzy complex numbers $z_f^n$ and $nz_f$ of the fuzzy complex number $z_f$, for any complex number $z \in z_f$ and $n \in R$ .

**Definition 1.3.** Fuzzy complex numbers $z_f^n$ and $nz_f$ are defined as

$$\mu(z \mid z_f^n) = \mu(z^n \mid z_f)$$

and

$$\mu(z \mid nz_f) = \mu(n.z \mid z_f).$$

In particular when $n = 2$, we have

$$\mu(z \mid z_f^2) = \mu(z^2 \mid z_f) \quad \text{and} \quad \mu(z \mid 2z_f) = \mu(2.z \mid z_f)\ .$$

It can be easily verified that

$$z_f^2 \neq z_f.z_f \qquad \text{and} \qquad 2z_f \neq z_f + z_f \text{ but } 2(z_{f_1} + z_{f_2}) = 2z_{f_1} + 2z_{f_2}.$$

From the definition of fuzzy complex number one may easily verify that $z_f^n$ and $nz_f$ are also fuzzy complex numbers when $z_f$ is a fuzzy complex number. It should be noted that the significance of Definition 1.3 is completely different from the definitions of additions and multiplications of fuzzy complex numbers as mentioned in (Buckley, 1989).

In this paper we wish to establish a few standard inequalities related to fuzzy complex numbers.

## 2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

**Lemma 2.1.** *(Buckley, 1989) Let $z_{f_1}$ and $z_{f_2}$ be any two fuzzy complex numbers. Suppose $A = z_{f_1} + z_{f_2}$ and $M = z_{f_1}.z_{f_2}$ respectively. Then for $0 \leq \alpha \leq 1$, $A^\alpha = S^\alpha$ and $M^\alpha = P^\alpha$ holds where*

$$S^\alpha = \left\{ z_{f_1} + z_{f_2} \mid (z_1, z_2) \in z_{f_1}^\alpha \times z_{f_2}^\alpha \right\}$$

*and*

$$P^\alpha = \left\{ z_{f_1}.z_{f_2} \mid (z_1, z_2) \in z_{f_1}^\alpha \times z_{f_2}^\alpha \right\}.$$

*Also $z_{f_1} + z_{f_2}$ and $z_{f_1}.z_{f_2}$ are fuzzy complex numbers.*

The following lemma may be deduced in the line of Lemma 2.1 and so its proof is omitted.

**Lemma 2.2.** *Let $z_{f_1}, z_{f_2}, z_{f_3}, ..., z_{f_n}$ be any n number of fuzzy complex numbers. Also let $A = z_{f_1} + z_{f_2} + z_{f_3} + ... + z_{f_n}$ and $M = z_{f_1}.z_{f_2}.z_{f_3}...z_{f_n}$ respectively. Then for $0 \leq \alpha \leq 1$, $A^\alpha = S^\alpha$ and $M^\alpha = P^\alpha$ holds where*

$$S^\alpha = \left\{ z_{f_1} + z_{f_2} + z_{f_3} + ... + z_{f_n} \mid (z_1, z_2, z_3, ..., z_n) \in z_{f_1}^\alpha \times z_{f_2}^\alpha \times z_{f_3}^\alpha \times ... \times z_{f_n}^\alpha \right\}$$

*and*

$$P^\alpha = \left\{ z_{f_1}.z_{f_2}.z_{f_3}...z_{f_n} \mid (z_1, z_2, z_3, ..., z_n) \in z_{f_1}^\alpha \times z_{f_2}^\alpha \times z_{f_3}^\alpha \times ... \times z_{f_n}^\alpha \right\}.$$

**Lemma 2.3.** *(Buckley, 1989) If $z_f$ is any fuzzy complex number then*

$$\left| z_f \right|^\alpha = \left| z_f^\alpha \right|$$

*where $0 \leq \alpha \leq 1$ and $\left| z_f \right|$ is a truncated real fuzzy number.*

**Lemma 2.4.** *(Kaufmann & Gupta, 1985) If M and N be any two real fuzzy numbers then*

$$(M + N)^\alpha = M^\alpha + N^\alpha$$

*and if $M \geq 0, N \geq 0$ then*

$$(M \cdot N)^\alpha = M^\alpha \cdot N^\alpha.$$

**Lemma 2.5.** *(Buckley, 1989) Let $\bar{z}_f$ be a fuzzy complex conjugate number of a fuzzy complex number $z_f$. Then*

$$\bar{z}_f^\alpha = \overline{z_f^\alpha}$$

*where $0 \leq \alpha \leq 1$.*

## 3. Theorems

In this section we present the main results of the paper.

**Theorem 3.1.** *Let $z_{f_1}$ and $z_{f_2}$ be any two fuzzy complex numbers. Then*

$$\left|z_{f_1} - z_{f_2}\right| \geq \left|z_{f_1}\right| - \left|z_{f_2}\right| .$$

*Proof.* The meaning of the inequality is that the interval $\left|z_{f_1} - z_{f_2}\right|^{\alpha}$ is greater that or equal to the interval $\left(\left|z_{f_1}\right| - \left|z_{f_2}\right|\right)^{\alpha}$ for $0 \leq \alpha \leq 1$.

Now from Lemma 2.1 and Lemma 2.3, we get that

$$\left|z_{f_1} - z_{f_2}\right|^{\alpha} = \left|\left(z_{f_1} - z_{f_2}\right)^{\alpha}\right| = \left|z_{f_1}^{\alpha} - z_{f_2}^{\alpha}\right| = \left\{|z_1 - z_2| \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}. \tag{3.1}$$

Again in view of Lemma 2.4, we obtain from Lemma 2.3 that

$$\left(\left|z_{f_1}\right| - \left|z_{f_2}\right|\right)^{\alpha} = \left|z_{f_1}\right|^{\alpha} - \left|z_{f_2}\right|^{\alpha} = \left|z_{f_1}^{\alpha}\right| - \left|z_{f_2}^{\alpha}\right| = \left\{|z_1| - |z_2| \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}. \tag{3.2}$$

Hence the result follows from (3.1) and (3.2) and in view of

$$|z_1 - z_2| \geq |z_1| - |z_2| .$$

This proves the theorem. □

J. J. Buckley (Buckley, 1989) proved the following results:

**Theorem A** (Buckley, 1989) *Let $z_{f_1}$ and $z_{f_2}$ be any two fuzzy complex numbers. Then*

$$(1).\quad \left|z_{f_1} - z_{f_2}\right| \leq \left|z_{f_1}\right| + \left|z_{f_2}\right| \quad \textit{and} \quad (2).\quad \left|z_{f_1}.z_{f_2}\right| = \left|z_{f_1}\right|\left|z_{f_2}\right| .$$

But he (Buckley, 1989) remained silent about the question when the equality holds in the inequality (1) of Theorem A. In the next two theorems, we wish to generalise the results of Theorem A and find out the condition for which $\left|z_{f_1} - z_{f_2}\right| = \left|z_{f_1}\right| + \left|z_{f_2}\right|$ holds respectively.

**Theorem 3.2.** *Let $z_{f_1}, z_{f_2}, z_{f_3}, ..., z_{f_n}$ be any n number of fuzzy complex numbers. Then*

$$(i).\quad \left|z_{f_1} + z_{f_2} + z_{f_3} + ... + z_{f_n}\right| \leq \left|z_{f_1}\right| + \left|z_{f_2}\right| + \left|z_{f_3}\right| + ... + \left|z_{f_n}\right| \ \textit{and}$$

$$(ii).\quad \left|z_{f_1}.z_{f_2}.z_{f_3}...z_{f_n}\right| = \left|z_{f_1}\right|\left|z_{f_2}\right|\left|z_{f_3}\right|...\left|z_{f_n}\right| .$$

*Proof.* In view of Lemma 2.1, it follows from Theorem A that

$$\begin{aligned}
\left|z_{f_1} + z_{f_2} + z_{f_3} + ... + z_{f_n}\right| &\leq \left|z_{f_1}\right| + \left|z_{f_2} + z_{f_3} + ... + z_{f_n}\right| \\
&\leq \left|z_{f_1}\right| + \left|z_{f_2}\right| + \left|z_{f_3} + ... + z_{f_n}\right| \\
&\leq \left|z_{f_1}\right| + \left|z_{f_2}\right| + \left|z_{f_3}\right| + \left|z_{f_4} + ... + z_{f_n}\right| \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\leq \left|z_{f_1}\right| + \left|z_{f_2}\right| + \left|z_{f_3}\right| + ... + \left|z_{f_n}\right| .
\end{aligned}$$

This proves the first part of the theorem.

Similarly with the help of Lemma 2.1 and the equality $\left|z_{f_1}.z_{f_2}\right| = \left|z_{f_1}\right|\left|z_{f_2}\right|$, one can easily establish the second part of the theorem. □

*Remark.* In view of Lemma 2.2, Lemma 2.3 and Lemma 2.4 it can also be shown that the intervals $\left|z_{f_1} + z_{f_2} + z_{f_3} + ... + z_{f_n}\right|^{\alpha}$ and $\left|z_{f_1}.z_{f_2}.z_{f_3}...z_{f_n}\right|^{\alpha}$ are less than or equal to the intervals $\left(\left|z_{f_1}\right| + \left|z_{f_2}\right| + \left|z_{f_3}\right| + ... + \left|z_{f_n}\right|\right)^{\alpha}$ and $\left(\left|z_{f_1}\right|\left|z_{f_2}\right|\left|z_{f_3}\right|...\left|z_{f_n}\right|\right)^{\alpha}$ respectively in Theorem 3.2 for $0 \leq \alpha \leq 1$.

**Theorem 3.3.** *Let $z_{f_1}$ and $z_{f_2}$ be any two fuzzy complex numbers such that $\left|z_{f_1} + z_{f_2}\right| = \left|z_{f_1}\right| + \left|z_{f_2}\right|$ then either $\arg z_1 - \arg z_2$ is an even multiple of $\pi$ or $\frac{z_1}{z_2}$ is a positive real number where $z_1$ and $z_2$ are any two members of $z_{f_1}$ and $z_{f_2}$ respectively.*

*Proof.* The meaning of the equality is that the interval $\left|z_{f_1} + z_{f_2}\right|^{\alpha}$ is equal to the interval $\left(\left|z_{f_1}\right| + \left|z_{f_2}\right|\right)^{\alpha}$ for $0 \leq \alpha \leq 1$.
Thus $\left|z_{f_1} + z_{f_2}\right| = \left|z_{f_1}\right| + \left|z_{f_2}\right|$ i.e, $\left|z_{f_1} + z_{f_2}\right|^{\alpha} = \left(\left|z_{f_1}\right| + \left|z_{f_2}\right|\right)^{\alpha}$ i.e., $\left|z_{f_1} + z_{f_2}\right|^{\alpha} = \left(\left|z_{f_1}\right|^{\alpha} + \left|z_{f_2}\right|^{\alpha}\right)$ i.e, $\left|z_{f_1}^{\alpha} + z_{f_2}^{\alpha}\right| = \left|z_{f_1}^{\alpha}\right| + \left|z_{f_2}^{\alpha}\right|$ i.e., $|z_1 + z_2| = |z_1| + |z_2| \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2$; which is only possible when either $\arg z_1 - \arg z_2$ is an even multiple of $\pi$ or $\frac{z_1}{z_2}$ is a positive real number. Hence the theorem follows. □

**Theorem 3.4.** *If $z_{f_1}$ and $z_{f_2}$ are any two fuzzy complex numbers with $\left|z_{f_1} + z_{f_2}\right| = \left|z_{f_1} - z_{f_2}\right|$, then $\arg z_1$ and $\arg z_2$ differ by $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ where $z_1$ and $z_2$ are any two members of $z_{f_1}$ and $z_{f_2}$ respectively.*

*Proof.* The meaning of the equality is that the $\alpha$-cuts of $\left|z_{f_1} + z_{f_2}\right|$ is equal to the corresponding $\alpha$-cuts of $\left|z_{f_1} - z_{f_2}\right|$ for $0 \leq \alpha \leq 1$.
Now in view of Lemma 2.1 and Lemma 2.3, we obtain that

$$\left|z_{f_1} + z_{f_2}\right|^{\alpha} = \left|\left(z_{f_1} + z_{f_2}\right)^{\alpha}\right| = \left|z_{f_1}^{\alpha} + z_{f_2}^{\alpha}\right| = \left\{|z_1 + z_2| \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}. \tag{3.3}$$

Similarly,

$$\left|z_{f_1} - z_{f_2}\right|^{\alpha} = \left|\left(z_{f_1} - z_{f_2}\right)^{\alpha}\right| = \left|z_{f_1}^{\alpha} - z_{f_2}^{\alpha}\right| = \left\{|z_1 - z_2| \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}. \tag{3.4}$$

Therefore from (3.3) and (3.4) it follows that $\left|z_{f_1} + z_{f_2}\right| = \left|z_{f_1} - z_{f_2}\right|$ which implies that $|z_1 + z_2| = |z_1 - z_2| \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2$ which is only possible when $\arg z_1$ and $\arg z_2$ differ by $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.
Thus the theorem is established. □

**Theorem 3.5.** *Let $z_{f_1}$ and $z_{f_2}$ be any two fuzzy complex numbers. Then*

$$\left|z_{f_1} \pm z_{f_2}\right| \geq \left|\left|z_{f_1}\right| - \left|z_{f_2}\right|\right| .$$

*Proof.* For $0 \leq \alpha \leq 1$, we have

$$\left|z_{f_1} \pm z_{f_2}\right|^{\alpha} = \left|\left(z_{f_1} \pm z_{f_2}\right)^{\alpha}\right| = \left|z_{f_1}^{\alpha} \pm z_{f_2}^{\alpha}\right| = \left\{|z_1 \pm z_2| \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}. \tag{3.5}$$

We also deduce that

$$\left|\left|z_{f_1}\right| - \left|z_{f_2}\right|\right|^{\alpha} = \left|\left(\left|z_{f_1}\right| - \left|z_{f_2}\right|\right)^{\alpha}\right| = \left|\left|z_{f_1}\right|^{\alpha} - \left|z_{f_2}\right|^{\alpha}\right| = \left|\left|z_{f_1}^{\alpha}\right| - \left|z_{f_2}^{\alpha}\right|\right| = \left\{||z_1| - |z_2|| \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}. \tag{3.6}$$

Hence the theorem follows from (3.5) and (3.6) and in view of the following inequality :

$$|z_1 \pm z_2| \geq ||z_1| - |z_2|| .$$

□

**Theorem 3.6.** *If $z_{f_1}$ and $z_{f_2}$ are any two fuzzy complex numbers, then*

$$2\left|z_{f_1}+z_{f_2}\right| \geq \left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)\left|\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right| .$$

*Proof.* In order to prove this theorem, we wish to show that the interval $\left(2\left|z_{f_1}+z_{f_2}\right|\right)^{\alpha}$ is greater than or equal to the interval $\left\{\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)\left|\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right|\right\}^{\alpha}$ for $0 \leq \alpha \leq 1$.

From Lemma 2.3 and Lemma 2.4, we get that

$$\left\{\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)\left|\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right|\right\}^{\alpha} = \left\{\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)^{\alpha}\left|\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right|^{\alpha}\right\} = \left\{\left(\left|z_{f_1}\right|^{\alpha}+\left|z_{f_2}\right|^{\alpha}\right)\left|\left(\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right)^{\alpha}\right|\right\}$$

$$= \left\{\left(\left|z_{f_1}^{\alpha}\right|+\left|z_{f_2}^{\alpha}\right|\right)\left|\left(\frac{z_{f_1}}{\left|z_{f_1}\right|}\right)^{\alpha}+\left(\frac{z_{f_2}}{\left|z_{f_2}\right|}\right)^{\alpha}\right|\right\} = \left\{\left(\left|z_{f_1}^{\alpha}\right|+\left|z_{f_2}^{\alpha}\right|\right)\left|\left(z_{f_1}\left|z_{f_1}\right|^{-1}\right)^{\alpha}+\left(z_{f_2}\left|z_{f_2}\right|^{-1}\right)^{\alpha}\right|\right\}$$

$$= \left\{\left(\left|z_{f_1}^{\alpha}\right|+\left|z_{f_2}^{\alpha}\right|\right)\left|\left(z_{f_1}^{\alpha}\left|z_{f_1}^{\alpha}\right|^{-1}\right)+\left(z_{f_2}^{\alpha}\left|z_{f_2}^{\alpha}\right|^{-1}\right)\right|\right\} = \left\{\left(|z_1|+|z_2|\right)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right| \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\}. \quad (3.7)$$

Since

$$2\left|z_1+z_2\right| \geq \left(|z_1|+|z_2|\right)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|,$$

in view of Definition 1.2 and Definition 1.3, the theorem follows from (3.3) and (3.7). □

**Theorem 3.7.** *Let $z_{f_1}$ and $z_{f_2}$ be any two fuzzy complex numbers. Then*

$$\left|\left(z_{f_1}+z_{f_2}\right)^2\right|+\left|\left(z_{f_1}-z_{f_2}\right)^2\right| = \left(2\left|z_{f_1}^2\right|-2\left|z_{f_2}^2\right|\right).$$

*Proof.* In view of Lemma 2.1, Lemma 2.3 and Lemma 2.4, we get for $0 \leq \alpha \leq 1$ that

$$\left(\left|\left(z_{f_1}+z_{f_2}\right)^2\right|+\left|\left(z_{f_1}-z_{f_2}\right)^2\right|\right)^{\alpha} = \left(\left|\left(z_{f_1}+z_{f_2}\right)^2\right|^{\alpha}+\left|\left(z_{f_1}-z_{f_2}\right)^2\right|^{\alpha}\right) = \left|\left(\left(z_{f_1}+z_{f_2}\right)^2\right)^{\alpha}\right|+\left|\left(\left(z_{f_1}-z_{f_2}\right)^2\right)^{\alpha}\right|$$

$$= \left(\left|\left(z_{f_1}^{\alpha}+z_{f_2}^{\alpha}\right)^2\right|+\left|\left(z_{f_1}^{\alpha}-z_{f_2}^{\alpha}\right)^2\right|\right) = \left\{\left|(z_1+z_2)^2\right|+\left|(z_1-z_2)^2\right| \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\}$$

$$= \left\{|z_1+z_2|^2+|z_1-z_2|^2 \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\} . \quad (3.8)$$

Analogously we also see that

$$\left(2\left|z_{f_1}^2\right|-2\left|z_{f_2}^2\right|\right)^{\alpha} = \left(2\left|z_{f_1}^2\right|^{\alpha}-2\left|z_{f_2}^2\right|^{\alpha}\right) = \left(2\left|\left(z_{f_1}^2\right)^{\alpha}\right|-2\left|\left(z_{f_2}^2\right)^{\alpha}\right|\right) = \left(2\left|\left(z_{f_1}^{\alpha}\right)^2\right|-2\left|\left(z_{f_2}^{\alpha}\right)^2\right|\right)$$

$$= \left\{2\left|z_1^2\right|-2\left|z_2^2\right| \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\} = \left\{2|z_1|^2-2|z_2|^2 \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\} . \quad (3.9)$$

Now in the line of Definition 1.3, it follows from (3.8)and (3.9) that the corresponding $\alpha$-cuts are equal. Hence the theorem follows as we obtain the equality of the two real fuzzy numbers. □

In the next theorem we establish a few properties of fuzzy complex conjugate numbers depending on the concept of it.

**Theorem 3.8.** *Let $\bar{z}_f$ be a fuzzy complex conjugate number of a fuzzy complex number $z_f$. Then*
(1) . $\bar{\bar{z}}_f = z_f$, (2) . $\overline{(z_{f_1} \pm z_{f_2})} = \bar{z}_{f_1} \pm \bar{z}_{f_2}$, (3). $\overline{(z_{f_1}.z_{f_2})} = \bar{z}_{f_1} \cdot \bar{z}_{f_2}$,
(4). $\overline{\left(\frac{z_{f_1}}{z_{f_2}}\right)} = \frac{\bar{z}_{f_1}}{\bar{z}_{f_2}}$ *and* (5) . $|z_f| = |\bar{z}_f|$ .

*Proof.* In view of Lemma 2.5 and for $0 \le \alpha \le 1$, we obtain that

$$\left(\bar{\bar{z}}_f\right)^\alpha = \overline{\left(\bar{z}_f\right)^\alpha} = \overline{\overline{z_f^\alpha}} = \left\{\bar{\bar{z}} \mid \text{ for all } z \in z_f^\alpha\right\}.$$

Again

$$z_f^\alpha = \left\{z \mid \mu\left(z \mid z_f\right) > \alpha\right\} = \left\{z \mid \text{ for all } z \in z_f^\alpha\right\}.$$

Since $\bar{\bar{z}} = z$, the first part of the theorem follows from above.

For the second part of the theorem, we have to prove that the $\alpha$-cuts of $\overline{\left(z_{f_1} \pm z_{f_2}\right)}$ are equal to the corresponding $\alpha$-cuts of $\bar{z}_{f_1} \pm \bar{z}_{f_2}$.
Now it follows from Lemma 2.1 and Lemma 2.5 that

$$\left(\overline{\left(z_{f_1} \pm z_{f_2}\right)}\right)^\alpha = \overline{\left(z_{f_1} \pm z_{f_2}\right)^\alpha} = \overline{\left(z_{f_1}^\alpha \pm z_{f_2}^\alpha\right)} = \left\{\overline{z_1 \pm z_2} \mid z_i \in z_{f_i}^\alpha, i = 1, 2\right\}$$

and

$$\left(\bar{z}_{f_1} \pm \bar{z}_{f_2}\right)^\alpha = \left(\bar{z}_{f_1}\right)^\alpha \pm \left(\bar{z}_{f_2}\right)^\alpha = \left(\overline{z_{f_1}^\alpha}\right) \pm \left(\overline{z_{f_2}^\alpha}\right) = \left\{\bar{z}_1 \pm \bar{z}_2 \mid z_i \in z_{f_i}^\alpha, i = 1, 2\right\}.$$

Thus the second part of the theorem is established in view of $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.
We also observe that

$$\overline{\left(z_{f_1}.z_{f_2}\right)}^\alpha = \overline{\left(z_{f_1}.z_{f_2}\right)^\alpha} = \overline{\left(z_{f_1}^\alpha.z_{f_2}^\alpha\right)} = \left\{\overline{z_1.z_2} \mid z_i \in z_{f_i}^\alpha, i = 1, 2\right\}. \tag{3.10}$$

We may also see that

$$\left(\bar{z}_{f_1}.\bar{z}_{f_2}\right)^\alpha = \left(\bar{z}_{f_1}\right)^\alpha.\left(\bar{z}_{f_2}\right)^\alpha = \overline{z_{f_1}^\alpha}.\overline{z_{f_2}^\alpha} = \left\{\bar{z}_1.\bar{z}_2 \mid z_i \in z_{f_i}^\alpha, i = 1, 2\right\}. \tag{3.11}$$

Now from (3.10) and (3.11), we obtain that the corresponding $\alpha$-cuts are equal. This proves the third part of the theorem.

For the fourth part of the theorem, we deduce that

$$\left(\overline{\left(\frac{z_{f_1}}{z_{f_2}}\right)}\right)^\alpha = \overline{\left(\frac{z_{f_1}}{z_{f_2}}\right)^\alpha} = \overline{\left(z_{f_1}.z_{f_2}^{-1}\right)^\alpha} = \overline{z_{f_1}^\alpha.\left(z_{f_2}^{-1}\right)^\alpha} = \overline{\left(z_{f_1}^\alpha.\left(z_{f_2}^\alpha\right)^{-1}\right)} = \left\{\overline{\frac{z_1}{z_2}} \mid z_i \in z_{f_i}^\alpha, i = 1, 2\right\}$$

and

$$\left(\frac{\bar{z}_{f_1}}{\bar{z}_{f_2}}\right)^\alpha = \left(\bar{z}_{f_1}.\bar{z}_{f_2}^{-1}\right)^\alpha = \left(\bar{z}_{f_1}\right)^\alpha.\left(\bar{z}_{f_2}^{-1}\right)^\alpha = \overline{z_{f_1}^\alpha}.\left(\left(\overline{z_{f_2}}\right)^\alpha\right)^{-1} = \overline{z_{f_1}^\alpha}\cdot\left(\left(\overline{z_{f_2}^\alpha}\right)\right)^{-1} = \left\{\frac{\bar{z}_1}{\bar{z}_2} \mid z_i \in z_{f_i}^\alpha, i = 1, 2\right\}.$$

Hence the $\alpha$-cuts of $\overline{\left(\frac{z_{f_1}}{z_{f_2}}\right)}$ are equal to the corresponding $\alpha$-cuts of $\frac{\bar{z}_{f_1}}{\bar{z}_{f_2}}$ which implies that the two fuzzy complex numbers are equal. Thus the fourth part of the theorem follows.
Again we have from Lemma 2.3 and Lemma 2.5 that

$$\left|z_f\right|^{\alpha} = \left|z_f^{\alpha}\right| = \left\{|z| \mid \text{ for all } z \in z_f\right\}$$

and

$$\left(\left|\bar{z}_f\right|\right)^{\alpha} = \left|\left(\bar{z}_f\right)^{\alpha}\right| = \left|\bar{z}_f^{\alpha}\right| = \left\{\left|\bar{z}\right| \mid \text{ for all } z \in z_f\right\}.$$

Consequently the last part of the theorem follows in view of $|z| = \left|\bar{z}\right|$. □

## 4. Open Problem

As open problems, there are several scopes to investigate the theory of analyticity and singularity in case of functions of fuzzy complex variables; and analogously entire or meromorphic functions of fuzzy complex variables may be defined. Naturally, the theory of different aspects of growth properties of entire and meromorphic functions, comparative growth estimates of iterated entire functions, results related to exponent of convergence of zeros of entire functions of fuzzy complex variables etc. may also be studied afresh.

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# Banach Frames, Double Infinite Matrices and Wavelet Coefficients

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**Abstract**

In this paper we study the action of a double infinite matrix $A$ on $f \in H_v^p$ (weighted Banach space, $1 \leq p \leq \infty$) and on its wavelet coefficients. Also, we find the frame condition for $A-$transform of $f \in H_v^p$ whose wavelet series expansion is known.

*Keywords:* Frames, Riesz basis, wavelet coefficients, Banach space and frame operators.
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## 1. Introduction

The mathematical background for today's signal processing applications are Gabor (Feichtinger & Strohmer, 1998), wavelet (Daubechies, 1992) and sampling theory (Benedetto & Ferreira, 2001). Without signal processing methods several modern technologies would not be possible, like mobile phone, UMTS, xDSL or digital television. In other words, we can say that any advance in signal processing sciences directly leads to an application in technology and information processing. A signal is sampled and then analyzed using a Gabor respectively wavelet system. Many applications use a modification on the coefficients obtained from the analysis and synthesis operations. If the coefficients are not changed, the result of synthesis should be the original signal, i.e., perfect reconstruction is needed. One way is to analyze the signal using orthonormal basis. For practical point of view it is noted that the concept of an orthonormal basis is not always useful. Sometimes it is more important for a decomposing set to have other special properties rather than guaranteing unique coefficients. This led to the concept of frames introduced by Duffin and Schaeffer (Duffin & Schaeffer, 1952). Now a days it is one of most important foundations of Gabor (Moricz & Rhoades, 1989), wavelet (S.T. Ali & Gazeau, 2000) and sampling theory (Aldroubi & Gröchenig, 2001). In signal processing applications frames have received more and more attention

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(H. Bölcskei & Feichtinger, 1998; Kronland-Martinet & Grossmann, 1991; Munch, 1992; Sheikh & Mursaleen, 2004).

Frame provide stable expansions in Hilbert spaces, but they may be over complete and the coefficients in the frame expansion need not be unique unlike in orthogonal expansions. This redundancy is useful for the application point of view that is to noise reduction or for the reconstruction from lossy data (Daubechies, 1992; Duffin & Schaeffer, 1952; Matz & Hlawatsch, 2002). The construction of stable orthonormal basis are often difficult in a numerical efficient way than the construction of frames which are more flexible. Sometimes it is reasonable to use the frames to analyze additional properties of functions beyond the Hilbert space. These properties are encoded in the frame coefficients. Wavelet frames encode information on the smoothness and decay properties or phase space localization of functions by means of the magnitudes of the frame coefficients. The aim is to study these properties in Banach space norms. Moreover, to characterize an associated family of Banach spaces of functions by the values of the frame coefficients which play an important role in non-linear approximation and in compression algorithms (DeVore & Temlyakov, 1996). However, in (Gröchenig, 2004) Gröchenig showed that certain frames for Hilbert spaces extend automatically to Banach frames. Using this theory he derived some results on the construction of non-uniform Gabor frames and solved a problem about non-uniform sampling in shift-invariant spaces. Recently, Kumar (Kumar, 2013) studied the convergence of wavelet expansions associated with dilation matrix in the variable $L^p$ spaces using the approximate identity. In an another paper Kumar (Kumar, 2009) studied the convergence of non-orthogonal wavelet expansions in $L^p(R), 1 < p < \infty$.

The space $L^2(R)$ of measurable function $f$ is defined on the real line $R$, that satisfies $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$. The inner product of two square integrable functions $f, g \in L^2(R)$ is defined as

$$< f, g >= \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx, \quad \|f\|^2 =< f, f >^{1/2} .$$

Every function $f \in L^2(R)$ can be written as $f(x) = \sum_{j,k \in z} c_{j,k}\varphi_{j,k}(x)$ ($z$ `is the set of integers`).

This series representation of $f$ is called wavelet series. Analogous to the notation of Fourier coefficients, the wavelet coefficients $c_{j,k}$ are given by $c_{j,k} = \int_{-\infty}^{\infty} f(x)\overline{\varphi_{j,k}(x)}dx =< f, \varphi_{j,k} >, \quad \varphi_{j,k} = 2^{j/2}\varphi(2^j x - k)$.

Now, if we define continuous wavelet transform as $(W_\varphi(f))(b, a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x)\overline{\varphi\left(\frac{x-b}{a}\right)}dx, f \in L^2(R)$ then the wavelet coefficients are given by $c_{j,k} = (W_\varphi(f))\left(\frac{k}{2^j}, \frac{1}{2^j}\right)$.

A sequence $\{x_n\}$ in a Hilbert space $H$ is a frame if there exist constant $c_1$ and $c_2, 0 < c_1 \le c_2 < \infty$, such that $c_1\|f\|^2 \le \sum_{n \in z} | < f, x_n > |^2 \le c_2\|f\|^2$, for all $f \in H$. The supremum of all such numbers $c_1$ and infimum of all such numbers $c_2$ are called the frame bounds of the frame. The frame is called tight frame when $c_1 = c_2 = 1$. Any orthonormal basis in a Hilbert space $H$ is a normalized tight frame. The connection between frames and numerically stable reconstruction from discretized wavelet was pointed out by (Grossmann *et al.*, 1985). In 1985, they defined that a wavelet function $\varphi \in L^2(R)$, constitutes a frame with frame bounds $c_1$ and $c_2$, if any $f \in L^2(R)$ such that $c_1\|f\|^2 \le \sum_{j,k \in z} | < f, \varphi_{j,k} > |^2 \le c_2\|f\|^2$. Again, it is said to be tight if $c_1 = c_2$ and is said to be exact if it ceases to be frame by removing any of its element. For more details see (Chui, 1992; Daubechies *et al.*, 1986).

## 2. Notations and Auxiliary Results

Let $N$ and $\chi$ be countable index sets in some $R^2$ and both $\chi$ and $N$ are separated i.e., $\inf_{m,n\in\chi;m\neq n}|m-n|\geq\delta>0$, and likewise for $N$.

**Weight Functions of Polynomial Growth.** A weight is a non-negative continuous function on $R^d$. An $s-$moderate weight $v$ is called polynomially grows, if there are constants $C, s\geq 0$ such that $v(t)\leq C(1+|t|)^s$.

**Lemma 2.1.** *If $f(x)=\sum_{j,k\in N}c_{j,k}\varphi_{j,k}(x)$ is a wavelet expansion of $f\in L^2(R^d)$ with wavelet coefficients $c_{j,k}=\int_{-\infty}^{\infty}f(x)\overline{\varphi_{j,k}(x)}dx=<f,\varphi_{j,k}>$ and $A(a_{mnjk})=[(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon}$ for some $\varepsilon>0$ and $j,k\in N, m,n\in\chi$, then the operator $A$ defined on finite sequences $(c_{j,k})_{j,k\in N}$ by matrix multiplication $(Ac)_{m,n}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{mnjk}c_{j,k}$ extends to a bonded operator from $l_v^p(N)$ to $l_v^p(\chi)$ for all $p\in[1,\infty]$ and all $s-$moderate weights $v$.*

*Proof.* To prove the result we have to show the boundedness of $A$ from $l_v^1(N)$ to $l_v^1(\chi)$ and from $l_v^\infty(N)$ to $l_v^\infty(\chi)$. Then using the interpolation technique of [4] for weighted $L^p-$space, the lemma holds for all $p\in[1,\infty]$.

First we consider

$$\begin{aligned}
\|Ac_{j,k}\|_{l_v^1(\chi)} &= \sum_{m,n\in\chi}\left|\sum_{j,k\in N}a_{mnjk}c_{j,k}\right|v(m,n)\leq\sum_{m,n\in\chi}\sum_{j,k\in N}[(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon}|c_{j,k}|v(m,n)\\
&\leq \sup_{j,k\in N}\left(\sum_{m,n\in\chi}[(1+|m-j|)(1+|n-k|)]^{-d-\varepsilon}\right)\times\\
&\quad\left(\sup_{m,n\in\chi;j,k\in N}[(1+|m-j|)(1+|n-k|)]^{-s}[v(j,k)]^{-1}v(m,n)\right)\times\sum_{j,k\in N}|c_{j,k}|v(j,k).
\end{aligned}$$

Using (Gröchenig, 2004), Lemma 2.2 in above inequality we obtain

$$\begin{aligned}
&\leq \sup_{j,k\in N}\left(C(1+|j-k|)^{-d-\varepsilon}\right)\left(\sup_{j,k\in N}C(1+|j-k|)^{-s}\right)\times\\
&\quad[v(j,k)]^{-1}v(m,n)\times\sum_{j,k\in N}|c_{j,k}|v(j,k)=C\|c_{j,k}\|_{l_v^1(N)}.
\end{aligned}$$

The first supremum in right hand side of above inequality is finite by (Gröchenig, 2004), Lemma 2.1] and second supremum in finite due to $s-$moderate and sub multiplicavity of the weights. Now we have

$$\begin{aligned}
\|Ac_{j,k}\|_{l_v^\infty(\chi)} &= \sup_{m,n\in\chi}\left|\sum_{j,k\in N}a_{mnjk}c_{j,k}\right|v(m,n)\\
&\leq \sup_{m,n\in\chi}\sum_{j,k\in N}[(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon}|c_{j,k}|v(m,n)\\
&\leq \left(\sup_{m,n\in\chi}\sum_{j,k\in N}[(1+|m-j|)(1+|n-k)]^{-d-\varepsilon}\right)\times
\end{aligned}$$

$$\left( \sup_{\substack{m,n\in\chi \\ j,k\in N}} [(1+|m-j|)(1+|n-k|)]^{-s}.\nu(m,n)\nu(j,k)^{-1} \right) \times \left( \sup_{j,k\in N} |c_{j,k}|\nu(j,k) \right).$$

Again, using (Gröchenig, 2004), Lemma 2.2 in above inequality we get

$$\leq \left( C \sup \sum_{j,k\in N} (1+|j-k|)^{-d-\varepsilon} \right) \left( \sup(1+|j-k|)^{-s}\nu(m,n)\nu(j,k)^{-1} \right) \times \left( \sup_{j,k\in N} |c_{j,k}|\nu(j,k) \right) \leq CC' \|c_{j,k}\|_{l_\nu^\infty(N)}.$$

Let $\{\phi_{j,k} : j,k \in N\}$ be a Riesz basis of $H$ with dual basis $\{\widetilde{\phi}_{j,k} : j,k \in N\}$ and $\nu$ be a weight function on $R^d$ of polynomial type. □

**Definition 2.1.** Assume that $l_\nu^p(N) \subseteq l_\nu^2(N)$. Then the Banach space $H_\nu^p$ is defined to be

$$H_\nu^p = \{f \in H : f = \sum_{j,k\in N} c_{j,k}\phi_{j,k} \texttt{ for } c_{j,k} \in l_\nu^p(N)\}$$

with norm $\|f\|_{H_\nu^p} = \|c_{j,k}\|_{l_\nu^p}$. It should be noted that $c_{j,k}$ is uniquely determined, in fact, $c_{j,k} =< f, \widetilde{\phi}_{j,k} >$.

By assumption $l_\nu^p(N) \subseteq l_\nu^2(N)$, it means $H_\nu^p$ is a (dense) subset of $H$. On the other hand, if $l_\nu^p \nsubseteq l_\nu^2$ and $p < \infty$, we define $H_\nu^p$ to be the completion of the subspace $H_0$ of finite linear combinations, i.e., $H_0 = \{f = \sum_{j,k\in N} c_{j,k}\phi_{j,k} : supp$ c is finite $\}$, with respect to the norm $\|f\|_{H_\nu^p} = \|c\|_{l_\nu^p}$. If $p = \infty$ and $l_\nu^p \nsubseteq l^2$, we take the weak completion of $H_0$ to define $H_\nu^\infty$. In these cases $H_\nu^p \nsubseteq H$.

**Frame Operators and Localization of Frames.** Let $\mathcal{F} = \{\varphi_{m,n} : m,n \in \chi\}$ be a frame for $H$ and $Sf = \sum_{m,n\in\chi} < f, \varphi_{m,n} > \varphi_{m,n}$ be the corresponding frame operator. Each frame element has a natural expansion with respect to the given Riesz basis as

$$\varphi_{m,n} = \sum_{j,k\in N} \left\langle \varphi_{m,n}, \widetilde{\phi}_{j,k} \right\rangle \phi_{j,k} = \sum_{j,k\in N} < \varphi_{m,n}, \phi_{j,k} > \widetilde{\phi}_{j,k}.$$

The frame operator $S$ is invertible on $H$. Our problem is how to extend the mapping properties of $S$ on Banach spaces $H_\nu^p$. For this purpose we take $f = \sum_{j,k} f_{j,k}\phi_{j,k}$ such that

$$\begin{aligned} Sf &= \sum_{m,n\in\chi} < f, \varphi_{m,n} > \varphi_{m,n} = \sum_{m,n\in\chi} \sum_{j,k\in N} f_{j,k} < \phi_{j,k}, \varphi_{m,n} > \varphi_{m,n} \\ &= \sum_{m,n\in\chi} \sum_{i,l\in N} \sum_{j,k\in N} f_{j,k} < \phi_{j,k}, \varphi_{m,n} > \left\langle \varphi_{m,n}, \widetilde{\phi}_{i,l} \right\rangle \phi_{i,l} \\ &= \sum_{i,l} \left( \sum_{j,k} \left( \sum_{m,n} < \phi_{j,k}, \varphi_{m,n} > \left\langle \varphi_{m,n}, \widetilde{\phi}_{i,l} \right\rangle \right) f_{j,k} \right) \phi_{i,l}. \end{aligned}$$

Now let $A = a_{iljk}$ be infinite matrix defined as

$$a_{iljk} = \sum_{m,n\in\chi} < \phi_{j,k}, \varphi_{m,n} > \left\langle \varphi_{m,n}, \widetilde{\phi}_{i,l} \right\rangle = \left\langle S\phi_{j,k}, \widetilde{\phi}_{i,l} \right\rangle. \tag{2.1}$$

Define a mapping $\Gamma$ such that $\Gamma : H \to l^2(N), (\Gamma f)_{j,k} = \left\langle f, \widetilde{\phi}_{j,k} \right\rangle$.

Since $\{\phi_{j,k}\}$ is a Riesz basis, $\Gamma$ is invertible and an isometric isomorphism between $H^p_\nu$ and $l^p_\nu(N)$. Therefore, $S = \Gamma^{-1}A\Gamma$ carries over to the Banach spaces $H^p_\nu$. To study the behavior of frame operator $S$ on $H^p_\nu$, it is sufficient to study the infinite matrix $A$ on sequence space $l^p_\nu(N)$. For this purpose we will use the following fundamental theorem of Jaffard [14].

**Theorem A.** Assume that the matrix $G = (G_{k,l})_{k,l\in N}$ satisfies the following properties:

(a) $G$ is invertible as an operator on $l^2(N)$, and

(b) $|G_{kl}| \leq C(1 + |k - l|)^{-l}, k, l \in N$ for some constant $C > 0$ and some $r > d$. Then the inverse matrix $H = G^{-1}$ satisfies the same off-diagonal decay, that is

$$|H_{kl}| \leq C'(1 + |k - l|)^{-r}, k, l \in N.$$

Using above theorem we can prove:

**Theorem 2.1.** *Assume that the matrix $A = (a_{iljk})_{i,l,j,k\in N}$ satisfies the following properties:*

*(a) $A$ is invertible as an operator on $l^2(N)$, and*

*(b) $|a_{iljk}| \leq C[(1 + |i - j|)(1 + |l - k|)]^{-r}, i, l, j, k \in N$ for some constant $C > 0$ and some $r > d$.*

*Then the inverse matrix $T = A^{-1}$ satisfies the same off-diagonal decay, i.e.,*

$$\left|T_{iljk}\right| \leq C'[(1 + |i - j|)(1 + |l - k|)]^{-r}, i, l, j, k \in N.$$

**Definition 2.2.** The frame $\mathcal{F} = \{\varphi_{m,n} : m, n \in \chi\}$ is said to be polynomially localized with respect to Riesz basis $\{\phi_{j,k}\}$ with decay $s > 0$ (or simply $s-$localized), if

$$| < \varphi_{m,n}, \phi_{j,k} > | \leq C[(1 + |m - j|)(1 + |n - k|)]^{-s}$$

and

$$| < \varphi_{m,n}, \widetilde{\phi}_{j,k} > | \leq C[(1 + |m - j|)(1 + |n - k|)]^{-s} \forall i, k \in N \text{ and } m, n \in \chi.$$

Now we prove:

**Proposition 2.1.** *Let $\mathcal{F} = (\varphi_{m,n} : m, n \in \chi)$ is an $(s + d + \varepsilon)-$localized frame for $\varepsilon > 0, r \geq 0$ and $1 \leq p \leq \infty$. Then*

*(i) the analysis operator defined by $C_\varepsilon f = (< f, \varphi_{m,n} >)_{m,n\in\chi}$ is bounded from $H^p_\nu$ to $l^p_\nu(\chi)$.*

*(ii) the synthesis operator defined on finite sequences by $D_\varepsilon c = \sum_{m,n\in\chi} c_{m,n}\varphi_{m,n}$ extends to a bounded mapping from $l^p_\nu(\chi)$ to $H^p_\nu$.*

*(iii) the frame operator* $S = S_\varepsilon = D_\varepsilon C_\varepsilon = \sum_{m,n\in\chi} < f, \varphi_{m,n} > \varphi_{m,n}$ *maps* $H^p_\nu$ *into* $H^p_\nu$*, and the series converges unconditionally for* $1 \leq p \leq \infty$.

*Proof.* (i) Assume that $f = \sum_{j,k\in N} f_{j,k}\phi_{j,k}$, $| < f, \varphi_{m,n} > | = \left|\sum_{j,k\in N} f_{j,k} < \phi_{j,k}, \varphi_{m,n} >\right|$. In view of Definition 2.4, we get $\leq C \sum_{j,k\in N} \left|f_{j,k}\right| [(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon} \leq CC' \sum_{j,k\in N} |f_{j,k}|(1+|j-k|)^{-s-d-\varepsilon}$.

If $f \in H^p_\nu$, then $\|f\|_{H^p_\nu} = \left\|(f_{j,k})_{j,k\in N}\right\|_{l^p_\nu(N)}$ and Lemma 2.1 gives that $\|C_\varepsilon f\|_{l^p_\nu(\chi)} \leq CC' \left\|(f_{j,k})_{j,k\in N}\right\|_{l^p_\nu(N)} = CC'\|f\|_{H^p_\nu}$.

(ii) Now we have $(D_\varepsilon c)_{j,k\in N} = \left\langle \sum_{m,n\in\chi} c_{m,n}\varphi_{m,n}, \widetilde{\phi}_{j,k}\right\rangle$ or

$$
\begin{aligned}
|(D_\varepsilon c)_{j,k\in N}| &\leq \sum_{m,n\in\chi} |c_{m,n}| \left|\left\langle \varphi_{m,n}, \widetilde{\phi}_{j,k}\right\rangle\right| \leq C \sum_{m,n\in\chi} |c_{m,n}|[(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon} \\
&\leq CC' \sum_{m,n\in\chi} |c_{m,n}|(1+|j-k|)^{-s-d-e} = CC'(A^*|c|)_{j,k}.
\end{aligned}
$$

Now Lemma 2.1 (by interchanging $N$ and $\chi$) gives $\|D_\varepsilon c\|_{H^p_\nu} = \|A^*|c|\|_{l^p_\nu(N)} \leq \|A^*\|_{op}\|c\|_{l^p_\nu(\chi)}$.

(iii) The boundlessness of frame operator $S$ follows by combining (1) and (ii). For unconditional convergence of the series defining $S$, let $\varepsilon > 0$, choose $N_0 = N_0(\varepsilon)$, such that $\| < f, \varphi_{m,n} >_{m,n(not\in N_0)} \|_{l^p_\nu} \leq \varepsilon$. Then for any finite set $N_1 \supseteq N_0$, from (i) and (ii), we obtain

$$
\left\| S f - \sum_{m,n\in N} < f, \varphi_{m,n} > \varphi_{m,n} \right\|_{H^p_\nu} \leq \|C_\varepsilon\|_{op} \| < f, \varphi_{m,n} > \| \leq \|C_\varepsilon\|_{op}.\varepsilon.
$$

Which implies that the series $\sum_{m,n\in\chi} < f, \varphi_{m,n} > \varphi_{m,n}$ converges unconditionally in $H^p_\nu$. □

**Proposition 2.2.** *Assume that* $\mathcal{F} = \{\varphi_{m,n} : m, n \in \chi\}$ *is polynomially localized with respect to the Riesz basis* $\{\phi_{j,k}\}$ *with decay* $s > d$*. Then*

$$
|A| = |a_{iljk}| \leq C(1+|j-k|)^{-s} \quad for \ i, l, j, k \in N.
$$

**Proposition 2.3.** *From (2.1) we get*

$$
\begin{aligned}
|a_{iljk}| &\leq C \sum_{m,n\in\chi} [(1+|m-j|)(1+|n-k|)(1+|i-m|)(1+|l-n|)]^{-s} \\
&\leq CC' \sum_{i,l\in N} [(1+|i-j|)(1+|l-k|)]^{-s} \leq CC'C''(1+|j-k|)^{-s}.
\end{aligned}
$$

## 3. Main Results

The following definition is due to Moricz and Rhoades (Moricz & Rhoades, 1989).

**Definition 3.1.** Let $A = (a_{iljk})$ be a double non-negative infinite matrix of real numbers. Then, $A-$transform of a double sequence $x = \{x_{j,k}\}$ is $\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{mnjk}x_{j,k}$ which is called $A-$means or $A-$transform of the sequence $x = \{x_{j,k}\}$.

Sheikh and Mursaleen (Sheikh & Mursaleen, 2004) study the frame condition by using the action of frame operator $A$ on non-negative infinite matrix in Hilbert space. In this paper our aim is to extend these results on weighted Banach space in $R^d$.

Now we prove our main results:

**Theorem 3.1.** *Let $A = (a_{iljk})$ be a double non-negative infinite matrix. If $f(x) = \sum_{m,n\in\chi} c_{m,n}\varphi_{m,n}(x)$ is a wavelet expansion of $f \in H^p_\nu$ with wavelet coefficients $c_{m,n} =< f, \varphi_{m,n} >$, then the frame condition for $A-$transform of $f \in H^p_\nu$ is*

$$c_1\|f\|_{H^p_\nu} \leq \left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \leq c_2\|f\|_{H^p_\nu}$$

*where $\{\varphi_{m,n} : m, n \in \chi\}$ is an $(s + d + \varepsilon)-$localized frame for $\varepsilon > 0$, $s \geq 0$ and $1 \leq p \leq \infty$.*

*Proof.* We take $f = \sum_{j,k\in N} f_{j,k}\phi_{j,k}$, then

$$\sum_{m,n\in\chi} | < Af, \varphi_{m,n} > | \leq \left| \sum_{j,k\in N} \sum_{m,n\in\chi} Af_{j,k} < \phi_{j,k}, \varphi_{m,n} > \right| \leq \sum_{j,k\in N} |Af_{jk}|| < \phi_{j,k}, \varphi_{m,n} > |$$
$$\leq c \sum_{j,k\in N} |Af_{j,k}|((1 + |m - j|)(1 + |n - k|))^{-s-d-\varepsilon} \leq CC' \sum_{j,k\in N} |Af_{j,k}|(1 + |j - k|)^{-s-d-\varepsilon}.$$

If $f \in H^p_\nu$, then $\|f\|_{H^p_\nu} = \|(f_{j,k})_{j,k\in N}\|_{l^p_\nu}$. Hence we get

$$\left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \leq CC'\|A\|_{op}\|f\|_{H^p_\nu} \leq c_2\|f\|_{H^p_\nu}.$$

Now, for any $f \in H^p_\nu$, define

$$\widetilde{f} \;=\; \left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|^{-1}_{l^p_\nu} f < A\widetilde{f}, \varphi_{m,n} >= \left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|^{-1}_{l^p_\nu} < Af, \varphi_{m,n} >$$

then

$$\left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \leq 1.$$

Hence, if there exists a positive constant $\alpha$, such that

$$\|Ac_{m,n}\|_{l^p_\nu} \leq \alpha \left[ \left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \right]^{-1} \|Ac_{m,n}\|_{l^p_\nu} \leq \alpha \left[ \left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \right]^{-1} \|f\|_{H^p_\nu} \leq \left( \frac{\alpha}{\|A\|_{op}} \right)$$

it follows that $\left[ \left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \right] \geq c_1\|f\|_{H^p_\nu}$.

Hence the proof is completed. □

**Theorem 3.2.** *If* $f = \sum_{j,k\in N} c_{j,k}\phi_{j,k}$ *and* $\{\varphi_{m,n} : m, n \in \chi\}$ *forms a frame with respect to Riesz basis* $\{\phi_{j,k}\}$*, then the* $\alpha_{j,k}$ *are the wavelet coefficients of* $Af$*, where* $\{d_{i,l}\}$ *is defined as the* $A-$*transform of* $\{c_{j,k}\}$ *such that*

$$\begin{aligned} d_{i,l} &= \sum_{j,k\in N} a_{iljk} c_{j,k}, \\ \alpha_{j,k} &= \sum_{i,l\in\chi} d_{i,l} < \phi_{j,k}, \varphi_{i,l} > . \end{aligned}$$

*Proof.* Using the definition of $A-$transform of $f = \sum_{il\in\chi} c_{i,l}\varphi_{i,l}$ by assumption we get

$$< Af, \varphi_{i,l} >= \sum_{j,k\in N} a_{iljk} c_{j,k} < \phi_{j,k}, \varphi_{i,l} >$$

or

$$\sum_{i,l\in\chi} < Af, \varphi_{i,l} >= \sum_{i,l\in\chi} (Ac)_{i,l} < \phi_{j,k}, \varphi_{i,l} >= \sum_{i,l\in\chi} d_{i,l} < \phi_{j,k}, \varphi_{i,l} > .$$

Therefore, the wavelet coefficients of $Af$ with respect to Riesz basis $\{\phi_{j,k}\}$ are given by

$$\alpha_{j,k} = \sum_{i,l\in\chi} d_{i,l} < \phi_{j,k}, \varphi_{i,l} > .$$

Hence the proof is completed. □

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# Fence-like Quasi-periodic Texture Detection in Images

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## Abstract

The focus of this article is on automatic detection of fence or wire mesh (a form of quasi-periodic texture) in images through frequency domain analysis. Textures can be broadly classified in to two general classes: quasi-periodic and random. For example, a fence has a repetitive geometric pattern, which can be classified as a quasi-periodic texture. Quasi-periodic textures can be easily detected in the frequency spectrum of an image as they result in peaks in the frequency spectrum. This article explores a novel way of de-fencing viewed as a quasi-periodic texture segmentation by filtering in frequency domain to segregate the fence from the background. A resulting de-fenced image is followed by support vector machine classification. An interesting application of the proposed approach is the removal of occluding structures such as fence or wire mesh in animal enclosure photography.

*Keywords:* Frequency spectrum, quasi-periodic texture, texture segmentation
*2010 MSC No:* Primary 35B36, Secondary 03C13, 03C45, 93C80.

## 1. Introduction

This article introduces an algorithm to detect automatically fence or wire mesh structures, which typically present in the foreground of the image. A region in an image has a constant texture, provided a set of local statistics or other local properties of the picture function are constant, slowly varying, or approximately periodic (Tuceryan & Jain, 1993). A fence can be classified as a texture in an image. Textures can be broadly classified in to two general classes: *periodic* or more generally *quasi-periodic textures* and *random textures*.

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According to (Rangayyan, 2004), if there is a repetition of a texture element at almost regular or quasi-periodic intervals, such textures can be classified as quasi-periodic or ordered and the smallest repetitive element is called a texton or a texel. In contrast if no such repetitive element can be identified, those textures can be classified as random.

(Ohm, 2004) classifies textures as *regular* and *irregular* textures. Regular textures refer to textures, which exhibits strong periodic or quasi-periodic behavior. According to (Ohm, 2004), exact periodicity is a very rare case mostly found in synthetic images. The regular structures in natural images are often quasi-periodic, which means that periodic pattern can clearly be recognized, but have slight variations of periods. As it will be shown in section 2, quasi-periodic textures are a generalization of periodic textures.

Based on the above classifications, a fence structure, which has a texture element repeating at quasi-periodic intervals can be categorized as a quasi-periodic texture. Hence, a fence-like texture can be modeled as a quasi-periodic signal, which shows peaks in its power spectrum. It is mentioned in (Chang & Kuo, 1993) that these kinds of quasi-periodic signals possess dominant frequencies located in the middle frequency channels.

The perception of texture has numerous dimensions. Thus, a number of different texture representations were introduced from time to time in order to accommodate a variety of textures. These representations are categorized in (Tuceryan & Jain, 1993) as statistical methods, which involves co-occurrence matrices and autocorrelation features, geometric methods, model based methods and signal processing methods. Signal processing methods are subdivided into spatial domain filtering (Malik & Perona, 1990) and frequency filtering.

Frequency analysis of the textured image is close to human perception of texture as human visual system analyzes the textured image by decomposing the image into its frequency and orientation components (Campbell & Robson, 1968). (Turner, 1986) and (Clark *et al.*, 1987) proposed to use the Gabor filters in texture analysis. The Gabor filter is a frequency and orientation selective filter. Another model, which is widely used for texture analysis is wavelet transform (Chang & Kuo, 1992, 1993; Wilscy & Sasi, 2010).

The focus of this article is on images, which are occluded with fence textures as shown in figure 1. In such cases, it is challenging to segment the fence from the rest of the image, especially when the image background is regular. Simple colour segmentations and edge detection does not work in this case.

The traditional frequency filters used for texture analysis, Gabor and Wavelet cannot be directly applied to extract fence texture in our scenario as the frequencies correspond to both fence and the background are present in the spectrum. Thus, we first perform frequency domain processing to isolate fence texture from the background and subsequently apply Wavelet transform.

An interesting application of the proposed algorithm can be detection and removal of fence-like textures obstructing the images in zoo photography. According to many web articles on photography (Stalking, 2010; Masoner, 2013), wire mesh and fences are a major challenge in zoo photography. The algorithm proposed in this article was tested for fences with different shapes, sizes, colours and orientations.

The rest of the article is organized as follows. Section 2 introduces quasi-periodic signals and provides the mathematical background to analyze quasi-periodic signals in images. Section 3 discusses the implementation of the quasi-periodic texture detection algorithm in three steps: (1)

frequency domain filtering for quasi-periodic texture detection, (2) multiresolution processing for fence mask formation and (3) fence segmentation through SVM classification. The experimental results of the proposed algorithm are given in Section 4 for some zoo images as well as for some challenging images from PSU NRT Database (Liu, 2007). A comparison of the proposed method with existing fence detection techniques is given in section 5 followed by future work and conclusion in sections 6 and 7 respectively.

**Figure 1.** Images Occluded with Fence Textures.

## 2. Quasi-periodic Signals

Before going into details of quasi-periodic texture detection in images, understanding the mathematical background of quasi-periodic signals is important.

**Definition 2.1. Continuous-time Periodic Signal** ((Proakis & Manolakis, 2006, §1, p. 13))
By definition, A continuous signal f(t) locally defined on the set $L^2(\mathfrak{R})$ of finite energy signals is fully periodic with period T, when the signal exactly satisfies

$$f(t) = f(t + T).$$

**Definition 2.2. Continuous-time Quasi-periodic Signal** ((Martin *et al.*, 2010))
A signal $f_{qp}(t)$ is quasi-periodic with k periods $T_1, ..., T_k$ when

$$f_{qp}(t) = g\{f_1(t), f_2(t), ..., f_k(t)\},$$

where the k signals $f_i(t)$ are continuous periodic signals with respect to each period $T_i$.

In the case of continuous functions locally defined on the set $L^2(\mathfrak{R})$ of finite energy signals, quasi-periodic signals are a generalization of periodic signals. All the periods are required to be strictly positive and to be rationally linearly independent (Martin *et al.*, 2010).

**Definition 2.3. Discrete-time Periodic Signal**((Proakis & Manolakis, 2006, §1, p. 15))
A discrete-time signal f(n) is periodic with period N, if and only if,

$$f(n) = f(n + N) \text{ for all n.}$$

Based on the definition of continuous-time quasi-periodic signals, the definition for discrete-time quasi-periodic signals can be derived.

**Definition 2.4. Discrete-time Quasi-periodic Signal**
A discrete-time signal $f_{qp}(n)$ is quasi-periodic with k periods $N_1, ...N_k$ when

$$f_{qp}(n) = g\{f_1(n), f_2(n), ..., f_k(n)\},$$

where $g : \mathbb{Z}^k \rightarrow \mathbb{Z}$ and the k signals $f_i(n)$ are discrete-time periodic signals with respect to each period $N_i$.

In the context of this paper, an image is considered as a 2D discrete-time signal. If we extend the definition of 1D quasi-periodic signal to 2D quasi-periodic signal;

**Definition 2.5. 2D Discrete-time Periodic Signal**((Woods, 2006, §1, p. 7))
A 2D discrete-time signal f(x,y) is periodic with period (M,N), if and only if,

$$f(x, y) = f(x + M, y) = f(x, y + N), \forall n, m \in \mathbb{Z}.$$

**Definition 2.6. 2D Discrete-time Quasi-periodic Signal**
A 2D discrete-time signal $f_{qp}(x, y)$ is quasi-periodic with k periods $(M_1, ...M_k, N_1, ...N_k)$ when

$$f_{qp}(x, y) = g\{f_1(x, y), f_2(x, y), ..., f_k(x, y)\},$$

where the k signals $f_i(x, y)$ are discrete-time periodic signals with respect to periods $(M_i, N_i)$. Hence, a quasi-periodic signal can be defined as a combination of periodic signals with incommensurate (not rationally related) frequencies (Battersby & Porta, 1996). If the frequencies are commensurate, then $f_{qp}$ becomes a periodic signal (Regev, 2006).

A discrete-time quasi-periodic signal can be expressed with a Fourier series as given in definition 2.8 as a generalization of definition 2.7. 1D case will be considered for simplicity and it can be extended to 2D.

**Definition 2.7. Fourier Series of a Discrete-time Periodic Signal** ((Proakis & Manolakis, 2006, §4, p. 242))

$$f(n) = \sum_{k=0}^{N-1} c_k exp\left(\frac{j2\pi kn}{N}\right).$$

**Definition 2.8. Fourier Series of a Discrete-time Quasi-periodic Signal** ((Regev, 2006, p. 156))
The Fourier series of a r-quasi-periodic signal is given by (Regev, 2006):

$$f_{qp}(n) = \sum_{k_1} \sum_{k_2} ... \sum_{k_r} c_{k_1 k_2 ... k_r} exp\left[j\left(\frac{2\pi k_1 n}{N_1} + \frac{2\pi k_2 n}{N_2} + ... + \frac{2\pi k_r n}{N_r}\right)\right],$$

where k=1,2,...,r and the frequencies $\omega_k = 2\pi/N_k$ are incommensurate.

**Theorem 2.1.** *Let $f_{qp}(n)$ be a discrete-time quasi-periodic signal. Then the frequency spectrum of $f_{qp}(n)$ consists of a set of peaks determined by the fundamental frequencies of each discrete periodic signal component in the signal.*

*Proof.* With $\omega_i = 2\pi/N_i$, $f_{qp}(n)$ in definition 2.8 can be re-written as

$$f_{qp}(n) = \sum_K c_K exp\left[jK\Omega n\right],$$

where $K = (k_1, k_2, ..., k_r)$ and $\Omega = (\omega_i, \omega_2, ..., \omega_r)$. Thus, the frequency spectrum contains numerous peaks at all frequencies $\nu$, satisfying

$$2\pi\nu = |K \cdot \Omega| = |k_1\omega_1 + k_2\omega_2 + ... + k_r\omega_r|,$$

for any combination of integers $k_1, k_2, ..., k_r$. □

## 3. Quasi-periodic Texture Detection in Frequency Domain

### *3.1. Frequency Domain Filtering for Quasi-periodic Texture Detection*

As proven by theorem 2.1, the Fourier spectrum of a quasi-periodic signal consists of a discrete set of spikes or peaks at a number of frequencies depending on the number of periodic signals it is comprised of. Hence, based on theorem 2.1, the fence-like quasi-periodic structure should result in peaks in the frequency spectrum of the image. The objective of this section is to filter those spikes in the frequency spectra relevant to the quasi-periodic signal in order to extract the fence texture corresponding to the quasi-periodic signal from the rest of the image.

To achieve this, first start with the frequency domain representation of the 2D image. We will be considering the DFT of an image.

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)exp\left[-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right] \text{u=0,1,...M-1, v=0,1,...N-1.} \tag{3.1}$$

To filter the frequencies showing spikes in the frequency spectra, it is necessary to perform thresholding based on the magnitude of each frequency component. A filter function $H_1(u,v)$ in frequency domain can be defined for this purpose as given below.

$$H_1(u,v) = \begin{cases} 1 & \text{if } |F(u,v)| > T, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

where T is a threshold to filter spikes in frequency.

Once the thresholding is applied to the frequency components:

$$F'(u,v) = H_1(u,v)F(u,v)$$

Although, we filtered the frequency components corresponding to peaks in the frequency spectra, it is necessary to filter peaks in frequencies resulted by other details in the image. For an example, the DC component F(0,0), which can be derived by substituting u=0 and v=0 in equation 3.1. $|F(0,0)|$ typically is the largest component of the spectrum.

$$F(0,0) = MN\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) = MN\overline{f}(x,y).$$

The quasi-periodic signal in our case is the fence. Fence-like textures typically result in qausi-periodic signals whose dominant frequencies are located in the middle frequency channels (Chang & Kuo, 1993). Therefore, by using a bandpass filter in frequency domain, the frequencies corresponding to the fence can be filtered.

$$H_2(u,v) = \begin{cases} 1 & \text{if } D_1 \leq D(u,v) \leq D_2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

where $D_1$ and $D_2$ are constants and D(u,v) is the distance between a point (u,v) in the frequency domain and the center of the frequency spectrum.

Thus, the final result in frequency domain after applying the second filter would be:

$$\begin{aligned} F''(u,v) &= H_2(u,v)F'(u,v), \\ &= H_2(u,v)H_1(u,v)F(u,v), \\ &= H(u,v)F(u,v), \end{aligned}$$

where $H = H_2H_1$, since the application of $H_1$ and $H_2$ can be considered as a cascade system. When $F''(u,v)$ is transferred back into spatial domain, the resulting image is given by:

$$g(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F''(u,v)exp\left[j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right] \text{x=0,1,...M-1, y=0,1,...N-1.}$$

It is important to note that $H_1$ and $H_2$ are zero phase shift filters, which affect the magnitude of the frequency spectra, but do not alter the phase angle. These filters affect the real (Re(u,v)) and imaginary (Im(u,v)) parts equally, thus cancels out when calculating phase angle $\phi(u,v) = arctan[Im(u,v)/Re(u,v)]$.

Figure 2(d) illustrates the final result of frequency domain filtering explained above. It can be clearly seen that the fence texture is emphasized and other image details have been suppressed.

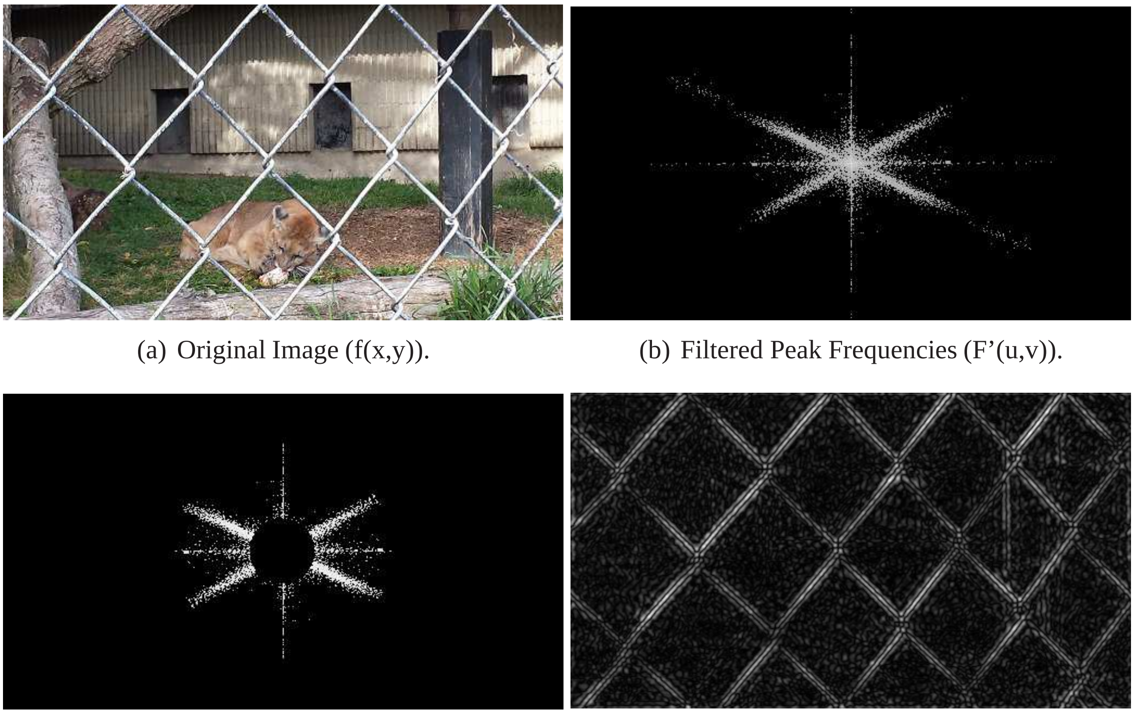
### 3.2. Multiresolution Processing for Fence Mask Formation

The human visual system analyzes the textured images by decomposing the image into its frequency and orientation components (Campbell & Robson, 1968). Wavelet transformation provides the ability to analyze images through multiresolution processing.

Wavelet transform in two dimension provides the two dimensional scaling function $\phi(x,y)$ and three two dimensional directionally sensitive wavelets $\psi^H(x,y)$, $\psi^V(x,y)$,$\psi^D(x,y)$ as given in (Gonzalez & Richard, 2002).

$$\phi_{j,m,n}(x,y) = 2^{\frac{j}{2}}\phi(2^jx-m, 2^jy-n).$$

$$\psi^i_{j,m,n}(x,y) = 2^{\frac{j}{2}}\psi^i(2^jx-m, 2^jy-n), i = \{H,V,D\}.$$



(a) Original Image (f(x,y)). (b) Filtered Peak Frequencies (F'(u,v)).

(c) Bandpassed Spectrum (F"(u,v)). (d) Spatial Domain (g(x,y)).

**Figure 2.** Frequency Domain Filtering for Fence Texture Segregation from Image Background.

These wavelets measure intensity variations for images along different directions: $\psi^H$ measures variations along horizontal direction (along columns), $\psi^V$ measures variations along vertical direction (along rows) and $\psi^D$ corresponds to variations along diagonals.

The discrete transform of image f(x,y) is:

$$W_\phi(j_0, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)\phi_{j_0,m,n}(x, y).$$

$$W_\psi^i(j, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)\psi^i_{j,m,n}(x, y), i = \{H, V, D\},$$

where $j_0$ is an arbitrary starting scale and the $W_\phi(J_0, m, n)$ coefficients define an approximation of f(x,y) at scale $j_0$. The $W_\psi^i(j, m, n)$ coefficients add horizontal, vertical and diagonal details for scales $j \geq j_0$. $W_\psi^i(j_0, m, n)$ coefficients are called detail coefficients. Usually $j_0$ is set to zero.

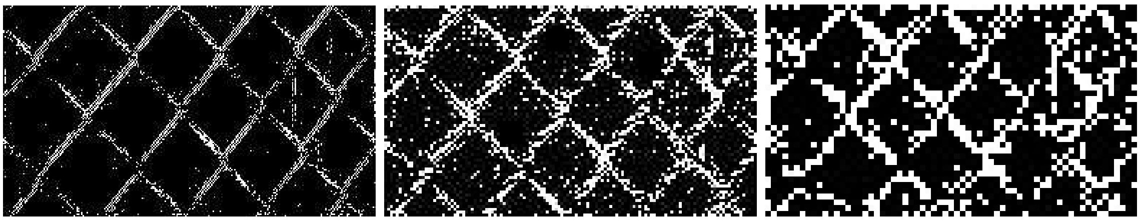
For each level j, thresholding is performed on the details coefficients $W_\psi^i(j, m, n)$ to extract the fence masks $M^i(j, m, n)$ at each level j.

$$M^i(j, m, n) = \begin{cases} 1 & \text{if } W_\psi^i(j, m, n) > T_j, \text{ where } T_j \text{ is the threshold for level } j, \\ 0 & \text{otherwise.} \end{cases}$$

The final fence mask at level j is obtained by performing *OR* operation of the vertical, horizontal and diagonal fence masks at level j.

$$M(j, m, n) = M^V(j, m, n) \oplus M^H(j, m, n) \oplus M^D(j, m, n).$$

The detected fence masks at 3 consecutive levels are shown in figure 3.



(a) M(1,m,n). (b) M(2,m,n). (c) M(3,m,n).

**Figure 3.** Detected Fence Masks at Three Different Levels.

Next, the fence masks at different levels of wavelet pyramid were combined by using a coarser to finer strategy. The objective is to reduce noise and extract pixels, which fall exactly on the fence. In order to make the resultant mask in the same size as the original image, a mask was created at the zero level by just thresholding the spatial domain result of frequency filtering (g(x,y)). Hence, altogether we have fence masks at 4 different levels in the pyramid.

First, the highest level fence mask (level 3) was considered and if a pixel belongs to the mask then we move to the next lower level (level 2) and check for the neighbouring children of the original pixel. If any of the neighbouring children are mask pixels, then recursively go and check for their neighbouring children in the subsequent lower levels. Finally, when the algorithms reaches the bottom most level (zero level), it marks the mask pixels as 1, given that the neighbouring children in the lowest level are mask pixels as well. The resultant fence mask is shown in Figure 4.

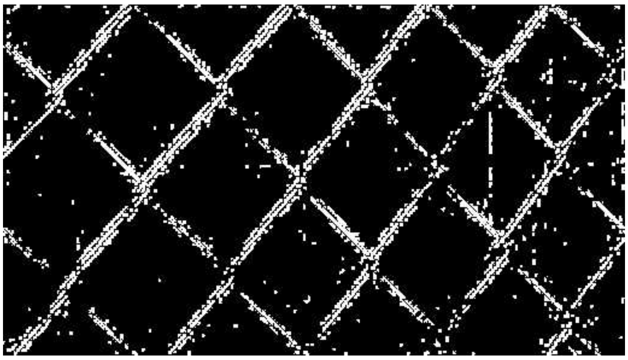


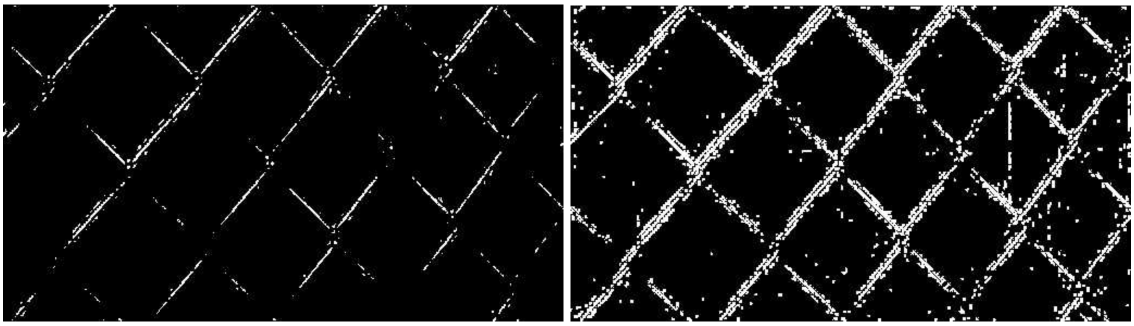
**Figure 4.** Fence Mask Formed by Combining Wavelet Decomposition Levels.

### 3.3. Fence Segmentation through SVM Classification

Although the noise is minimized and the fence is emphasized in the detected fence mask, it is not perfectly detected yet. However, the detected fence mask classifies a good number of pixels, which exactly falls on the fence in the image. This knowledge on fence pixels can be used to segment the fence. Hence, it was decided to pick some samples from the fence mask and use the features of those sample pixels to train a *Support Vector Machine (SVM) classifier* in order to segment the fence texture. A SVM classifier with a linear kernel is used in this case.

In addition to the samples from fence, it is necessary to pick samples from background to train the SVM classifier. For this purpose two root level fence masks were generated. One root level mask was generated by selecting a very high threshold and the other one is generated by using a very low threshold. These masks were used as the root level mask in the process of combining wavelet decomposition levels as explained in section 3.2 separately in order to generate two different final fence masks as shown in Figure 5.

As it can be clearly seen, the root level mask with high threshold generates a very thin final mask, resulting points, which exactly lie on the fence. On the other hand the root level mask with low threshold generates a thick fence mask, which has some points fall on the background as well.



(a) Thin Mask with High Threshold. (b) Thick Mask with Low Threshold.

**Figure 5.** Two Fence Masks used for SVM Classification.

The thin mask was used to pick random samples, which represent fence class and the negation of the thick mask (1-thick mask) is used to pick random samples, which represent the background class. The use of negation of thick mask for background sample selection reduces the chance of picking fence pixels as background pixels and hence improves the accuracy of classification.

The feature vector selected for classification plays a very important role in this case as it affects the overall performance of the classification. The RGB colour channels and the gradient direction of the samples were used as the feature set for classification. The resultant fence mask can be further improved with the help of morphological operations.

The algorithm to achieve fence-like quasi-periodic texture detection in digital images is given in Algorithm 1.

**Algorithm 1** Algorithm for fence-like quasi-periodic texture detection in images

1: Read the fenced image $I$

2: Convert $I$ into frequency domain using Discrete Fourier Transform (let the output be $F$)

3: Filter $F$ using the peak frequency filter $H_1$ defined in equation 3.2 (let the output be $F1$)

4: Filter $F1$ using the band pass filter $H_2$ defined in equation 3.3 (let the output be $F2$)

5: Convert $F2$ back into spatial domain (let the output be *filtI*)

6: Perform Wavelet decomposition on *filtI* with three decomposition levels

7: **for** each Wavelet decomposition level **do**

8:     Find vertical (V), horizontal (H) and Diagonal (D) components

9:     Threshold V, H and D with the same threshold

10:     Combine thresholded V, H and D components using logical OR operation

11: **end for**

▹ %comment: Obtain fence mask by combining all three levels of the wavelet pyramid (let the output be fenceMask)%

12: Start from the highest Wavelet decomposition level (level 3)

13: **for** each pixel in level 3 **do**

14:     **if** a pixel belongs to the mask **then**

15:         Move to next lower level

16:         **if** current level == lowest level **then**

17:             Mark the pixel as mask pixels

18:             Mark the neighbouring children as mask pixels

19:         **else**

20:             Check neighbouring children

21:             **if** neighbouring children are mask pixels **then**

22:                 Go back to step 14

23:             **end if**

24:         **end if**

25:     **end if**

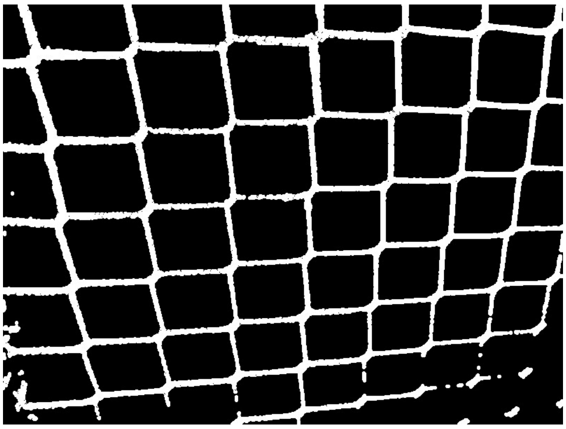
26: **end for**

27: Prepare the training data matrix using feature vectors of sample pixels fall on fence (fenceMask==1) and background (fenceMask==0).

28: Train the SVM classifier by using training data matrix of step 25.

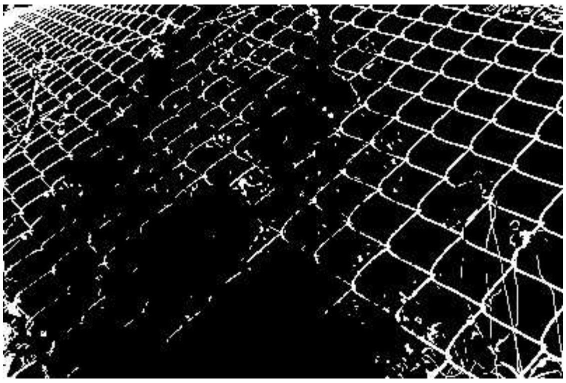
29: Perform SVM classification by using the trained classifier in step 26 by giving original image as the input to obtain final fence mask.

(a) Child Image.



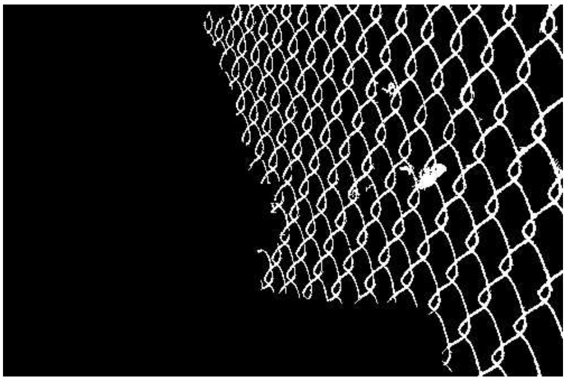
(b) Fence Mask for Child Image.

(c) Building Image.



(d) Fence Mask for Building Image.

(e) Flower Image.



(f) Fence Mask for Flower Image.

**Figure 6.** Results of Fence-like Texture Detection in Images from PSU NRT Database (Liu, 2007).

## 4. Experimental Results

The frequency domain-based fence-like quasi-periodic texture detection algorithm proposed in this article was implemented in Matlab R2013a and it was tested with a number of images with fence-like texture. Some test images were obtained from PSU Near-regular Texture database (Liu, 2007). Images with fences of different shapes (square and diagonal), sizes, colours and orientations were used for this experiment. Figure 6 illustrates results of some of the challenging cases encountered during experiments.

For the completion of the sample application chosen in this paper, once the fence texture was successfully detected and removed, the region, which belonged to the fence, should be filled with relevant information in order to obtain the final image. One of the techniques, which can serve this purpose is *inpainting*. According to (Bertalmio *et al.*, 2000), inpainting is the *modification of images in a way that is non-detectable for an observer who does not know the original image*. There are numerous inpainting techniques introduced in past literature.

For examples region filling and object removal by exemplar-based image inpainting by Criminisi et al. (Criminisi *et al.*, 2004), Fields of experts by Roth et al. (Roth & Black, 2009) and Image completion with structure propagation by Sun et al. (Sun *et al.*, 2005). Among these techniques, the exemplar based image inpainting technique (Criminisi *et al.*, 2004) was used to fill the fence region in this approach. The results are given in figure 7.

Interestingly, some image distortions can be observed after performing inpainting for some images. The region belonged to the fence texture is much more difficult to texture fill than large, circular regions of similar area. The fence texture in this case is usually wide spread in the whole image. Thus, it requires the inpainting algorithm to correctly propagate and join different types of structures in order to fill this wide spread fence region. Hence, mistakes in structure propagation can be quiet frequent in this case. The high ratio of foreground area to background area and the fragmented background source textures may become challenging for the inpainting technique.

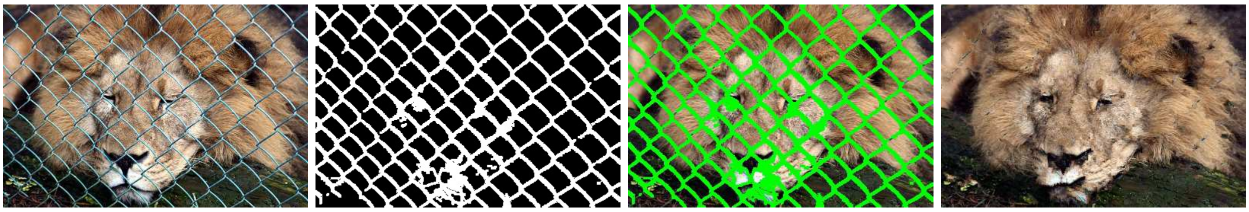
## 5. Comparison with Existing Fence Detection Techniques

Most of the articles, which investigated the image de-fencing problem, have used a texture based approach to detect the fence, based on the assumption that a fence is a near regular structure. (Liuy *et al.*, 2008) introduced an image de-fencing technique based on lattice structure of regular textures in their article. The de-fencing algorithm proposed in (Liuy *et al.*, 2008) consists of three steps. *(1) automatically finding the skeleton structure of a potential frontal layer in the form of a deformed lattice; (2) classifying pixels as foreground or background using appearance regularity as the dominant cue, and (3) inpainting the foreground regions using the background texture which is typically composed of fragmented source regions to reveal a complete, non-occluded image* (Liuy *et al.*, 2008).

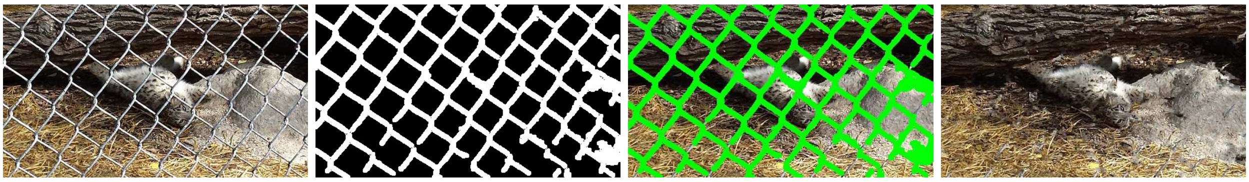
In the first step, to automatically detect the lattice of the fence, (Liuy *et al.*, 2008) uses the iterative algorithm explained in (Hays *et al.*, 2006), which tries to find the most regular lattice for a given image by assigning the neighbour relationships such that neighbors have maximum visual similarity. Step one results in a mesh of quadratiles, which contains repeated elements or texels. In the second step standard deviation of each colour channel and the color features are used for k-means clustering for background foreground separation. In order to obtain the standard deviation,

(a) Lynx. (b) Fence Mask. (c) Fence Mask Applied Image. (d) Inpainted Image.

(e) Puma. (f) Fence Mask. (g) Fence Mask Applied Image. (h) Inpainted Image.



(i) Lion (Schneider, 2010). (j) Fence Mask. (k) Fence Mask Applied Image. (l) Inpainted Image.



(m) Leopard. (n) Fence Mask. (o) Fence Mask Applied Image. (p) Inpainted Image.

(q) Puppies (Liu, 2007). (r) Fence Mask. (s) Fence Mask Applied Image. (t) Inpainted Image.

**Figure 7.** Results of Fence Removal from Zoo Images.

the texels were aligned and arranged in a stack and standard deviation is calculated along each vertical column of pixels. Finally, texture based inpainting technique introduced by Criminisi et al. (Criminisi *et al.*, 2004, 2003) is used to obtain the final de-fenced image.

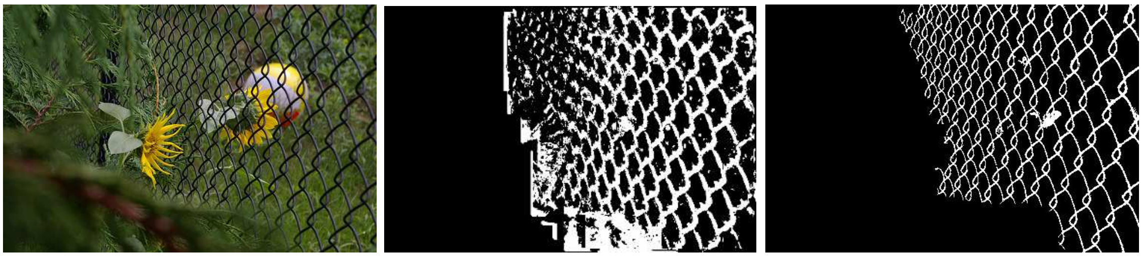
Park et al. revisits the image de-fencing problem in their paper (Park *et al.*, 2011). They no longer uses the lattice detection algorithm introduced in (Hays *et al.*, 2006), as they states its performance is far from practical due to inaccuracy and slowness. Rather the implementation of lattice detection algorithm in (Park *et al.*, 2011) is similar to (Park *et al.*, 2009). In their method, once the type of the repeating pattern is learnt, the irregularities are removed and the learned regularity is used in evaluating the foreground appearance likelihood during the lattice growth. They have improved the lattice detection algorithm by introducing an online learning and classification.

In essence, the de-fencing algorithms introduced in both of these articles uses a lattice detection algorithm in order to find the fence mask. Thus, the success of both algorithms depends on finding the repeated element or texel in the fence structure. The lattice detection algorithm used by (Liuy *et al.*, 2008) has no measures against irregularities in the lattice while the lattice detection algorithm used by (Park *et al.*, 2011) takes some measures to remove irregularities during lattice growth. However, both these approaches depend on the regularity of the fence as well as the irregularity of the background of the image. Although (Park *et al.*, 2011) takes measures against irregularities in the fence, it does not take in to account the possibility of regularities in the background. Furthermore, the lattice detection process itself is very complex and time consuming.

In contrast to the two methods discussed above, the method explained in this article uses a frequency domain approach to address the fence detection problem. Due to the uncertainty principle, the global wide spread fence texture in spatial domain becomes local to a set of frequencies in the frequency domain. So the processing required to extract the fence texture in frequency domain is simpler and faster compared to spatial domain processing. This becomes advantageous in the proposed method compared to the existing techniques. Moreover, the band pass filtering in frequency domain used in the proposed method helps to avoid other periodic structures (regularities) in the background, which is not possible in existing techniques. The proposed method is robust against deformations and irregularities in the fence texture due to SVM classification used in fence segmentation phase.

The existing near regular lattice detection approaches work well for some images and on the other hand fail for some cases. They have observed that the failure cases are often accompanied by sudden changes of colors in the background and obscuring objects in front of the fence. For examples in (Liuy *et al.*, 2008) method, the lattice detection fails for images (a) and (c) in Figure 6 and for image (q) in Figure 7. The proposed method is successful in detecting fence texture in all those images. A comparison of fence mask detected in Flower image by (Liuy *et al.*, 2008) method and proposed method is given in Figure 8.

However, the proposed method fails to provide satisfactory results for blurred images, especially when the fence is very much blurred. In such cases preprocessing to sharpen the fence may give better results. Furthermore, fence segmentation becomes challenging when the visual similarity between fence pixels and background pixels becomes high. Feature set used for segmentation has to be tuned to overcome such problems. Determining the correct feature set is challenging in such scenarios.



(a) Flower Image. (b) (Liuy *et al.*, 2008). (c) Proposed Method.

**Figure 8.** Comparison of Fence Mask Detected for Flower Image.

## 6. Future Work

Fence texture segmentation becomes challenging, when there are pixels with features similar to fence pixels in the background. SVM classification used for final segmentation of the fence texture in this article can be replaced with descriptive motif pattern generation described in (Peters & Hettiarachichi, 2013). The accuracy of this phase can be further improved with help of near set theory (Peters, 2013; Peters & Naimpally, 2012; Peters, 2014; Peters *et al.*, 2014).

## 7. Conclusion

Fence-like texture present in the foreground of the image occludes the points of interest in an image and is difficult to segment by directly applying conventional frequency filters used for texture analysis. The proposed approach in this article segregates each fence texture by frequency domain processing prior to wavelet transformation and the segmentation is achieved through support vector machine classification.

The proposed method works well for fence texture with different shapes, sizes, colours and orientations. Fence texture detection was successful not only for images having fence in the foreground but also for images having fence in the background.

As a sample application of the proposed approach, removal of fences from zoo animal enclosure images is presented. In addition to this, the proposed approach to de-fencing can be used for any application, where the images are occluded with fence-like texture.

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# Counting Sets of Lattice Points in the Plane with a Given Diameter under the Manhattan and Chebyshev Distances

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**Abstract**

In this paper we present new algorithms for counting the sets of lattice points in the plane whose diameter is a given value $D$, under the Manhattan ($L_1$) and Chebyshev ($L_\infty$) distances. We consider two versions of the problem: counting all sets within a given lattice $U \times V$, and counting all sets that are not equivalent under translations.

*Keywords:* lattice points, Chebyshev distance, Manhattan distance, diameter.
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## 1. Introduction

In this paper we present new algorithms for counting the sets of lattice points in the plane with a given diameter, under the Manhattan ($L_1$) and Chebyshev ($L_\infty$) distances. We consider two versions of the problem. In the first version we assume that a fixed size 2D grid is given and the sets must be placed inside this grid. Two sets are different if they have a different number of points or the positions of their points inside the grid are not all identical. In the second version we assume that two sets are considered identical (and, thus, need to be counted only once) if one can be obtained from another by translation operations.

The rest of this paper is structured as follows. In Section 2 we present the problems in more details, together with some preliminaries required by the algorithms presented in the other sections. In Section 3 we present an algorithm with $O(D \cdot log(D))$ arithmetic operations for the Chebyshev ($L_\infty$) distance which can solve both versions of the problem. In Section 4 we present a more efficient algorithm, with only $O(log(D))$ arithmetic operations, for the Chebyshev distance, but only for the second version of the problem. In Sections 5 and 6 we present algorithms with a similar number of arithmetic operations for the Manhattan distance and for the same versions of

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the problem. In Section 7 we present experimental results regarding the two algorithms for the Manhattan distance. In Section 8 we discuss related work. In Section 9 we conclude and discuss future work.

## 2. Problem Statement and Preliminaries

In this paper we consider sets of lattice points in the plane. A lattice point is a point with integer coordinates. The diameter of a set of points is the maximum distance between any two points in the set. In this paper we will consider two distances. The Manhattan distance (also called the $L_1$ distance) between two points $(x_1, y_1)$ and $(x_2, y_2)$ is defined as $|x_1 - x_2| + |y_1 - y_2|$. The $L_\infty$ distance (also called the Chebyshev distance) between two points $(x_1, y_1)$ and $(x_2, y_2)$ is defined as $max\{|x_1 - x_2|, |y_1 - y_2|\}$. When the coordinates of the points are integer (i.e. when we consider only lattice points) both the $L_1$ and the $L_\infty$ distances are integers.

We consider two versions of the problem for counting sets of lattice points having exactly a given diameter $D$ (under the $L_1$ or $L_\infty$ distances). The first version assumes that a 2D grid of fixed size $U \times V$ is given ($U$ is the number of points along the $OX$ axis and $V$ is the number of points along the $OY$ axis). We may assume that the points of the grid have coordinates $(x, y)$ with $0 \le x \le U - 1$ and $0 \le y \le V - 1$. In this case two sets of points are considered different if they consist of a different number of points or if the positions of their points are not all identical. The dimensions of the grid ($U$ and $V$) are part of the input of the algorithms presented for this version.

For the second version we assume that two sets $A$ and $B$ are identical if one can be obtained from another by translation operations. To be more precise, set $A$ is considered identical to $B$ if there exist the integer numbers $TX$ and $TY$ such that by adding $TX$ to the x-coordinate of each point of $A$ and $TY$ to the y-coordinate of each point of $A$ we obtain exactly the set $B$ (note that this automatically implies that $A$ and $B$ have the same number of points). In this case the sets are not constrained to be located within a fixed size grid, so the parameters $U$ and $V$ from the first version of the problem do not exist here.

We are interested in computing the number of sets of points with a given value of the diameter $D$ ($D \ge 1$) under both versions of the problem and considering either the $L_1$ or the $L_\infty$ distance. Let's consider, for instance, the second version of the problem. For $D = 1$ there are two sets of lattice points for the Manhattan distance, each consisting of two adjacent lattice points. In the first set the two points are horizontally adjacent and in the second set the two points are vertically adjacent. On the other hand, there are 9 sets of lattice points for $D = 1$ and the Chebyshev distance.

In order for a set of points in the plane to have diameter $D$ under the $L_\infty$ distance all the points must be located inside a square of side length $D$ and for at least one pair of opposite sides there must be at least one point from the set located on each of the two sides.

The diameter of a set of points $A$ under the Manhattan distance is equivalent to the diameter under the $L_\infty$ distance of a modified set of points $B$ (Indyk, 2001). $B$ is obtained by transforming each point $(x, y)$ of $A$ into the point $(x - y, x + y)$ in $B$. Thus, the two problems considered in this paper are strongly connected to each other. The transformed coordinates correspond to *diagonal coordinates*.

In a 2D plane we have two types of diagonals: *main diagonals* (running from north-east to south-west) and *secondary diagonals* (running from north-west to south-east). All the points $(x, y)$

on the same *main diagonal* have the same value of $x - y$ and all the points $(x, y)$ on the same *secondary diagonal* have the same value of $x + y$. The *index of a main diagonal* is the difference $x - y$ of all the points $(x, y)$ on it. Similarly, the *index of a secondary diagonal* is the sum $x + y$ of all the points $(x, y)$ on it. The *parity* of a diagonal (main or secondary) is defined as the parity of its index. The distance between two diagonals of the same type (main or secondary) is defined as the absolute difference of their indices.

After the transformation to *diagonal coordinates* we can easily see that in order for a set of points to have exactly diameter $D$ under the Manhattan distance one of the following conditions must hold:

1. It should have at least two points located on main diagonals at distance $D$ apart, while the other pairs of main diagonals of all the points are located at distance at most $D$ apart and the pairs of secondary diagonals of all the points are at distance strictly less than $D$ or
2. It should have at least two points located on secondary diagonals at distance $D$ apart while the other pairs of secondary diagonals of all the points are located at distance at most $D$ apart and the pairs of main diagonals of all the points are at distance strictly less than $D$ or
3. It should have at least two points located on main diagonals at distance $D$ apart and at least two points located on secondary diagonals at distance $D$ apart and all the other pairs of main and secondary diagonals of the points are at distance at most $D$ apart.

The three cases correspond to different types of sets of points (i.e. each set of points having diameter $D$ under the Manhattan distance belongs to exactly one of the three cases). Note that there is a bijection between the sets of points corresponding to cases 1 and 2. Each set of points corresponding to case 1 can be transformed into a set of points corresponding to case 2 (by switching the order of the diagonals). Similarly, each set of points corresponding to case 2 can be transformed into a set of points corresponding to case 1.

Thus, for the second version of the problem, it will be enough to count the number of sets of points corresponding to case 1 ($C_1$) and the number of sets of points corresponding to case 3 ($C_3$) in order to obtain the total number of sets of points having diameter $D$ under the Manhattan distance. Then, the total number of sets of lattice points having diameter $D$ (under the Manhattan distance) is equal to $2 \cdot C_1 + C_3$.

## 3. Algorithm 1 for Counting Sets of Lattice Points of Diameter $D$ under the Chebyshev ($L_\infty$) Distance

In this section we will present an algorithm which computes the number of sets of lattice points having diameter $D$ under the Chebyshev distance for both versions of the problem. The algorithm will make use of a function denoted by $CNTSETS(LX, LY)$ which will compute the number of sets of lattice points contained in a rectangle having side length $LX$ along the $OX$ axis and side length $LY$ along the $OY$ axis and such that each counted set has at least one point on each of the 4 sides of the rectangle. Moreover, the corners of the rectangle are lattice points.

We will start with some simple cases. We have $CNTSETS(0, 0) = 1$ and $CNTSETS(P, 0) = CNTSETS(0, P) = 2^{P-1}$. When both $LX$ and $LY$ are greater than or equal to 1 we will use the following approach. We will first identify the 4 corners of the rectangle. We will consider each of the $2^4$ binary configurations of 4 bits. Let $BC$ denote the current binary configuration and $BC(i)$ will denote bit $i$ in the configuration ($0 \leq i \leq 3$). Each bit will correspond to one of the 4 corners. If $BC(i)$ is 1 we will assume that the corresponding corner is selected to be part of the set; otherwise we will assume that it is not selected. Lets consider now each of the horizontal sides of the rectangle. If at least one corner located on the considered side was selected, then there are $2^{LX-1}$ possibilities left for selecting the remaining points (non-corners) of the horizontal side (note that the side contains $LX + 1$ points overall, out of which 2 are corners). If none of the corners of the side are selected, then there are only $2^{LX-1} - 1$ possibilites left for selecting the remaining points of the horizontal side. The situation is similar for the vertical sides: if at least one corner is selected from a vertical side, there are $2^{LY-1}$ possibilities of selecting the remaining points of the vertical side; otherwise the number of possibilities is only $2^{LY-1} - 1$. After considering all the 4 sides of the rectangle we need to consider the points located strictly inside the rectangle. There are $NIN = (LX - 1) \cdot (LY - 1)$ points located strictly inside the rectangle. Each of these inner points may be selected or not, meaning that there are $2^{NIN}$ possibilities of selecting these points. For a given binary configuration $BC$ the number of possibilities of selecting points according to it is equal to the product of five terms: four of which are the number of possibilities corresponding to each of the 4 sides of the rectangle and the $5^{th}$ term corresponds to the number of possibilities of selecting the inner points of the rectangle. The value returned by $CNTSETS(LX, LY)$ is equal to the sum of the numbers of possibilities of selecting points corresponding to each of the $2^4$ binary configurations.

We will use a variable $C$ ranging from 0 to $D$. For each value of $C$ we will first compute $CNTSETS(D, C)$. Note that this value corresponds to the number of sets of lattice points having diameter $D$ and which are contained in a minimum bounding rectangle of side lengths $D$ (along the $OX$ axis) and $C$ (along the $OY$ axis). For the first version of the problem, each set counted by $CNTSETS(C, D)$ may appear multiple times inside the grid - in fact, it may appear $(U-D)\cdot(V-C)$ times (as that's the number of possibilities of placing a $D \cdot C$ rectangle inside the grid). Thus, we will add the term $CNTSETS(D, C) \cdot (U - D) \cdot (V - C)$ to the final answer for the first version of the problem (or 0, if $D > U$ or $C > V$). In the second version of the problem we simply need to add $CNTSETS(D, C)$ to the final answer for the second version of the problem. This is because all the sets counted by $CNTSETS(D, C)$ are different under translation operations.

If $C < D$ we will also compute $CNTSETS(C, D)$ (which is identical in value to $CNTSETS(D, C)$). For the first version of the problem we will add to the final answer the value $CNTSETS(C, D) \cdot (U - C) \cdot (V - D)$ (or 0, if $C > U$ or $D > V$). For the second version of the problem we will add to the final answer the value $CNTSETS(C, D)$.

The algorithm presented in this section uses $O(D \cdot log(D))$ arithmetic operations, because it considers $O(D)$ cases and for each case it needs to perform a constant number of exponentiations where the base 2 logarithm of the exponent is of the order $O(log(D))$. All the exponentiations raise 2 to a given exponent. If $D$ is not very large we may consider precomputing all the powers of 2 from 0 to $D$ (we may achieve this with only $O(D)$ multiplications because we can write $2^P = 2^{P-1} \cdot 2$ for $P \geq 1$ and we can consider the values of $P$ in ascending order). However, $NIN$ is of the order $O(D^2)$. If $D$ is sufficiently small then we may precompute powers of 2 up to $D^2$ (using $O(D^2)$ multiplications). If, however, $D^2$ is too large, then we need to notice that, as $C$ increases from 0 to $D$, $NIN$ also increases. We will assume that our algorithm considers the values of $C$ in ascending order (note that $NIN$ is the same for both $CNTSETS(D, C)$ and $CNTSETS(C, D)$). Let's assume that $PREVNIN$ is equal to the value of $NIN$ for the case $C - 1$ and $RESPREVNIN = 2^{PREVNIN}$. We will initially (for $C = 0$) have $PREVNIN = 0$ and $RESPREVNIN = 1$. When we need to compute $2^{NIN}$ for a case we will first compute the difference $DIFNIN = NIN - PREVNIN$. We will always have $DIFNIN = D - 1$. Thus, we can compute $2^{NIN}$ with only one multiplication, as $RESPREVNIN \cdot 2^{DIFNIN}$ (note that $2^{DIFNIN}$ is taken from the table of precomputed powers of two). After handling the current value of $C$ we will update $PREVNIN = NIN$ and $RESPREVNIN = 2^{NIN}$ (where $2^{NIN}$ was just computed by the method we presented). Using this approach we only need $O(D)$ arithmetic operations instead of $O(D \cdot log(D))$.

So far we assumed that we want to compute the number of sets of lattice points exactly. In this case we will need to work with numbers which have $O(4 \cdot D)$ bits. However, there are many situations when the exact numbers are not required. For instance, if we are only interested in computing the number of sets modulo a given number $M$, then we only need numbers having $O(2 \cdot log(M))$ bits for storing intermediate and final results. If $M$ is sufficiently small (e.g. a 32-bit number) then we can practically assume that on the current machine architectures the numbers we use have a constant number of bits. However, the exponents to which 2 is raised can still be pretty large numbers (having $O(log(D))$ bits). This may not necessarily be a problem, but we may inadvertently face some challenging algorithmic problems. For instance, when multiplying $(LX - 1)$ by $(LY - 1)$ in the $CNTSETS$ function we need to multiply together two numbers having $O(log(D))$ bits. The naive algorithm would use $O(log^2(D))$ time for computing the result. In order to speed up the multiplication we may need to use more complicated algorithms (Schonhage & Strassen, 1971), (Furer, 2009) which reduce the time complexity to $O(log(D) \cdot log(log(D)) \cdot log(log(log(D))))$ or slightly better. Nevertheless, there is a simple situation when all these complications are not needed: when $M$ is an odd prime. In this case we know that $A^{M-1} = 1$ (modulo $M$) for any natural number $1 \leq A \leq M - 1$. Since we only need to raise 2 at some powers (modulo $M$), we notice that we only need the remainder of the exponent when divided by $M - 1$ in order to compute the required result. Thus, instead of using exact exponents we will only use the exponents modulo $M - 1$. This way we can avoid the complicated multiplication of $(LX - 1)$ by $(LY - 1)$ and replace it with the multiplication of *((LX-1) mod (M-1))* by *((LY-1) mod (M-1))*. This way we will need to spend $O(log(D))$ time in order to compute the remainders of numbers having $O(log(D))$ bits when divided by $M - 1$, but we do not need to multiply together two large numbers.

## 4. Algorithm 2 for Counting Sets of Lattice Points of Diameter $D$ under the Chebyshev ($L_\infty$) Distance

The algorithm presented in this section can only solve the second version of the problem (i.e. when two sets are identical if one can be obtained from another by using translation operations). We will first define the following function: $NSETS(LX, LY)$=the number of sets of lattice points contained in a rectangle of horizontal side length $LX$ and vertical side length $LY$ such that at least one point is located on each of the opposite vertical sides (for this function we will ignore the fact the two sets are identical if one can be obtained from another by translation operations). We assume $LX \geq 1$ and $LY \geq 0$, both numbers are integers and the corners of the rectangle are lattice points. Such a rectangle contains $(LX + 1) \cdot (LY + 1)$ lattice points inside of it or on its borders. It is easy to see that $NSETS(LX, LY) = (2^{LY+1} - 1)^2 \cdot 2^{(LX+1)\cdot(LY+1)-2\cdot(LY+1)}$. This formula corresponds to the following cases. On each of the two opposite vertical sides we must have one selected point. Thus, there are $2^{LY+1} - 1$ possibilities of choosing lattice points on each of these two sides. Each of the remaining $(LX + 1) \cdot (LY + 1) - 2 \cdot (LY + 1)$ lattice points can be selected or not to be part of the set. Thus, we have $2^{(LX+1)\cdot(LY+1)-2\cdot(LY+1)}$ possibilities for selecting these points. If $LY < 0$, by definition, we will have $NSETS(LX, LY) = 0$.

In order for a set of points in the plane to have diameter $D$ under the Chebyshev distance all the points must be located inside a square of side length $D$, such that at least one pair of opposite sides has at least one point from the set on each side from the pair. We will consider three cases:

1. both of the vertical opposite sides of the square contain points from the set on them, but not both horizontal sides of the square contain points from the set: the number of sets corresponding to this case is $NSETS(D, D-1) - NSETS(D, D-2)$ (this forces every set to have a point selected on the bottom side of the square of side length $D$)
2. both of the horizontal opposite sides of the square contain points from the set on them, but not both vertical sides of the square contain points from the set: the number of sets corresponding to this case is also $NSETS(D, D - 1) - NSETS(D, D - 2)$.
3. both of the horizontal opposite sides and both of the vertical opposite sides of the square contain points from the set on them: the number of sets corresponding to this case is $NSETS(D, D) - 2 \cdot NSETS(D, D - 1) + NSETS(D, D - 2)$. We actually made use of the inclusion-exclusion principle here. From all the sets of lattice points with points on both opposite vertical sides ($NSETS(D, D)$) we subtracted the sets of lattice points which do not have points on the top or bottom horizontal side ($2 \cdot NSETS(D, D - 1)$). In doing this we over-subtracted the sets of lattice points which do not have points on any of the horizontal sides ($NSETS(D, D - 2)$) thus, we need to add this number back.

By adding together the numbers corresponding to the cases 1, 2 and 3, we obtain the total number of sets of lattice points having diameter $D$ under the Chebyshev distance: $2 \cdot (NSETS(D, D - 1) - NSETS(D, D - 2)) + NSETS(D, D) - 2 \cdot NSETS(D, D - 1) + NSETS(D, D - 2)$, which simplifies to $NSETS(D, D) - NSETS(D, D - 2)$.

This method requires $O(log(D^2)) = O(log(D))$ arithmetic operations in order to compute the answer (this number corresponds to raising 2 to a power whose value is of the order $O(D^2)$). In case exact results are not needed, the same discussion from the previous section applies to this case, too, because in the $NSETS$ function we need to multiply two numbers of $O(log(D))$ bits each: $(LX + 1)$ and $(LY + 1)$.

## 5. Algorithm 1 for Counting Sets of Lattice Points of Diameter $D$ under the Manhattan ($L_1$) Distance

In this section we present an algorithm similar in essence to the one from section 3. The algorithm can compute the number of sets of lattice points having diameter $D$ under the Manhattan distance for both versions of the problem. The algorithm will make use of a function $CNTEQ(C, X)$ which computes the number of sets of lattice points such that:

- the main diagonals of at least two points are at distance exactly $D$ apart

- all the other pairs of main diagonals of the points are at distance at most $D$ apart
- the secondary diagonals of at least two points are at distance exactly $C$ apart
- all the other pairs of secondary diagonals of the points are at distance at most $C$ apart
- $X = 0$ means that the parity of the first secondary diagonal is equal to the parity of the first main diagonal, while $X = 1$ means that these parities differ (the first diagonal of each type is the one with the smallest index)

The algorithm will simply iterate through all the values of $C$ (from 0 to $D$), and for each value of $C$, through all the values of $X$ (from 0 to 1).

For the first version of the problem we will need to compute the minimum bounding rectangle for the sets counted by $CNTEQ(C, X)$ ($X = 0, 1$). Let's assume that the minimum bounding rectangle has side length $MBRX$ along the $OX$ axis and $MBRY$ along the $OY$ axis. We will add to the final answer the value $CNTEQ(C, X)\cdot(U - MBRX)\cdot(V - MBRY)$ ($X = 0, 1$), or 0 if $MBRX > U$ or $MBRY > V$. If $C < D$ then we have a set of symmetric sets of lattice points by switching the role of main and secondary diagonals. These sets have a minimum bounding rectangle with side length $MBRY$ along the $OX$ axis and $MBRX$ along the $OY$ axis. Thus, we will also add to the final answer the value $CNTEQ(C, X) \cdot (U - MBRY) \cdot (V - MBRX)$ ($X = 0, 1$), or 0 if $MBRX > V$ or $MBRY > U$.

For the second version of the problem $C_1$ will be equal to the sum of the values $CNTEQ(C, X)$ ($0 \le C \le D - 1$, $0 \le X \le 1$) and $C_3$ will be equal to $CNTEQ(D, 0) + CNTEQ(D, 1)$.

When computing $CNTEQ(C, *)$, we need to consider a figure containing lattice points enclosed by a pair of main diagonals at distance $D$ and a pair of secondary diagonals at distance $C$. We will denote the first main diagonal as the *left* diagonal, the second main diagonal as the *right* diagonal, the first secondary diagonal as the *bottom* diagonal and the second secondary diagonal as the *top* diagonal. We will need to compute the following numbers:

- NLEFT=the number of lattice points on the left diagonal of the figure
- NRIGHT=the number of lattice points on the right diagonal of the figure
- NUP=the number of lattice points on the top diagonal of the figure
- NDOWN=the number of lattice points on the bottom diagonal of the figure
- NTOTAL=the total number of lattice points inside the figure and on its borders

Then, we will need to identify the corners of the figure. A corner is a lattice point which belongs to two adjacent diagonals (a main diagonal and a secondary diagonal). Note that we may have 0, 2 or 4 corners. Let's assume that we have $NC$ corners. We will make sure to decrease the corresponding numbers ($NLEFT$, $NRIGHT$, $NUP$, $NDOWN$) by the number of corners among the set of lattice points which were counted (e.g. if the left diagonal has $Q$ corners on it, we will decrease $NLEFT$ by $Q$).

In Fig. 1, 2, 3, 4 we present all the cases which may occur during the computation of the $CNTEQ(C, X)$ function (the remaining cases are reducible to these 4 cases by symmetry). Lattice points on the main diagonals are drawn in green, lattice points on the secondary diagonals are drawn in red, corners are drawn in cyan and inner lattice points are drawn in yellow. In Fig. 1 we have $D = 10$, $C = 8$ and $X = 0$. Notice that we obtain $NC = 4$ corners. Note that two adjacent diagonals form a corner if they have the same parity. In Fig. 2 we have $D = 10$, $C = 7$ and $X = 0$. In this case we obtain only $NC = 2$ corners. This is because the top diagonal has a different parity from both the left and the right diagonals, thus forming no corners with them. In Fig. 3 we have $D = 9$, $C = 7$ and $X = 0$; we obtain $NC = 2$ corners. In Fig. 4 we have $D = 12$, $C = 8$ and $X = 1$; no corner is formed in this case.

The main algorithm for computing $CNTEQ(C, X)$ is as follows. We will consider each possible binary configuration of $NC$ bits. If bit $i$ $(0 \leq i \leq NC - 1)$ is set to 1 we will assume that the corresponding corner ($i$) belongs to the set of lattice points; otherwise, it doesn't belong to the set. After deciding the states of the corners we will check which of the first and second main and secondary diagonals have no selected corners on them. For each such diagonal we will have $2^{NP} - 1$ possibilities of choosing lattice points on it (where $NP$ is the number of lattice points on it, excluding the corners). This equation makes sure that at least one lattice point is selected on each such diagonal. For each of the other diagonals we will have $2^{NP}$ possibilities of choosing lattice points on them (for these diagonals it is possible to not select any of the lattice points on them, because they already have a selected corner). Then each of the interior lattice points of the figure can be selected as part of the set or not (there are $NIN = NTOTAL - (NUP + NDOWN + NLEFT + NRIGHT + NC)$ lattice points inside the figure and, thus, there are $2^{NIN}$ possibilities of choosing the inner points). The answer for each binary configuration of the corners is the product between the number of possibilities for each of the 4 diagonals and for the inner points of the figure. $CNTEQ(C, X)$ is the sum of all the answers for each binary configuration of corners. Note that this algorithm works even when $NC = 0$ (there is one binary configuration of 0 bits).

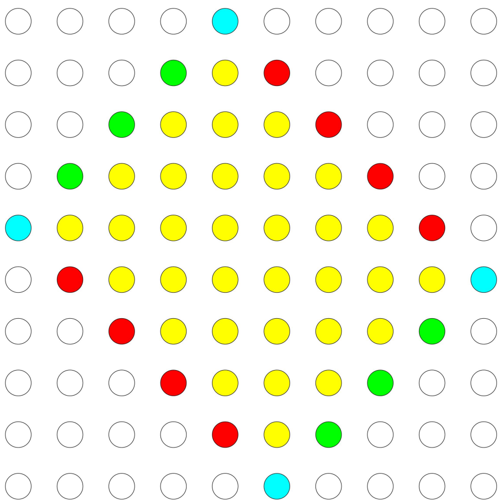

**Figure 1.** D=10, C=8, X=0, NC=4.

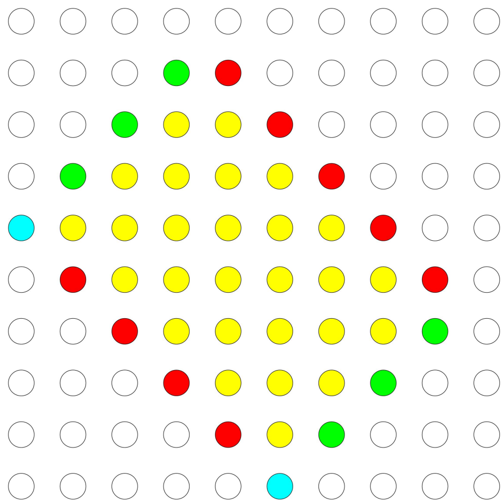

**Figure 2.** D=10, C=7, X=0, NC=2.

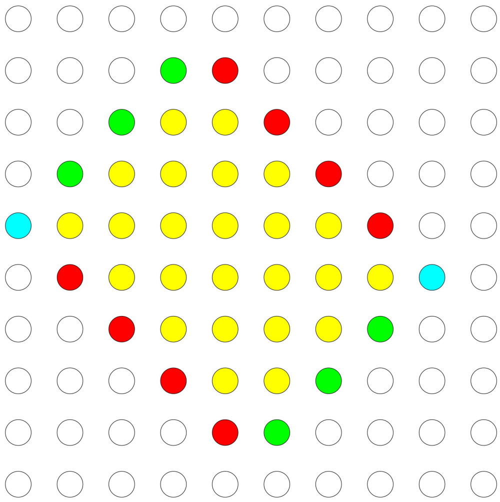

**Figure 3.** D=9, C=7, X=0, NC=2.

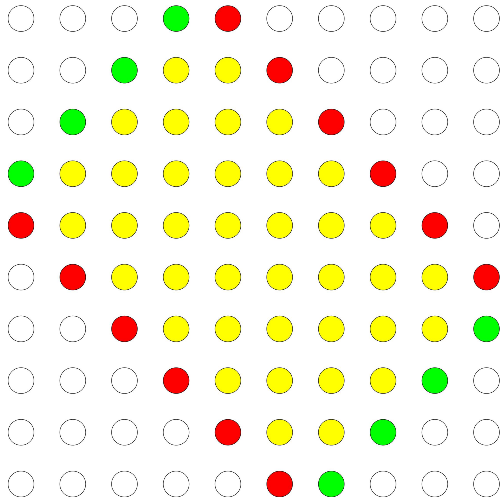

**Figure 4.** D=12, C=8, X=1, NC=0.

What is left is to identify the values $NUP$, $NDOWN$, $NLEFT$, $NRIGHT$, $NTOTAL$ and the corners depending on the values of $C$, $X$ and the parity of $D$. We will also define the parameter $Y$, which is defined similarly as $X$, but for the second secondary diagonal (i.e. $Y = 0$ if the second secondary diagonal has the same parity as the first main diagonal, and $Y = 1$ otherwise). Note that $Y = X$ if $C$ is even, and $Y = 1 - X$ if $C$ is odd. From now on we will assume that the value of $Y$ is computed when evaluating the function $CNTEQ(C, X)$.

We will first consider the case when $D$ is even. If $C = 0$ and $X = 0$ then $CNTEQ(0, 0) = 2^{D/2-1}$. If $C = 0$ and $X = 1$ then $CNTEQ(0, 1) = 0$. Let's consider now that $C \geq 1$. If $X = 0$ then $NDOWN = (D/2) + 1$ and if $X = 1$ then $NDOWN = D/2$. Note that whenever we use the division operator "/" in this paper we refer to integer division. Similarly, if $Y = 0$ then $NUP = (D/2) + 1$, and if $Y = 1$ then $NUP = D/2$. If $X = 0$ then we have $NLEFT = NRIGHT = (C/2) + 1$; otherwise, if $X = 1$ then we have $NLEFT = NRIGHT = (C + 1)/2$. $NTOTAL$ is equal to $NLEFT \cdot ((D/2) + 1) + (C + 1 - NLEFT) \cdot (D/2)$.

If $D$ is odd then we have the following values. $NUP = NDOWN = (D + 1)/2$. If $X = 0$ then $NLEFT = (C/2) + 1$ and $NRIGHT = (C + 1)/2$; otherwise, if $X = 1$ then $NLEFT = (C + 1)/2$ and $NRIGHT = (C/2) + 1$. $NTOTAL$ is equal to $(C + 1) \cdot ((D + 1)/2)$.

The exact formulas we presented for $NUP$, $NDOWN$, $NLEFT$, $NRIGHT$ and $NTOTAL$ can be easily derived by a careful analysis of all the relevant cases. Let's consider now the cases from Fig. 1, 2, 3, 4 and verify the formulas for those cases. In Fig. 1 we have $NLEFT = (4/2) + 1 = 5$, $NRIGHT = (4/2) + 1 = 5$, $NDOWN = (10/2) + 1 = 6$, $NUP = (10/2) + 1 = 6$ and $NTOTAL = 5 \cdot ((10/2) + 1) + (8 + 1 - 5) \cdot (10/2) = 50$. In Fig. 2 we have $NLEFT = (7/2) + 1 = 4$, $NRIGHT = (7/2) + 1 = 4$, $NDOWN = (10/2) + 1 = 6$, $NUP = 10/2 = 5$ and $NTOTAL = 4 \cdot ((10/2) + 1) + (7 + 1 - 4) \cdot (10/2) = 44$. In Fig. 3 we have $NLEFT = (7/2) + 1 = 4$, $NRIGHT = (7 + 1)/2 = 4$, $NDOWN = (9 + 1)/2 = 5$, $NUP = (9 + 1)/2 = 5$ and $NTOTAL = (7 + 1) \cdot ((9 + 1)/2) = 40$. In Fig. 4 we have $NLEFT = (8 + 1)/2 = 4$, $NRIGHT = (8 + 1)/2 = 4$, $NDOWN = 12/2 = 6$, $NUP = 12/2 = 6$ and $NTOTAL = 4 \cdot ((12/2) + 1) + (8 + 1 - 4) \cdot (12/2) = 58$.

We will show now how to compute the sizes $MBRX$ and $MBRY$ of the minimum bounding rectangle corresponding to the sets counted by $CNTEQ(C, X)$ ($X = 0, 1$). $MBRX = NLEFT - 1 + NUP - 1 + Y$ and $MBRY = NLEFT - 1 + NDOWN - 1 + X$. Let's verify now these formulas for the cases presented in Fig. 1, 2, 3, 4. In Fig. 1 we have $MBRX = 5 - 1 + 6 - 1 + 0 = 9$ and $MBRY = 5 - 1 + 6 - 1 + 0 = 9$. In Fig. 2 we have $MBRX = 4 - 1 + 5 - 1 + 1 = 8$ and $MBRY = 4 - 1 + 6 - 1 + 0 = 8$. In Fig. 3 we have $MBRX = 4 - 1 + 5 - 1 + 1 = 8$ and $MBRY = 4 - 1 + 5 - 1 + 0 = 7$. In Fig. 4 we have $MBRX = 4 - 1 + 6 - 1 + 1 = 9$ and $MBRY = 4 - 1 + 6 - 1 + 1 = 9$.

After initializing the $NUP$, $NDOWN$, $NLEFT$, $NRIGHT$ and $NTOTAL$ values, we need to identify the corners. We will consider each pair of (main diagonal, secondary diagonal) and check if they have the same parity (note that the parity of each main and secondary diagonal can be uniquely determined relative to the parity of the first main diagonal from the values $X$, $Y$ and $D$; for instance, if $X = 0$ the first main diagonal and the first secondary diagonal have the same parity, if $Y = 0$ the first main diagonal and the second secondary diagonal have the same parity, if ($X = 0$) and ($D$ is even) the second main diagonal and the first secondary diagonal have the same parity, if ($Y = 0$) and ($D$ is even) the second main diagonal and the second secondary diagonal have the same parity). Whenever a main diagonal and a secondary diagonal have the same parity, they form a corner. Whenever a corner is identified, the number of lattice points corresponding to the two diagonals is decremented by 1. For instance, if the first main diagonal and the first secondary diagonal form a corner then $NLEFT$ and $NDOWN$ are both decremented by 1. If the first main diagonal and the second secondary diagonal form a corner then both $NLEFT$ and $NUP$ are decremented by 1. If the second main diagonal and the first secondary diagonal form a corner then $NRIGHT$ and $NDOWN$ are both decremented by 1. If the second main diagonal and the second secondary diagonal form a corner then both $NRIGHT$ and $NUP$ are decremented by 1. $NC$ is set to the number of identified corners and the corners are placed in an array on positions 0 to $NC - 1$, so that we know exactly to which diagonals each corner $i$ ($0 \le i \le NC - 1$) belongs to.

The algorithm presented in this section uses $O(D \cdot log(D))$ arithmetic operations, because it considers $O(D)$ cases and for each case it needs to perform a constant number of exponentiations where the base 2 logarithm of the exponent is of the order $O(log(D))$. In order to reduce the number of arithmetic operations to $O(D)$ we can use the same approach as in section 3. We will assume that our algorithm considers the values of $C$ in ascending order and for each value of $C$ it first computes $CNTEQ(C, 0)$ and then $CNTEQ(C, 1)$. Let's assume that $PREVNIN$ is equal to the value of $NIN$ for the case $C - 1$ and $X = 1$ and $RESPREVNIN = 2^{PREVNIN}$. We will initially have $PREVNIN = 0$ and $RESPREVNIN = 1$. When we need to compute $2^{NIN}$ for a case we will first compute the difference $DIFNIN = NIN - PREVNIN$. We will always have $0 \le DIFNIN \le D$. Thus, we can compute $2^{NIN}$ with only one multiplication, as $RESPREVNIN \cdot 2^{DIFNIN}$ (note that $2^{DIFNIN}$ is taken from the table of precomputed powers of two). After handling the case ($C, X = 1$) we will update $PREVNIN = NIN$ and $RESPREVNIN = 2^{NIN}$ (where $2^{NIN}$ was just computed by the method we presented). Using this approach we only need $O(D)$ arithmetic operations instead of $O(D \cdot log(D))$. The same discussion as in Section 3, regarding the computation of exact numbers or of numbers modulo a given number $M$, applies here, too. In this case $NTOTAL$ is the value of order $O(D^2)$ which is obtained by multiplying together two numbers which are of the order $O(D)$.

## 6. Algorithm 2 for Counting Sets of Lattice Points of Diameter $D$ under the Manhattan ($L_1$) Distance

Our second algorithm for the Manhattan distance (and only for the second version of the problem) will use a function $CNTLEQ(C, X)$, where $C \leq D$ and $X = 0$ or 1. $CNTLEQ(C, X)$ computes the number of sets of lattice points such that:

- the main diagonals of at least two points are at distance exactly $D$ apart
- all the other pairs of main diagonals of the points are at distance at most $D$ apart
- all the pairs of secondary diagonals of the points are at distance at most $C$ apart
- $X = 0$ means that the parity of the first secondary diagonal is equal to the parity of the first main diagonal, while $X = 1$ means that these parities differ (the first diagonal of each type is the one with the smallest index)

For $C < 0$ we have $CNTLEQ(C, 0) = CNTLEQ(C, 1) = 0$, by definition. We will present solutions for $C \geq 0$ depending on the parity of $D$.

We will first consider the case when $D$ is even. In this case we have $CNTLEQ(0, 0) = 2^{D/2-1}$ and $CNTLEQ(0, 1) = 0$ (note that every time we use division we consider integer division). Let's consider now the case $C \geq 1$. All the points must be contained between two main diagonals located at distance $D$ apart and between two secondary diagonals located at distance $C$ apart.

For $X = 0$ this figure has $P = (C/2) + 1$ lattice points on each of the main diagonals and has $R = ((C/2) + 1) \cdot ((D/2) + 1) + (C - (C/2)) \cdot (D/2)$ lattice points in total inside of it and on its borders. $CNTLEQ(C, 0)$ is equal to $(2^P - 1)^2 \cdot 2^{R-2 \cdot P}$.

For $X = 1$ the figure has $P = (C + 1)/2$ lattice points on each of the main diagonals and has $R = ((C + 1)/2) \cdot ((D/2) + 1) + (C + 1 - (C + 1)/2) \cdot (D/2)$ lattice points in total inside of it and on its borders. $CNTLEQ(C, 1)$ is defined identically as $CNTLEQ(C, 0)$, except that we use these new values for $P$ and $R$.

Let's consider now the case when $D$ is odd. We have $CNTLEQ(C, 0) = CNTLEQ(C, 1)$. The figure defined by the main and secondary diagonals has $P = (C/2) + 1$ lattice points on the first main diagonal and $Q = (C + 1)/2$ lattice points on the second main diagonal. In total, the figure contains $R = (C + 1) \cdot ((D + 1)/2)$ lattice points inside of it and on its borders. We have $CNTLEQ(C, 0) = CNTLEQ(C, 1) = (2^P - 1) \cdot (2^Q - 1) \cdot 2^{R-P-Q}$.

Note that the $CNTLEQ$ function ignores the fact that two sets are identical if one can be obtained from another by translation operations. Instead, it considers two sets to be different if they correspond to different subsets of points belonging to the figure. However, this aspect will be considered when deriving the final formula for the number of sets of lattice points with a given diameter, by using the inclusion-exclusion principle.

The total number of sets of lattice points corresponding to case 1 is equal to $C_1 = CNTLEQ(D-1, 0) + CNTLEQ(D-1, 1) - CNTLEQ(D-2, 0) - CNTLEQ(D-2, 1)$. The total number of sets of lattice points corresponding to case 3 is equal to $C_3 = (CNTLEQ(D, 0) - CNTLEQ(D-1, 0) - CNTLEQ(D-1, 1) + CNTLEQ(D-2, 1)) + (CNTLEQ(D, 1) - CNTLEQ(D-1, 0) - CNTLEQ(D-$

$1,1) + CNTLEQ(D-2,0))$. Again we made use of the inclusion-exclusion principle when computing $C_1$ and $C_3$.

It is easy to see that this algorithm uses $O(log(D))$ arithmetic operations (from a constant number of exponentiations where the base 2 logarithm of the exponent is of the order $O(log(D))$). The same discussion as in Section 3, regarding the computation of exact numbers or of numbers modulo a given number $M$, applies to this case, too. In this case $R$ is the value which is obtained by multiplying together two numbers having $O(log(D))$ bits each.

## 7. Experimental Results

We implemented the two algorithms for the Manhattan distance and the second version of the studied problem, presented in Sections 5 and 6. For the algorithm from Section 5 we used its $O(D)$ optimized version. We computed the values modulo a prime number $M = 10^9 + 7$, in order to make use of all the computation optimizations possible. We also implemented a backtracking algorithm which generates every set independently (i.e. it enumerates all the valid sets of lattice points having diameter $D$). We used several values of $D$ in order to compare the running times of the three algorithms. Note that for some values of $D$ some of the algorithms were too slow and we stopped them after a running time of 5 minutes. The running times are presented in Table 1 (a ”-” is shown where the running time exceeded the 5 minutes threshold). All the three algorithms were implemented in C/C++ and the code was compiled using the G++ compiler version 3.3.1. The tests were run on a machine running Windows 7 with an Intel Atom N450 1.66 GHz CPU and 1 GB RAM.

As expected, the $O(log(D))$ algorithm is much faster than the other two algorithms. The $O(D)$ algorithm is faster for odd values of $D$ than for even values. This is because, when $D$ is odd, we can never obtain a figure with 4 corners (in order to have 4 corners both the main and secondary diagonals would need to have the same parity, but when $D$ is odd the main diagonals have different parities).

## 8. Related Work

There is a large body of work in the scientific literature concerned with counting lattice points in various multidimensional structures. In (Loera, 2005) the general problem of counting lattice points in polytopes was considered. The general problem of counting lattice points in a bounded subset of the Euclidean space was considered in (Widmer, 2012). Harmonic analysis is applied in (Chamizo, 2008) for counting lattice points in large parts of space.

A problem concerned with counting configurations of lattice points obtained when translating a convex set in the plane was considered in (Huxley & Zunic, 2009), (Huxley & Zunic, 2013). Two configurations were considered identical under similar conditions as the ones used in this paper. Counting arrangements of connected polyominoes (equivalent under translation) and other figures was considered in (Rechnitzer, 2000). The problem of counting directed lattice walkers in horizontal strips of finite width was considered in (Chan & Guttman, 2003). Counting lattice triangulations was studied in (Keibel & Ziegler, 2003).

As far as we are aware, the problems we considered in this paper have not been considered before in any other publication.

**Table 1.** Running time (in sec) of the three algorithms for several values of D.

| D | Backtracking Algorithm | O(D) Algorithm | O(log(D)) Algorithm |
|---|---|---|---|
| 1 | 0.002 | 0.002 | 0.002 |
| 2 | 0.002 | 0.002 | 0.002 |
| 3 | 0.004 | 0.002 | 0.002 |
| 4 | 0.065 | 0.002 | 0.002 |
| 5 | 3.91 | 0.002 | 0.002 |
| 6 | - | 0.002 | 0.002 |
| 10 | - | 0.002 | 0.002 |
| 11 | - | 0.002 | 0.002 |
| $10^4$ | - | 0.09 | 0.003 |
| $10^4 + 1$ | - | 0.08 | 0.003 |
| $10^5$ | - | 0.81 | 0.003 |
| $10^5 + 1$ | - | 0.54 | 0.003 |
| $10^6$ | - | 7.84 | 0.003 |
| $10^6 + 1$ | - | 5.2 | 0.003 |
| $10^7$ | - | 78.3 | 0.003 |
| $10^7 + 1$ | - | 51.9 | 0.003 |
| $10^8$ | - | - | 0.003 |
| $10^8 + 1$ | - | - | 0.003 |
| $10^9$ | - | - | 0.004 |
| $10^9 + 1$ | - | - | 0.004 |

## 9. Conclusions

In this paper we presented novel, efficient algorithms for computing the number of sets of lattice points in the plane whose diameter is exactly equal to $D$, when considering the Manhattan ($L_1$) or the Chebyshev ($L_\infty$) distance. We considered two versions for defining the equivalence of two such sets of lattice points. The first version forces the sets of points to be fully included inside a given 2D grid. The second version defines two sets of lattice points to be equivalent if one can be obtained from another by using translation operations. Our algorithms require $O(D \cdot log(D))$ or $O(D)$ arithmetic operations (additions, multiplications) for the first version of the problem and only $O(log(D))$ arithmetic operations for the second version of the problem for both distances. We also discussed the possibility of computing the results modulo a given number $M$, as a way of simplifying some parts of the algorithms (in particular, in order to use numbers with a number of bits independent of $D$).

As future work we intend to approach the same problems described in this paper but for a number of dimensions greater than 2. Note that in the 1D case the two problems are identical and very simple to solve (for instance, the answer is always $2^{D-1}$ for the second version of the problem, because we must have two points in the set at distance $D$ and all the other $D - 1$ points between them may be selected or not to be part of the set).

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# Proximate Growth and Best Approximation in $L^p$-norm of Entire Functions

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## Abstract

Let $0 < p \leq +\infty$ and $V_K = \sup\left\{\frac{1}{d}\ln|P_d|, P_d \quad \text{polynomial of degree} \leq d, \|P_d\|_K \leq 1\right\}$ the Siciak extremal function of a $L$-regular compact $K$. The aim of this paper is the characterization of the proximate growth of entire functions of several complex variables by means of the best polynomial approximation in $L_p$-norm on a L-regular compact $K$.

*Keywords:* Extremal function, L-regular, proximate growth, best approximation of entire function, $L^p$-norm.
*2010 MSC:* Primary 30E10; Secondary 41A21, 32E30.

## 1. Introduction

The classical growth have been characterized in term of approximation errors for a function continuous on $[-1, 1]$ by A.R. Reddy(see (Reddy, 1972*a*), and a compact $K$ of positive capacity by T. Winiarski (see (Winiarski, 1970) with respect to maximum norm. For a nonconstant entire function $f(z) = \sum_{k=0}^{+\infty} a_k.z^{\lambda_k}$ and $M(f, r) = \max_{|z|=r} |f(z)|$, it is well known that the function $r \to \log(M(f, r))$ is indefinitely increasing convex function of $\log(r)$. To estimate the growth of $f$ precisely, R.P. Boas, (see (Boas, 1954)), has introduced the concept of order, defined by the number $\rho$ $(0 \leq \rho \leq +\infty)$:

$$\rho = \limsup_{r\to+\infty} \frac{\log\log(M(f, r))}{\log(r)}.$$

The concept of type has been introduced to determine the relative growth of two functions of same nonzero finite order. An entire function, of order $\rho$ $(0 < \rho < +\infty)$, is said to be of type $\sigma$ $(0 \leq \sigma \leq +\infty)$ if

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$\sigma = \limsup_{r\to+\infty} \dfrac{\log(M(f,r))}{r^\rho}$.

If $f$ is an entire function of infinite or zero order, the definition of type is not valid and the growth of such function cannot be precisely measured by the above concept. However S.K. Bajpai, O.P. Juneja and G.P. Kapoor (see (Bajpai *et al.*, 1976)) have introduced the concept of index-pair of an entire function. Thus, for $p \geq q \geq 1$, they defined also the number

$$\rho(p,q) = \limsup_{r\to+\infty} \frac{\log^{[p]}(M(f,r))}{\log^{[q]}(r)}$$

$b \leq \rho(p,q) \leq +\infty$ where $b = 0$ if $p > q$ and $b = 1$ if $p = q$.

The function $f$ is said to be of index-pair $(p,q)$ if $\rho(p-1,q-1)$ is nonzero finite number. The number $\rho(p,q)$ is called the $(p,q)$-order of $f$.

S.K. Bajpai, O.P. Juneja and G.P. Kapoor defined also the concept of the $(p,q)$-type $\sigma(p,q)$, for $b < \rho(p,q) < +\infty$, by

$$\sigma(p,q) = \limsup_{r\to+\infty} \frac{\log^{[p-1]}((M(f,r)))}{\left(\log^{[q-1]}(r)\right)^{\rho(p,q)}}$$

In their works, the authors established the relationship of $(p,q)$-growth of $f$ in term of the coefficients $a_k$ in the Maclaurin series of $f$.

We have also many results in terms of polynomial approximation in classical case. Let $K$ be a compact subset of the complex plane $\mathbb{C}$, of positive logarithmic capacity and $f$ be a complex function defined and bounded on $K$. For $k \in \mathbb{N}$ put

$$E_k(K,f) = \left\|f - T_k\right\|_K$$

where the norm $\left\|.\right\|_K$ is the maximum on $K$ and $T_k$ is the $k-th$ Chebytchev polynomial of the best approximation to $f$ on $K$.

S.N. Bernstein showed (see (Bernstein, 1926), p. 14), for $K = [-1,1]$, that there exists a constant $\rho > 0$ such that

$$\lim_{k\to+\infty} k^{1/\rho}\sqrt[k]{K_k(K,f)}$$

is finite, if and only if, $f$ is the restriction to $K$ of an entire function of order $\rho$ and some finite type.

This result has been generalized by A.R. Reddy (see (Reddy, 1972*a*) and (Reddy, 1972*b*)) as follows:

$$\lim_{k\to+\infty} \sqrt[k]{E_k(K,f)} = (\rho.e.\sigma)2^{-\rho}$$

if and only if $f$ is the restriction to $K$ of an entire function $g$ of order $\rho$ and type $\sigma$ for $K = [-1,1]$.

In the same way T. Winiarski (see (Winiarski, 1970)) generalized this result for a compact $K$ of the complex plane $\mathbb{C}$, of positive logarithmic capacity noted $c = cap(K)$ as follows:

If $K$ be a compact subset of the complex plane $\mathbb{C}$, of positive logarithmic capacity then

$$\lim_{k\to+\infty} k^{\frac{1}{\rho}}\sqrt[k]{E_k(K,f)} = c(e\rho\sigma)^{\dfrac{1}{\rho}}$$

if and only if $f$ is the restriction to $K$ of an entire function of order $\rho$ $(0 < \rho < +\infty)$ and type $\sigma$.

Recall that the capacity of $[-1, 1]$ is $cap([-1, 1]) = \frac{1}{2}$ and the capacity of a unit disc is $cap(D(O, 1)) = 1$.

The authors considered respectively the Taylor development of $f$ with respect to the sequence $(z_n)_n$ and the development of $f$ with respect to the sequence $(W_n)_n$ defined by

$$W_n(z) = \prod_{j=1}^{j=n} (z - \eta_{nj}), \quad n = 1, 2, ...$$

where $\eta^{(n)} = (\eta_{n0}, \eta_{n1}, ..., \eta_{nn})$ is the $n - th$ extremal points system of $K$ (see (Winiarski, 1970), p. 260). We remark that the above results suggest that rate at which the sequence $\left(\sqrt[k]{K_k(K, f)}\right)_k$ tends to zero depends on the growth of the entire function (order and type). For a compact $K$ the Siciak's extremal function of $K$ (see (Siciak, 1962) and (Siciak, 1981)) is defined by:

$$V_K = \sup\left\{\frac{1}{d} \log |P_d|, P_d \text{ polynomial of degree} \leq d, \|P_d\|_K \leq 1\right\}.$$

It is known that the regularity of a compact $K$ (we say $K$ is L-regular) is equivalent to the continuity of $V_K$ in $\mathbb{C}^n$.

Let $K$ be a compact L-regular of $\mathbb{C}^n$. For an entire function $f$ in $\mathbb{C}^n$ developed according an extremal polynomial basis $(A_k)_k$ (see (Zeriahi, 1983)), M. Harfaoui (see (Harfaoui, 2010) and (Harfaoui, 2011)) generalized growth in term of coefficients with respect the sequence $(A_k)_k$. The growth used by M. Harfoui was defined according to the functions $\alpha$ and $\beta$ (see (Harfaoui, 2010), pp. 5, eq. (2.14)), with respect to the set:

$$\Omega_r = \left\{z \in \mathbb{C}^n,\ exp(V_K)(z) < r\right\}.$$

M. Harfaoui (see (Harfaoui, 2010) and (Harfaoui, 2011)) obtained a result of generalized order and generalized type ($(\alpha,\beta)$-order and $(\alpha,\beta)$-type) in term of approximation in $L^p$-norm for a compact of $\mathbb{C}^n$. Later M. Harfaoui an M. El Kadiri (see (Kadiri & Harfaoui, 2013)) obtained the results in term of $(p, q)$-order and $(p, q)$-type for the entire functions .

These results will bee used to establish the generalized growth in terms of best approximation in $L_p$-norm for $p \geq 1$.

Let $f$ be a function defined and bounded on $K$. For $k \in \mathbb{N}$ put

$$\pi_k^p(K, f) = \inf\left\{\left\|f - P\right\|_{L^p(K,\mu)},\ P \in \mathcal{P}_k(\mathbb{C}^n)\right\},$$

where $\mathcal{P}_k(\mathbb{C}^n)$ is the family of all polynomials of degree $\leq k$ and $\mu$ the well-selected measure (The equilibrium measure $\mu = (dd^c V_K)^n$ associated to a $L$-regular compact $K$) (see (Zeriahi, 1987)) and $L^p(K, \mu)$, $p \geq 1$, is the class of all functions such that:

$$\left\|f\right\|_{L^p(K,\mu)} = \left(\int_K |\ f\ |^p\ d\mu\right)^{1/p} < \infty.$$

For an entire function $f \in \mathbb{C}^n$, M. Harfaoui and M. El Kadiri established a precise relationship between the $(p, q)$-growth and the general growth ($(\alpha, \alpha)$-growth) with respect to the set (see

((Harfaoui, 2010), (Harfaoui, 2011), (Kadiri & Harfaoui, 2013) and (Harfaoui & Kumar, 2014)) and the coefficients of the development of $f$ with respect to the sequence $(A_k)_k$. He used these results to give the relationship between the generalized growth of $f$ and the sequence $(\pi_k^p(K,f))_k$.

To our knowledge no work is discussed in term of best approximation in $L_p$-norm with respect to the proximate growth.

The aim of this paper is to give the proximate growth and the $(m,1)$- proximate growth of entire functions in $\mathbb{C}^n$ $(m \in \mathbb{N}^*)$ by means of the best polynomial approximation in term of $L^p$-norm, with respect to the set

$$\Omega_r = \left\{z \in \mathbf{C}^n; \exp V_K(z) \leq r\right\}.$$

In the paper of A. R. Reddy and T. Winiarski (see (Reddy, 1972*a*), (Reddy, 1972*b*) and (Winiarski, 1970)) the authors use the development of $f$ in the basis $(z_n)_n$ and $(W_n)_n$ and used the Cauchy inequality.

In our work we use a new basis of extremal polynomial and we replace the the Cauchy inequality by an inequality given by A. Zeriahi (see (Zeriahi, 1983)).

So we establish relationship between the rate at which $\left(\pi_k^p(K,f)\right)^{1/k}$, for $k \in \mathbb{N}$, tends to zero in terms of best approximation in $L^p$-norm, and the proximate growth growth of entire functions of several complex variables for a $L$-regular compact $K$ of $\mathbb{C}^n$.

## 2. Notations aned auxiliary results

Before we give some definitions and results which will be frequently used.

For $p \in \mathbb{Z}$ put

$$\log^{[p]}(x) = \log(\log^{[p-1]}(x));\;\; \log^{[0]}(x) = x; \Lambda_{[p]} = \prod_{k=1}^{p} \log^{[k]}(x).$$

$$\exp^{[p]}(x) = \exp(\exp^{[p-1]}(x));\;\; \exp^{[0]}(x) = x \text{ and } \; E_{[p]}(x) = \prod_{k=0}^{p} \exp^{k}(x).$$

**Lemma 2.1.** *(see (Bajpai et al., 1976))*

*With the above notations we have the following results*

*(RR₁)* $$E_{[-p]}(x) = \frac{x}{\bigwedge_{[p-1]}(x)} \text{ and } \bigwedge_{[-p]}(x) = \frac{x}{E_{[p-1]}(x)}$$

*(RR₂)* $$\frac{d}{dx}\exp^{[p]}(x) = \frac{E_{[p]}(x)}{x} = \frac{1}{\bigwedge_{[-p-1]}(x)}$$

*(RR₃)* $$\frac{d}{dx}\log^{[p]}(x) = \frac{E_{[-p]}(x)}{x} = \frac{1}{\bigwedge_{[p-1]}(x)}$$

*(RR₄)* $$E_{[p]}^{-1}(x) = \begin{cases} x, & \text{if } p = 0 \\ \log^{[p-1]}\{\log(x) - \log^{[2]}(x) + o(\log_{[3]}(x))\}, & \text{if } p = 1,2,\ldots \end{cases}.$$

*(RR₅)* $\lim_{x\to+\infty} \exp\left(E_{[p-2]}(x)\right) = \begin{cases} e & if\ p = 2 \\ 1 & if\ p \geq 3 \end{cases}$

*(RR₆)* $\lim_{x\to+\infty} \left[\exp^{[p-1]}\left(E^{-1}_{[p-2]}(x)\right)\right]^{\frac{1}{x}} = \begin{cases} e & if\ p = 2 \\ 1 & if\ p \geq 3 \end{cases}$

It is known that if $K$ is a compact $L$-regular of $\mathbb{C}^n$, there exists a measure $\mu$, called extremal measure, having interesting properties (see (Siciak, 1962) and (Siciak, 1981)), in particular, we have:

$(P_1)$ Bernstein-Markov inequality:

$\forall\epsilon > 0$, there exists $C = C_\varepsilon$ is a constant such that

$$(BM) : \left\|P_d\right\|_K = C(1+\varepsilon)^{s_k}\left\|P_d\right\|_{L^2(K,\mu)}, \tag{2.1}$$

for every polynomial of $n$ complex variables of degree at most $d$.

$(P_2)$ Bernstein-Waish (B.W) inequality:

For every set $L$-regular $K$ and every real $r > 1$ we have:

$$\left\|f\right\|_K \leq M.r^{deg(f)}\Big(\int_K \mid f \mid^p .d\mu\Big)^{1/p} \tag{2.2}$$

Note that the regularity is equivalent to the Bernstein-Markov inequality.

For $s : \mathbb{N} \to \mathbb{N}^n, k \to s(k) = (s_1(k), ...., s_n(k))$ be a bijection such that

$$|s(k+1)| \geq |s(k)| \ \text{ where } \ |s(k)| = s_1(k) + .... + s_n(k),$$

A. Zeriahi (see (Zeriahi, 1983)) constructed according to the Hilbert Schmidt method a sequence of monic orthogonal polynomials according to a extremal measure (see (Siciak, 1962)), $(A_k)_k$, called extremal polynomial, defined by

$$A_k(z) = z^{s(k)} + \sum_{j=1}^{k-1} a_j z^{s(j)} \tag{2.3}$$

such that

$$\left\|A_k\right\|_{L^p(E,\mu)} = \left[\inf\left\{\left\|z^{s(k)} + \sum_{j=1}^{k-1} a_j z^{s(j)}\right\|_{L^2_{(E,\mu)}}, (a_1, a_2, ..., a_n) \in \mathbb{C}^n\right\}\right]^{1/s_k}.$$

We need the following notations which will be used in the sequel:

$(N_1)$ $\nu_k = \nu_k(K) = \left\|A_k\right\|_{L^2(K,\mu)}$.

$(N_2)$ $a_k = a_k(K) = \left\|A_k\right\|_K = \max_{z\in K}\left|A_k(z)\right|$ and $\tau_k = \left(a_k\right)^{1/s_k}$,

where $s_k = deg(A_k)$.

With that notations and (B.W) inequality we have

$$\left\|A_k\right\|_{\overline{\Omega}_r} \leq a_k.r^{s_k} \tag{2.4}$$

where $s_k = deg(A_k)$.

**Lemma 2.2.** *(see (Zeriahi, 1983))*
*Let $K$ be a compact L-regular subset of $\mathbb{C}^n$. Then*

$$\lim_{k\to+\infty}\left[\frac{|A_k(z)|}{\nu_k}\right]^{1/s_k} = \exp(V_K(z)), \tag{2.5}$$

*for every $z \in \mathbb{C}^n \setminus \widehat{K}$ the connected component of $\mathbb{C}^n \setminus K$,*

$$\lim_{k\to+\infty}\left[\frac{\|A_k\|_K}{\nu_k}\right]^{1/s_k} = 1. \tag{2.6}$$

## 3. Growth with respect to the proximate order and coefficient with respect to extremal polynomial.

Before we give some definitions and results which will be frequently used in this paper.

**Definition 3.1.**
Let $\rho$ be a positive real such that $0 < \rho < +\infty$. A proximate order for $\rho$ is a function $\rho(r)$ defined in $\mathbb{R}^+$ and verified:

1. $\lim\limits_{r\to+\infty} \rho(r) = \rho$;
2. $\lim\limits_{r\to+\infty} r\rho' \log(r) = 0$.

**Example 3.1.** The function $\rho(r)$ defined by

$$r^{\rho(r)} = r^{\rho}\big(\ln(r)\big)^{\beta_1}.\big(\ln^{[2]}(r)\big)^{\beta_2})...\big(\ln^{[m](r)}\big)^{\beta_m}$$

is a proximate order for $\rho$, where $\log^{[m](r)}$ is defined by:

$$\log^{[0]}(r) = r, \quad \log^{[m](r)} = \ln^+\big(\log^{[m-1]}(r)\big) \text{ and } \ln^+(t) = 1_{[1;+\infty[}\ln(t)$$

**Theorem 3.1.** *If $h(r)$ is a positive function for $r > 0$ such that*

$$\lim_{r\to+\infty}\frac{\log(h(r))}{\log(r)} = \rho < +\infty,$$

*then the proximate order $\rho(r)$ maybe chosen such that for every $r > 0$: $h(r) \leq r^{\rho(r)}$, and for some sequence $r_n^{\rho(r_n)}$, $h(r_n) \leq r_n^{\rho(r_n)}$, for n sufficiently large.*

For an entire function in $\mathbb{C}^n$ we define the $K$-type for the proximate order as follows:

**Definition 3.2.**

Let $K$ be a L-regular of $\mathbb{C}^n$. If for an entire function in $\mathbb{C}^n$

$$\limsup_{r\to+\infty} \frac{\log\left(M_K(f,r)\right)}{r^{\rho(r)}} \tag{3.1}$$

is finite not zero then the function $\rho(r)$ is called proximate order

$$\sigma_K = \lim_{r\to+\infty} \frac{\ln\left(M_K(f,r)\right)}{r^{\rho(r)}} \tag{3.2}$$

is called $K$-type of $f$ with respect to the proximate order $\rho(r)$, where

$$M_K(f,r) = \sup_{z\in\Omega_r} \left|f(z)\right|.$$

Let $K$ be a compact L-regular and $f$ an entire function of several variables and $f(z) = \sum_{k=0}^{+\infty} f_k.A_k$ the development of $f$ with respect to the sequence of extremal polynomials.

### 2.1. $K$-type of $f$ with respect to the proximate order

**Theorem 3.2.**

*If $\rho(r)$ is a proximate order for $\rho$ then the $K$-type of $f$ with respect to the proximate order is given by the formula:*

$$\sigma_K = \frac{1}{e.\rho} \limsup_{k\to+\infty} \left(\varphi(s_k)\tau_k\right)^{\rho}.\left|f_k\right|^{\rho/s_k}, \tag{3.3}$$

*where $\varphi$ is the inverse function of the function $r \to r^{\rho(r)} = \psi(r)$.*

*We have so $\psi(r) = y \Leftrightarrow \varphi(y) = r$.*

**Lemma 3.1.** *[7, p.42(1.58)]*

*For every $k > 0$ we have*

$$\limsup_{t\to+\infty} \frac{\varphi(k.t)}{\varphi(t)} = k^{1/\rho}.$$

**Proof of theorem 3.2.**

Put $\sigma = \frac{1}{e.\rho} \limsup_{k\to+\infty} \left(\varphi(s_k)\tau_k\right)^{\rho}.\left|f_k\right|^{\rho/s_k}$ and show that $\sigma = \sigma_K$.

Show that $\sigma \leq \sigma_K$.

We have for every $\theta > 1$ $\sigma_K = \lim_{r\to+\infty} \frac{\ln\left(M_K(f,r\theta\theta)\right)}{r^{\rho(r\theta)}}$, then for every $\varepsilon > 0$ there exists $r(\varepsilon)$ such that for every $r > r(\varepsilon)$

$$\log\left(\|f\|_{\overline{\Omega}_{r\theta}}\right) \leq (r\theta)^{r\theta}(\sigma(K,f)+\varepsilon). \tag{3.4}$$

But $(r+1)^{N_\theta}\|f\|_{\overline{\Omega}_{r\theta}} \leq \exp\left((\sigma_{K,f}+\varepsilon)(r\theta)^{r\theta}\right)$, where $N_\theta \in \mathbb{N}$ such that

$$\left|f_k\right|\nu_k \leq C_\theta.r^{-s_k}.\frac{(r+1)^{N_\theta}}{(r-1)^{2n-1}}\|f\|_{\overline{\Omega}_{r\theta}} \tag{3.5}$$

then

$$\left|f_k\right|\nu_k \leq C_\theta . r^{-s_k} . \exp\left((\sigma(K,f)+\varepsilon)(r\theta)^{r\theta}\right),$$

for $r > r(\varepsilon)$ and $k > k(\varepsilon)$ or

$$\log\left(\left|f_k\right|\nu_k\right) \leq \log(C_\theta) - s_k \log(r) + \left((\sigma(K,f)+\varepsilon)(r\theta)^{r\theta}\right), \tag{3.6}$$

for $r > r(\varepsilon)$ and $k > k(\varepsilon)$.

Chose $r$ such that $s_k = [(\sigma(K,f)+\varepsilon)(r\theta)^{r\theta}.]$, where $[x]$ means the integer part of $x$. Then $s_k \leq (\sigma(K,f)+\varepsilon)(r\theta)^{r\theta} < s_k + 1$. Replacing in the relation (3.6) we get

$$\log\left(\left|f_k\right|\nu_k\right) \leq \log(C_\theta) - s_k \log(r) + s_k \log(\theta) + \frac{s_k+1}{\rho}. \tag{3.7}$$

Since $\dfrac{s_k}{\rho(\sigma(K,f)+\varepsilon)} \leq (r\theta)^{r\theta}$, then $\varphi(\dfrac{s_k}{\rho(\sigma(K,f)+\varepsilon)}) \leq r\theta$, thus

$$\log\left[(\tau_k.\varphi(s_k))^{\rho}(\left|f_k\right|)^{\rho/s_k}\right] \leq \frac{\rho}{s_k}\log(C_\theta) + \rho\log\Bigg(\frac{\varphi(s_k)}{\varphi(\dfrac{s_k}{\rho.(\sigma(K,f)+\varepsilon)})}\Bigg) + 1 + \frac{1}{s_k}.$$

After passing to the upper limit and applying the lemma 2.1, the relation (2.6) of the lemma 2.2 and the lemma 3.1 we get

$$\limsup_{k\to+\infty} \log\left[(\varphi(s_k))^{\rho}(\nu_k.\left|f_k\right|)^{\rho/s_k}\right] \leq \log\big(\rho.\sigma(K,f)\big) + 1 = \log\big(e.\rho.(\sigma(K,f))\big). \tag{3.8}$$

which gives the result

$$\limsup_{r\to+\infty} (\tau_k.\varphi(s_k))^{\rho}(\left|f_k\right|)^{\rho/s_k} \leq e.\rho.(\sigma(K,f)). \tag{3.9}$$

Show that $\sigma \geq \sigma_K$. If $\sigma < \sigma_K$ let $\sigma_1$ and $\sigma_2$ such that $\sigma < \sigma_1 < \sigma_2 < \sigma_K$. There exists $k_1$such that for every $k > k_1$:

$$(\tau_k)^{s_k}.|f_k| \leq \frac{e.\rho.(\sigma_1)^{1/\rho}}{\varphi(s_k)} \tag{3.10}$$

as we have also for $k$ sufficiently large $(k > q_2)$, $(\sigma_1.\rho)^{1/\rho}.\dfrac{\varphi(\frac{s_k}{\sigma_1.\rho})}{\varphi(s_k)}$, then for $k_0 = \max(q_1, q_2)$ we have

$$M_K(f,r) \leq \sum_{k=0}^{k_0} |f_k|.\|A_k\|_{\overline{\Omega}_r} + \sum_{k=k_0+1}^{+\infty} |f_k|.\|A_k\|_{\overline{\Omega}_r}. \tag{3.11}$$

According to the Bernstein-Walsh inequality we have

$$\|A_k\|_{\overline{\Omega}_r} \leq a_k(K).r^{s_k},$$

and according to the Bernstein-Markov inequality we have

$$a_k(K) \leq A_\epsilon.(1+\epsilon)^{s_k} a_k(K).\tau_k^{s_k}.$$

Thus

$$M_K(f,r) \le C_0.r^{s_k} + A_\epsilon. \sum_{k=k_0+1}^{+\infty} \Big(\frac{e^{1/\rho}}{\varphi(s_k/\sigma_1.\rho)}\Big)^{s_k}.((1+\epsilon))^{s_k}. \tag{3.12}$$

If we put $\delta = \dfrac{\sigma_1}{\sigma_2}$ ($\delta < 1$) then

$$M_K(f,r) \le C_0.r^{s_k} + A_\epsilon. \sum_{k=k_0+1}^{+\infty} \Big(\frac{1+\epsilon}{1-\epsilon}\Big)^{s_k}. \sup_{k>k_0} e^{\Psi(s_k)}.$$

where

$$\Psi(x) = x\log(r) - \frac{x}{\rho} - x\log\Big(\varphi(x/\sigma_2.\rho)\Big).$$

If we choose $\epsilon$ such that $0 < \epsilon < \dfrac{1-\delta}{1+\delta}$ then $\dfrac{\delta(1+\epsilon)}{1-\epsilon} < 1$ and thus

$$M_K(f,r) \le C_0.r^{s_k} + C. \sup_{k>k_0} e^{\Psi(s_k)}.$$

We note that $\Psi(x) = 0$ is equivalent to

$$\log(r) + \frac{1}{\rho} - \frac{x}{\sigma_2.\rho}.\frac{\varphi'(x/\sigma_2.\rho)}{\varphi(x/\sigma_2.\rho)} - \log\Big(\varphi(x/\sigma_2.\rho)\Big) = 0, \tag{3.13}$$

then the solution $x_r$ of the equation (3.13) verify

$$\log(r) - \frac{\rho}{\epsilon} < \varphi(\frac{x_r}{\sigma_2.\rho}) < \log(r) + \frac{\rho}{\epsilon} \ \text{ for } r > r_1$$

and thus

$$\begin{cases} \Psi(x) \le \dfrac{x_r}{\rho} + x_r\Big(\log(r) - \log(\varphi(\dfrac{x_r}{\sigma_2.\rho}))\Big) \le (1+\epsilon)\dfrac{x_r}{\rho} \\ \dfrac{x_r}{\sigma_2} \le (\theta.r)^{\varphi(\theta)} \text{ where } \theta = e^{\epsilon/\rho} \end{cases}$$

Since for every $\theta > 1$ we have $(\theta.r)^{\varphi(\theta)} \le (\theta.r)^{\rho+\epsilon}.r^{\rho(r)}$ then

$$e^{\Psi(x_r)} \le e^{(1+\epsilon).\theta^{\rho+\epsilon}}.\sigma_2.r^{\rho(r)} \ \text{ for } r > r_1$$

and consequently, for $r > r_1$,

$$M_K(f,r) \le C_0.r^{s_{k_0}} + A.\theta^{\rho+\epsilon}.\sigma_2.r^{\rho(r)}.$$

whence

$$\frac{\log(M_K(f,r))}{r^{\rho(r)}} \le \sigma_1 + o(1),$$

passing to the upper limit we get $\sigma(K,f) \le \sigma_1$. Which leads a contradiction and this shows the result.

**2.2.$(K, m)$-type of $f$ with respect to the proximate order**

For the entire functions infinite order we introduce the notion of $m$-order defined by:

$$\rho_m = \limsup_{r\to+\infty} \frac{\log^{[m]}(M_K(f,r))}{\log(r)}, \tag{3.14}$$

for $m \geq 2$. The function $f$ is said to be of index-pair $(m,1)$ if $\rho_{m-1} = +\infty$ and $\rho_m < +\infty$. The number $\rho_m$ is called the $m$-order of $f$.

**Definition 3.3.**
If $\rho(r)$ is a proximate order associated to the $m$-order $\rho_m$, the $(K,m)$-type with respect to the proximate order $\rho(r)$ is defined by:

$$\sigma_m(K,f) = \limsup_{r\to+\infty} \frac{\log^{[m]}(M_K(f,r)}{r^{\rho(r)}} \tag{3.15}$$

Let $f = \sum_{k=0}^{+\infty} f_k(f).A_k$ the development of $f$ with respect to the sequence of extremal polynomials.

**Theorem 3.3.**
*The $(K,m)$-type of $f$ with respect to the proximate order is given by the formula:*

$$\sigma_m(K,f) = \limsup_{k\to+\infty} \Big(\varphi(\log^{[m-2]}(s_k))\tau_k\Big)^{\rho}.\big|f_k\big|^{\rho/s_k}, \tag{3.16}$$

*for $m > 2$.*

**Proof of theorem 3.3.**
Put $\rho_m = \rho$ and $\sigma = \limsup_{k\to+\infty} \Big(\varphi(\log^{[m]}(s_k))\tau_k\Big)^{\rho}.\big|f_k\big|^{\rho/s_k}$.
Show that $\sigma_m(K,f) \leq \sigma$.
We have for every $\epsilon > 0$ there exists $k_0$ such that for every $k > k_0$

$$\varphi(\log^{[m-2]}(s_k))\tau_k.\big|f_k\big|^{1/s_k} \leq \sigma^{1/\rho} + \epsilon, \tag{3.17}$$

thus

$$M_K(f,r) \leq C_0 r^{s_{k(r)}} + \sum_{k=0}^{k_0} |f_k|.\|A_k\|_{\overline{\Omega}_r} + \sum_{k=k_0+1}^{+\infty} \Big(\frac{\sigma^{1/\rho}+\epsilon}{\varphi(\log^{[m-2]}(s_k))}\Big)^{s_k}.r^{s_k}. \tag{3.18}$$

For $\sigma_1 > \sigma$ we have

$$\Big(\frac{\sigma^{1/\rho}+\epsilon}{\varphi(\log^{[m-2]}(s_k))}\Big)^{s_k}.r^{s_k} \leq \Big(\frac{\sigma^{1/\rho}+\epsilon}{\sigma_1^{1/\rho}+\epsilon}\Big)^{s}.\sup_{k>k_0} e^{\Psi(s_k)},$$

where

$$\Psi(x) = x\log(r) + x\log(\sigma^{1/\rho}+\epsilon) + x\log(\varphi(\log^{[m-2]}(x)).$$

The solution $x_r$ of the equation $\Psi'(x) = 0$ verify, for $r$ sufficiently large ($r > r_1$)

$$\Psi(x_r) \leq \epsilon.exp^{[m-2]}((1+\epsilon).\theta^{\rho+\epsilon}.r^{\rho(r)}), \text{ where } \theta = (\sigma^{1/\rho} + \epsilon).e^{\epsilon}$$

therefore

$$M_K(f,r) \leq C_0 r^{s_{k(r)}} + A.e^{\Psi(x_r)}, \text{ where A is a constant.}$$

This gives $\limsup_{r\to+\infty} \dfrac{\log^{[m]}(M_K(f,r)}{r^{\rho(r)}} \leq \sigma_1$ and since this is true for every $\sigma_1 > \sigma$ then

$$\limsup_{r\to+\infty} \frac{\log^{[m]}(M_K(f,r)}{r^{\rho(r)}} \leq \sigma.$$

Show now that $\sigma_m(K,f) \geq \sigma$.

By definition of $\sigma_m(K,f)$ we have for every $\epsilon > 0$ there exists $r_0(\epsilon)$ such that for every $r > r_0(\epsilon)$ ,

$$M_K(f,r) \leq \exp^{[m-2]}\left[(\sigma_m(K,f)+\epsilon)(r\theta)^{r\theta}\right], \text{ and } \theta > 1,$$

thus

$$\left|f_k\right|.\tau_k^{s_k} \leq C_0'.\sup_{k>k_0} \exp^{\Psi(s_k)}.$$

where

$$\Psi(x) = -s_k \log\left(\frac{r}{1+\epsilon}\right) + \exp^{[m-2]}\left[(\sigma_m(K,f)+\epsilon)(r\theta)^{r\theta}\right].$$

For $r$ sufficiently large the solution of the equation $\Psi'(x) = 0$ verify

$$E_{[m-2]}^{-1}\left(s_k(\frac{1}{\rho}-1)\right) \leq (\sigma_m(K,f)+\epsilon)(r_k\theta)^{r_k\theta} \leq E_{[m-2]}^{-1}(s_k(\frac{1}{\rho}+1)). \tag{3.19}$$

Using the relation (3.19) an elementary calculus gives

$$\left|f_k\right|.\tau_k^{s_k}.\varphi\left(E_{[m-2]}^{-1}(s_k(\frac{1}{\rho}-1))\right) \leq (\sigma_m(K,f)+\epsilon)^{1/\rho}.\exp^{[m-1]}\left[E_{[m-2]}^{-1}(s_k(\frac{1}{\rho}+1))\right]. \tag{3.20}$$

Therefore passing to the upper limit and using the propriety of the function $x \to E_{[m-2]}(x)$ we obtain the result.

## 4. Best polynomial approximation in terms of $L^p$-norm.

The object of this section is to study the relationship of the rate of the best polynomial approximation of $f$ in $L^p$-norm with the -growth with respect to the proximate order of an entire function $g$ such that $g_{/K} = f$.

More precisely we show the following theorem:

**Theorem 4.1.**

*If $\rho(r)$ is a proximate order for $\rho$and $f$ and let $f \in L^p(K,\mu)$ for $p > 0$. Then $f$ is $\mu$-almost-surely the restriction to $K$ of an entire function in $\mathbb{C}^n$, $f_1$, of finite nonzero order $\rho$ and $K$-type $\sigma(K, f_1) \in ]0, +\infty 5$ with respect to the proximate order $\rho(r)$ for $\rho$ if and only if*

$$\sigma(K, f_1) = \frac{1}{e.\rho} \limsup_{k\to+\infty} (\varphi(k))^{\rho}.(\mathcal{E}_k^p)^{\rho/k}, \tag{4.1}$$

*where $\varphi$ is the inverse function of the function $r \to r^{\rho(r)} = \psi(r)$.*

*We have so $\psi(r) = y \Leftrightarrow \varphi(y) = r$.*

**Proof of theorem 4.1.**

Suppose that $f$ is $\mu$-almost-surely the restriction to $K$ of an entire function in $\mathbb{C}^n$, $f_1$, of finite nonzero order $\rho$ and $K$-type $\sigma(K, f_1) \in ]0, +\infty 5$ with respect to the proximate order $\rho(r)$ for $\rho$. We have $f_1 \in L^2(K,\mu)$ and

$$f_1 = \sum_{k=0}^{+\infty} f_k.A_k.$$

Put $\sigma = \dfrac{1}{e.\rho} \limsup_{k\to+\infty} \Big(\varphi(s_k)\tau_k\Big)^{\rho}.\Big|f_k\Big|^{\rho/s_k}$

By the relation (92) for $p \geq 2$ and the relation (96) for $p \in [1, 2[$ of the paper of M. El Kadiri and M. Harfaoui (see (Kadiri & Harfaoui, 2013))

$$\Big(\varphi(s_k)\Big)^{\rho}.(\nu_k.|f_k|)^{\rho/s_k} \leq (A_\epsilon)^{\rho/s_k}\Big(\varphi(s_k)\Big)^{\rho}(1+\epsilon)^{\rho}.(\mathcal{E}_k^p)^{\rho/s_k} \tag{4.2}$$

then

$$\Big(\varphi(s_k)\tau_k\Big)^{\rho}.(\Big|f_k\Big|)^{\rho/s_k} \leq \Big(\varphi(s_k)\Big)^{\rho}(\Big|f_k\Big|.\nu_k)^{\rho/s_k}.\Big(\frac{\tau_k^{s_k}}{\nu_k}\Big)^{\rho/s_k} \tag{4.3}$$

By the relation 3.6 we have

$$\Big(\mathcal{E}_k^p\Big)^{1/s_k} \leq (A_\epsilon)^{\rho/s_k}.\Big[\Big|f_k\Big|.\nu_k.(1+\epsilon)^{s_k+1} + ...\Big]. \tag{4.4}$$

But

$$\sigma' = \limsup_{k\to+\infty} \Big(\varphi(s_k)\Big)^{\rho}.(\nu_k.\Big|f_k\Big|^{\rho/s_k} = e.\rho.\sigma.$$

Thus, for $k$ sufficiently large

$$\varphi(s_k).(\nu_k.\Big|f_k\Big|)^{1/s_k} \leq (\sigma')^{1/\rho} + \epsilon \Leftrightarrow \nu_k.\Big|f_k\Big| \leq \Big[\frac{(\sigma')^{1/\rho}+\epsilon}{\varphi(s_k)}\Big]^{s_k}.$$

Hence for every $j \in \mathbb{N}$;

$$\nu_{k+j}.\Big|f_{k+j}\Big| \leq \Big[\frac{(\sigma')^{1/\rho}+\epsilon}{\varphi(s_{k+j})}\Big]^{s_{k+j}}.$$

Then, if we put $S = (1+\epsilon)^{s_k}.\nu_k.\big|f_k + (1+\epsilon)^{s_{k+1}}.\nu_{k+1}.\big|f_{k+1}....$, we have

$$S \leq \sum_{j=0}^{+\infty} \epsilon)^{s_{k+j}}[\frac{(\sigma')^{1/\rho}+\epsilon}{\varphi(s_{k+j})}]^{s_{k+j}}$$

which is equivalent to

$$S \leq \Big[\frac{(\sigma)^{1/\rho}+\epsilon}{\varphi(s_{k+j})}\Big]^{s_k} \sum_{j=0}^{+\infty}(1+\epsilon)^{s_{k+j}} \frac{\big[(\sigma')^{1/\rho}+\epsilon\big]^{s_k+j}}{\big[(\sigma)^{1/\rho}+\epsilon\big]^{s_k}}\Big[\frac{[\varphi(s_k)]^{s_k}}{[\varphi(s_{k+j})]^{s_{k+j}}}\Big]^{s_{k+j}}$$

or

$$S \leq \Big[\frac{(\sigma)^{1/\rho}+\epsilon}{\varphi(s_k)}\Big]^{s_k} \sum_{j=0}^{+\infty}(1+\epsilon)^{s_{k+j}} \frac{\big[(\sigma')^{1/\rho}+\epsilon\big]^{s_k+j}}{\big[(\sigma)^{1/\rho}+\epsilon\big]^{s_k}} \frac{\big[\varphi(s_k)\big]^{s_k}}{\big[\varphi(s_k+j)\big]^{s_k}+j}$$

Since $\dfrac{\varphi(s_k)}{\varphi(s_k+j)} \leq 1$ we get also

$$S \leq (1+\epsilon)^{s_k}.\frac{\Big((\sigma')^{1/\rho}+\epsilon\Big)^{s_k}}{\varphi(s_k)}.\sum_{j=0}^{+\infty}\Big[(1+\epsilon).\frac{(\sigma')^{1/\rho}+\epsilon}{\varphi(j)}\Big]^j.$$

As for $k$ sufficiently large $\dfrac{(\sigma')^{1/\rho}+\epsilon}{\varphi(s_k+j)} < 1$ the series is convergent to a finite sum $L$ and we will get finally

$$\Big(\mathcal{E}_k^p\Big)^{1/s_k} \leq (1+\epsilon)^{\rho}(A_\epsilon)^{1/s_k}.L^{\rho/s_k}.\Big[\frac{(\sigma')^{1/\rho}+\epsilon}{\varphi(s_k)}\Big]^{\rho}$$

which equivalent to

$$(\varphi(s_k))^{\rho}.\Big(\mathcal{E}_k^p\Big)^{1/s_k} \leq (1+\epsilon)^{\rho}(A_\epsilon)^{\rho/s_k}.L^{\rho/s_k}.((\sigma')^{1/\rho}+\epsilon^{\rho}).$$

Passing to the upper limit get

$$\sigma(K,f_1) = \frac{1}{e.\rho}\limsup_{k\to+\infty}(\varphi(k))^{\rho}.(\mathcal{E}_k^p)^{\rho/k} \leq \sigma.$$

Conversely, suppose now that $f$ satisfies the relation 4.6. We show the result by three steps.

If $f \in L^p(K,\mu)$ with $p \geq 2$ then $f \in L^2(K,\mu)$ and we have $\sum_{k=0}^{+\infty} f_k A_k$ with convergence in $L^2(K,\mu)$, where

$$f_k = \frac{1}{\nu_k^2}\int_K f.\overline{A}_k \quad (k \geq 0).$$

We verify easily by the relations 3.3, 3.6 and the inequality (B.M):

$$\limsup_{k\to+\infty}\Big(\varphi(s_k)\tau_k\Big)^{\rho}.\big|f_k\big|^{\rho/s_k} = \limsup_{k\to+\infty}(\varphi(k))^{\rho}.(\mathcal{E}_k^p)^{\rho/k}. \tag{4.5}$$

By this inequality the series $\sum f_k A_k$ considered in $\mathbb{C}^n$ converges normally on every compact of $\mathbb{C}^n$ to a function denoted $f_1$ by the inequality (B.M) and the inequality of the coefficient of $|f_k|$. We have obviously $f_1 = f$ $\mu$-a.s on $K$ and the proof is completed by the theorem3.1.

If $p \in [0, 2[$ we take $p'$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, then $p' \geq 2$. Applying the previous arguments of the first step to $p'$ and Hölder and Bernstein inequality we obtain the result.

If $0 < p < 1$, of course, for $0 < p < 1$ the $L_p$-norm does not satisfy the triangle inequality. But our relations (4.2) and (4.3) are also satisfied for $0 < p < 1$ (see (Harfaoui & Kumar, 2014)), because using Holder's inequality we have, for some $M > 0$ and all $r > p$ ($p$ fixed)

$$\| f \|_{L^p(K,\mu)} \leq M. \| f \|_{L^r(K,\mu)} .$$

Using the inequality

$$\int_K | f |^p \, d\mu \leq \| f \|_K^{p-r} . \int_K | f |^r \, d\mu$$

we get

$$\| f \|_{L^p(K,\mu)} \leq \| f \|_K^{1-(r/p)} . \| f \|_{L^r(K,\mu)}^{r/p} .$$

We deduce that $(K, \mu)$ satisfies the Bernstein-Markov inequality. For $\epsilon > 0$ there is a constant $C = C(\epsilon, p) > 0$ such that, for all (analytic) polynomials $P$ we have

$$\| P \|_K \leq C(1 + \epsilon)_{deg(P)}. \ \| P \|_{L^p(K,\mu)} .$$

Thus if $(K, \mu)$ satisfies the Bernstein-Markov inequality for one $p > 0$ then (4.2) and (4.3) are satisfied for all$p > 0$.

The rest of proof is easily deduced using the same reasoning as in step.1 and step.2

**Theorem 4.2.**

*If $\rho(r)$ is a proximate order for $\rho_m \in ]0, +\infty[$ ($m > 2$), and $f$ and let $f \in L^p(K, \mu)$ for $p > 0$. Then $f$ is $\mu$-almost-surely ($\mu$-a.s) the restriction to $K$ of an entire function in $\mathbb{C}^n$, $f_1$, of finite nonzero $m$-order $\rho_m$ and $(K, m)$-type $\sigma_m(K, f_1) \in ]0, +\infty 5$ with respect to the proximate order $\rho(r)$ for $\rho$ if and only if*

$$\sigma_m(K, f_1) = \limsup_{k \to +\infty} \Big( \varphi(( \log^{[m-2]}(k))) \Big)^{\rho_m} . (\mathcal{E}_k^p)^{\rho_m / k}, \tag{4.6}$$

*where $\varphi$ is the inverse function of the function $r \to r^{\rho(r)} = \psi(r)$.*
*We have so $\psi(r) = y \Leftrightarrow \varphi(y) = r$.*

**Proof of theorem 4.2.**

The theorem can be proved on similar lines as those of the proof of the theorem 4.1 because the relations (4.2) and (4.3) are still valid by iteration of logarithm . Hence we omit the proof.

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# Some Results in Connection with the Bounds for the Zeros of Entire Functions in the Light of Slowly Changing Functions

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**Abstract**

A single valued function of one complex variable which is analytic in the finite complex plane is called an entire function. In this paper we would like to establish the bounds for the moduli of zeros of entire functions on the basis of slowly changing functions.

*Keywords:* Zeros of entire functions, proper ring shaped region, slowly changing functions.
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## 1. Introduction, Definitions and Notations.

Let

$$P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ...... + a_{n-1} z^{n-1} + a_n z^n ; |a_n| \neq 0$$

be a polynomial of degree $n$. Datt and Govil (Datt & Govil, 1978); Govil and Rahaman(Govil & Rahaman, 1968); Marden (Marden, 1966); Mohammad (Mohammad, 1967); Chattopadhyay, Das, Jain and Konwer (Chattopadhyay, 2005); Joyal, Labelle and Rahaman (Joyal, Labelle & Rahaman 1967) Jain (Jain, 1976), (Jain, 2006) Sun and Hsieh (Sun & Hsie, 1996); Zilovic, Roytman, Combettes and Swamy (Zilovic, Roytman); Das and Datta (Das & Datta, 2008) etc. worked in the theory of the distribution of the zeros of polynomials and obtained some newly developed results.

In this paper we intend to establish some sharper results concerning the theory of distribution of zeros of entire functions on the basis of slowly changing functions.

The following definitions are well known :

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**Definition 1.1.** (Valiron, 1949)The order $\rho$ and lower order $\lambda$ of an entire function $f$ are defined as

$$\rho = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, \ldots$ and $\log^{[0]} x = x$.

Let $L \equiv L(r)$ be a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant $a$. Singh and Barker (Singh & Barker, 1977) defined it in the following way:

**Definition 1.2.** (Singh & Barker, 1977) A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon$ ($> 0$),

$$\frac{1}{k^{\varepsilon}} \le \frac{L(kr)}{L(r)} \le k^{\varepsilon} \text{ for } r > r(\varepsilon) \quad \text{and}$$

uniformly for $k (\ge 1)$.

If further, $L(r)$ is differentiable, the above condition is equivalent to $\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0$.

Somasundaram and Thamizharasi (Somasundaram & Thamizharasi, 1988) introduced the notions of $L$-order and $L$-lower order for entire functions defined in the open complex plane $\mathbb{C}$ as follows:

**Definition 1.3.** (Somasundaram & Thamizharasi, 1988) The $L$-order $\rho^L$ and the $L$-lower order $\lambda^L$ of an entire function $f$ are defined as

$$\rho^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]} \quad \text{and} \quad \lambda^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}.$$

The more generalised concept for $L$-order and $L$-lower order are $L^*$-order and $L^*$-lower order respectively. Their definitions are as follows:

**Definition 1.4.** The $L^*$-order $\rho^{L^*}$ and the $L^*$-lower order $\lambda^{L^*}$ of an entire function $f$ are defined as

$$\rho^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]} \quad \text{and} \quad \lambda^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]}.$$

## 2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

**Lemma 2.1.** *If $f(z)$ is an entire function of $L$-order $\rho^L$, then for every $\varepsilon > 0$ the inequality*

$$N(r) \le [rL(r)]^{\rho^L + \varepsilon}$$

*holds for all sufficiently large $r$ where $N(r)$ is the number of zeros of $f(z)$ in $|z| \le [rL(r)]$.*

*Proof.* Let us suppose that $f(0) = 1$. This supposition can be made without loss of generality because if $f(z)$ has a zero of order $'m'$ at the origin then we may consider $g(z) = c \cdot \frac{f(z)}{z^m}$ where $c$ is so chosen that $g(0) = 1$. Since the function $g(z)$ and $f(z)$ have the same order therefore it will be unimportant for our investigations that the number of zeros of $g(z)$ and $f(z)$ differ by $m$.

We further assume that $f(z)$ has no zeros on $|z| = 2[rL(r)]$ and the zeros $z_i$'s of $f(z)$ in $|z| < [rL(r)]$ are in non decreasing order of their moduli so that $|z_i| \leq |z_{i+1}|$. Also let $\rho^L$ suppose to be finite.

Now we shall make use of Jenson's formula as state below

$$\log|f(0)| = -\sum_{i=1}^{n} \log \frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(R\, e^{i\phi}\right)\right| d\phi. \tag{2.1}$$

Let us replace $R$ by $2r$ and $n$ by $N(2r)$ in (2.1).

$$\therefore \log|f(0)| = -\sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(2r\, e^{i\phi}\right)\right| d\phi.$$

Since $f(0) = 1, \therefore \log|f(0)| = \log 1 = 0$.

$$\therefore \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(2r\, e^{i\phi}\right)\right| d\phi. \tag{2.2}$$

$$\text{L.H.S.} = \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} \geq \sum_{i=1}^{N(r)} \log \frac{2r}{|z_i|} \geq N(r)\log 2 \tag{2.3}$$

because for large values of $r$, $\log \frac{2r}{|z_i|} \geq \log 2$.

$$\text{R.H.S} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(2r\, e^{i\phi}\right)\right| d\phi \leq \frac{1}{2\pi}\int_0^{2\pi} \log M(2r)\, d\phi = \log M(2r). \tag{2.4}$$

Again by definition of order $\rho^L$ of $f(z)$ we have fore every $\varepsilon > 0$, and as $L(2r) \sim L(r)$,

$$\log M(2r) \leq [2rL(2r)]^{\rho^L+\varepsilon/2} \log M(2r) \leq [2rL(r)]^{\rho^L+\varepsilon/2}. \tag{2.5}$$

Hence from (2.2) by the help of (2.3), (2.4) and (2.5) we have

$$N(r)\log 2 \leq [2rL(r)]^{\rho^L+\varepsilon/2}$$

$$N(r) \leq \frac{2^{\rho^L+\varepsilon/2}}{\log 2} \cdot \frac{[rL(r)]^{\rho^L+\varepsilon}}{[rL(r)]^{\varepsilon/2}} \leq [rL(r)]^{\rho^L+\varepsilon}.$$

This proves the lemma. □

In the line of Lemma 2.1, we may state the following lemma:

**Lemma 2.2.** *If $f(z)$ is an entire function of $L^*$-order $\rho^{L^*}$, then for every $\varepsilon > 0$ the inequality*

$$N(r) \leq [re^{L(r)}]^{\rho^{L^*}+\varepsilon}$$

*holds for all sufficiently large $r$ where $N(r)$ is the number of zeros of $f(z)$ in $|z| \leq [re^{L(r)}]$.*

*Proof.* With the initial assumptions as laid down in Lemma 1, let us suppose that $f(z)$ has no zeros on $|z| = 2[re^{L(r)}]$ and the zeros $z_i$'s of $f(z)$ in $|z| < [re^{L(r)}]$ are in non decreasing order of their moduli so that $|z_i| \leq |z_{i+1}|$. Also let $\rho^{L^*}$ supposed to be finite.

In view of (2.1), (2.2), (2.3) and (2.4), by definition of $\rho^{L^*}$ and as $L(2r) \sim L(r)$, we get for every $\varepsilon > 0$ that

$$\log M(2r) \leq [2re^{L(2r)}]^{\rho^{L^*}+\varepsilon/2}, i.e., \log M(2r) \leq [2re^{L(r)}]^{\rho^{L^*}+\varepsilon/2}.$$

Hence by the help of (2.3), (2.4) and (2) we obtain from (2.2) that

$$N(r)\log 2 \leq [2re^{L(r)}]^{\rho^{L^*}+\varepsilon/2}, N(r) \leq \frac{2^{\rho^{L^*}+\varepsilon/2}}{\log 2} \cdot \frac{[re^{L(r)}]^{\rho^{L^*}+\varepsilon}}{[rL(r)]^{\varepsilon/2}} \leq [re^{L(r)}]^{\rho^{L^*}+\varepsilon}.$$

Thus the lemma is established. □

## 3. Theorems

In this section we present the main results of the paper.

**Theorem 3.1.** *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

*with L-order $\rho^L$. Also for all sufficiently large $r$ in the disc $|z| \leq [rL(r)]$, $|a_{N(r)}| \neq 0$, $|a_0| \neq 0$. and also $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region*

$$\frac{1}{t_0'} \leq |z| \leq t_0$$

*where $t_0$ is the greatest positive root of*

$$g(t) \equiv |a_{N(r)}| t^{N(r)+1} - \left(|a_{N(r)}| + M\right) t^{N(r)} + M = 0$$

*and $t_0'$ is the greatest positive root of*

$$f(t) \equiv |a_0| t^{N(r)+1} - (|a_0| + M') t^{N(r)} + M' = 0$$

$$\text{where } M = max\left\{|a_0|, |a_1|, .... |a_{N(r)-1}|\right\}$$

$$\text{and } M' = max\left\{|a_1|, |a_2|, .... |a_{N(r)}|\right\}.$$

*Proof.* Now

$$P(z) \approx a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)} z^{N(r)}$$

because $N(r)$ exists for $|z| \leq [rL(r)]$; $r$ is sufficiently large and $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region given in Theorem 3.1 which we are to prove.

Now

$$\begin{aligned} |P(z)| &\approx \left|a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)} z^{N(r)}\right| \\ &\geq \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)-1} z^{N(r)-1}\right|. \end{aligned}$$

Also

$$\begin{aligned} &\left|a_0 + a_1 z + a_2 z^2 + ..... + a_{N(r)-1} z^{N(r)-1}\right| \leq |a_0| + ... + \left|a_{N(r)-1}\right| |z|^{N(r)-1} \leq M\left(1 + |z| + ... + |z|^{N(r)-1}\right) \\ &= M\frac{|z|^{N(r)} - 1}{|z| - 1} \text{ if } |z| \neq 1. \end{aligned} \tag{3.1}$$

Therefore using (3.1) we obtain that

$$|P(z)| \geq \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)-1} z^{N(r)-1}\right| \geq \left|a_{N(r)}\right| |z|^{N(r)} - M\frac{|z|^{N(r)} - 1}{|z| - 1}.$$

Hence

$$|P(z)| > 0 \text{ if } \left|a_{N(r)}\right| |z|^{N(r)} - M\frac{|z|^{N(r)} - 1}{|z| - 1} > 0$$

$$\begin{aligned} &\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)} > M\frac{|z|^{N(r)} - 1}{|z| - 1} \\ &\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)+1} - \left|a_{N(r)}\right| |z|^{N(r)} > M\ \left(|z|^{N(r)} - 1\right) \\ &\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)+1} - \left|a_{N(r)}\right| |z|^{N(r)} - M\ |z|^{N(r)} + M > 0 \\ &\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) |z|^{N(r)} + M > 0. \end{aligned}$$

Therefore on $|z| \neq 1$, $|P(z)| > 0$ if $\left|a_{N(r)}\right| |z|^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) |z|^{N(r)} + M > 0$. Now let us consider

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) t^{N(r)} + M = 0. \tag{3.2}$$

Clearly the maximum number of changes in sign in (3.2) is two. So the maximum number of positive roots of $g(t) = 0$ is two and by Descartes' rule of sign if it is less, less by two. Clearly $t = 1$ is one positive root of (3.2). So $g(t) = 0$ must have another positive root $t_1$(say).
Let us take $t_0 = max\{1, t_1\}$. Clearly for $t > t_0$, $g(t) > 0$. If not, for some $t = t_2 > t_0$, $g(t_2) < 0$.
Now $g(t_2) < 0$ and $g(\infty) > 0$ imply that $g(t) = 0$ has another positive root in $(t_2, \infty)$ which gives a contradiction.
Therefore for $t > t_0$, $g(t) > 0$ and so $t_0 > 1$.
Hence $|P(z)| > 0$ for $|z| > t_0$.

$$\text{Therefore all the zeros of } P(z) \text{ lie in the disc } |z| \leq t_0. \tag{3.3}$$

Again let us consider

$$Q(z) = z^{N(r)}P\left(\frac{1}{z}\right) \approx z^{N(r)}\left\{a_0 + \frac{a_1}{z} + ... + \frac{a_{N(r)}}{z^{N(r)}}\right\} = a_0 z^{N(r)} + a_1 z^{N(r)-1} + ... + a_{N(r)}$$

$$\text{i.e., } |Q(z)| \geq |a_0|\,|z|^{N(r)} - \left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| \text{ for } |z| \neq 1.$$

Now

$$\left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| \leq |a_1|\,|z|^{N(r)-1} + ... + \left|a_{N(r)}\right| \leq M'\left(|z|^{N(r)-1} + ... + 1\right)$$

$$= M'\left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right) \text{ for } |z| \neq 1. \tag{3.4}$$

Using (3.4) we get that

$$|Q(z)| \geq |a_0|\,|z|^{N(r)} - \left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| \geq |a_0|\,|z|^{N(r)} - M'\left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right) \text{ for } |z| \neq 1.$$

Therefore for $|z| \neq 1$,

$$|Q(z)| > 0 \text{ if } |a_0|\,|z|^{N(r)} - M'\left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right) > 0$$

$$\text{i.e., if } |a_0|\,|z|^{N(r)} > M'\left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right)$$

$$\text{i.e., if } |a_0|\,|z|^{N(r)+1} - |a_0|\,|z|^{N(r)} - M'\,|z|^{N(r)} + M' > 0$$

$$\text{i.e., if } |a_0|\,|z|^{N(r)+1} - (|a_0| + M')\,|z|^{N(r)} + M' > 0.$$

So for $|z| \neq 1$, $|Q(z)| > 0$ if $|a_0|\,|z|^{N(r)+1} - (|a_0| + M')\,|z|^{N(r)} + M' > 0$. Let us consider

$$f(t) \equiv |a_0|\,t^{N(r)+1} - (|a_0| + M')\,t^{N(r)} + M' = 0.$$

Since the maximum number of changes of sign in $f(t)$ is two, the maximum number of positive roots of $f(t) = 0$ is two and by Descartes' rule of sign if it is less, less by two. Clearly $t = 1$ is one positive root of $f(t) = 0$. So $f(t) = 0$ must have another positive root $t_2$ (say).
Let us take $t'_0 = max\{1, t_2\}$. Clearly for $t > t'_0$, $f(t) > 0$. If not, for some $t_3 > t'_0$, $f(t_3) < 0$. Now $f(t_3) < 0$ and $f(\infty) > 0$ implies that $f(t) = 0$ have another positive root in the interval $(t_3, \infty)$ which is a contradiction.
Therefore for $t > t'_0$, $f(t) > 0$.
Also $t'_0 \geq 1$. So $|Q(z)| > 0$ for $|z| > t'_0$.
Therefore $Q(z)$ does not vanish in $|z| > t'_0$.

Hence all the zeros of $Q(z)$ lie in $|z| \leq t'_0$.
Let $z = z_0$ be a zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.
Putting $z = \frac{1}{z_0}$ in $Q(z)$ we get that$Q\left(\frac{1}{z_0}\right) = \left(\frac{1}{z_0}\right)^{N(r)}.P(z_0) = \left(\frac{1}{z_0}\right)^{N(r)}.0 = 0$. Therefore $Q\left(\frac{1}{z_0}\right) = 0$. So

$z = \frac{1}{z_0}$ is a root of $Q(z) = 0$. Hence $\left|\frac{1}{z_0}\right| \leq t_0'$ implies that $|z_0| \geq \frac{1}{t_0'}$.
As $z_0$ is an arbitrary root of $P(z) = 0$.

$$\text{Therefore all the zeros of } P(z) \text{ lie in } |z| \geq \frac{1}{t_0'}. \tag{3.5}$$

From (3.3) and (3.5) we get that all the zeros of $P(z)$ lie in the proper ring shaped region $\frac{1}{t_0'} \leq |z| \leq t_0$ where $t_0$ and $t_0'$ are the greatest positive roots of the equations $g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) t^{N(r)} + M = 0$ and $f(t) \equiv |a_0|\, t^{N(r)+1} - (|a_0| + M')\, t^{N(r)} + M' = 0$ where $M$ and $M'$ are given in the statement of Theorem 3.1. This proves the theorem. □

In the line of Theorem 3.1, we may state the following theorem in view of Lemma 2.2:

**Theorem 3.2.** *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + .... + a_n z^n + ...$$

*with $L^*$-order $\rho^{L^*}$. Also for all sufficiently large $r$ in the disc $|z| \leq [re^{L(r)}]$, $\left|a_{N(r)}\right| \neq 0$, $|a_0| \neq 0$. and also $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region*

$$\frac{1}{t_0'} \leq |z| \leq t_0$$

*where $t_0$ is the greatest positive root of*

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) t^{N(r)} + M = 0$$

*and $t_0'$ is the greatest positive root of*

$$f(t) \equiv |a_0|\, t^{N(r)+1} - (|a_0| + M')\, t^{N(r)} + M' = 0$$
$$\text{where } M = \max\left\{|a_0|, |a_1|, ... \left|a_{N(r)-1}\right|\right\}$$
$$\text{and } M' = \max\left\{|a_1|, |a_2|, ... \left|a_{N(r)}\right|\right\}.$$

The proof is omitted.

**Theorem 3.3.** *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

*with L-order $\rho^L$, $a_{N(r)} \neq 0$, $a_0 \neq 0$ and also $a_n \to 0$ for $n > N(r)$ for the disc $|z| \leq [rL(r)]$ when $r$ is sufficiently large. Further, for $\rho^L > 0$,*

$$|a_0|\, (\rho^L)^{N(r)} \geq |a_1|\, (\rho^L)^{N(r)-1} \geq ... \geq \left|a_{N(r)-1}\right| \rho^L \geq \left|a_{N(r)}\right|.$$

*Then all the zeros of $P(z)$ lie in the ring shaped region*

$$\frac{1}{\rho^L\left(1 + \frac{|a_1|}{|a_0|\rho^L}\right)} < |z| < \frac{1}{\rho^L}\left(1 + \frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^L)^{N(r)}\right).$$

*Proof.* For the given entire function

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

with $a_n \to 0$ as $n > N(r)$, where $r$ is sufficiently large, $N(r)$ exists and $N(r) \le [rL(r)]^{\rho^L+\epsilon}$.
Therefore

$$P(z) \approx a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)} z^{N(r)}$$

as $a_0 \neq 0, a_{N(r)} \neq 0$ and $a_n \to 0$ for $n > N(r)$.
Let us consider

$$\begin{aligned} R(z) &= (\rho^L)^{N(r)} P\left(\frac{z}{\rho^L}\right) \approx (\rho^L)^{N(r)} \left(a_0 + a_1 \frac{z}{\rho^L} + a_2 \frac{z^2}{(\rho^L)^2} + ... + a_{N(r)} \frac{z^{N(r)}}{(\rho^L)^{N(r)}}\right) \\ &= \left(a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + ... + a_{N(r)} z^{N(r)}\right). \end{aligned}$$

Therefore

$$|R(z)| \ge \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + ... + a_{N(r)-1} \rho^L z^{N(r)-1}\right|. \tag{3.6}$$

Now by the given condition $|a_0| (\rho^L)^{N(r)} \ge |a_1| (\rho^L)^{N(r)-1} \ge ...$ provided $|z| \neq 0$,we obtain that

$$\begin{aligned} &\left|a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + ... + a_{N(r)-1} \rho^L z^{N(r)-1}\right| \le |a_0| (\rho^L)^{N(r)} + ... + \left|a_{N(r)-1}\right| \rho^L |z|^{N(r)-1} \\ &\le |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + ...... + \frac{1}{|z|^{N(r)}}\right). \end{aligned}$$

Therefore on $|z| \neq 0$,

$$-\left|a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + ... + a_{N(r)-1} \rho^L z^{N(r)-1}\right| \ge -|a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right). \tag{3.7}$$

Therefore using (3.7) we get from (3.6) that

$$\begin{aligned} |R(z)| &\ge \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right) \\ &\ge \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}} + ...\right) \\ &= |z|^{N(r)} \left[\left|a_{N(r)}\right| - |a_0| (\rho^L)^{N(r)} \left\{\sum_{k=1}^{\infty} \frac{1}{|z|^k}\right\}\right]. \end{aligned}$$

Clearly $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is a geometric series which is convergent for $\frac{1}{|z|} < 1$ i.e., for $|z| > 1$ and converges to $\frac{1}{|z|} \frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}$. Therefore $\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z|-1}$ if $|z| > 1$. Hence we get from above that for $|z| > 1 |R(z)| >$

$|z|^{N(r)}\left(\left|a_{N(r)}\right| - (\rho^L)^{N(r)}|a_0|\frac{1}{|z|-1}\right)$. Now for $|z| > 1$,

$$\begin{aligned}|R(z)| > 0 \text{ if } &|z|^{N(r)}\left(\left|a_{N(r)}\right| - (\rho^L)^{N(r)}|a_0|\frac{1}{|z|-1}\right) \geq 0\\ \text{i.e., if } &\left|a_{N(r)}\right| - (\rho^L)^{N(r)}|a_0|\frac{1}{|z|-1} \geq 0\\ \text{i.e., if } &\left|a_{N(r)}\right| \geq (\rho^L)^{N(r)}\frac{|a_0|}{|z|-1}\\ \text{i.e., if } &|z|-1 \geq (\rho^L)^{N(r)}\frac{|a_0|}{\left|a_{N(r)}\right|}\\ \text{i.e., if } &|z| \geq 1 + (\rho^L)^{N(r)}\frac{|a_0|}{\left|a_{N(r)}\right|} > 1.\end{aligned}$$

Therefore $|R(z)| > 0$ if $|z| \geq 1 + (\rho^L)^{N(r)}\frac{|a_0|}{\left|a_{N(r)}\right|}$. So all the zeros of $R(z)$ lie in $|z| < 1 + \frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^L)^{N(r)}$. Let $z_0$ be an arbitrary zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \rho^L z_0$ in $R(z)$ we have $R(\rho^L z_0) = (\rho^L)^{N(r)}.P(z_0) = (\rho^L)^{N(r)}.0 = 0$.

Hence $z = \rho^L z_0$ is a zero of $R(z)$. Therefore $\left|\rho^L z_0\right| < 1+\frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^L)^{N(r)}$ i.e., $|z_0| < \frac{1}{\rho^L}\left(1 + \frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^L)^{N(r)}\right)$. Since $z_0$ is any zero of $P(z)$ therefore all the zeros of $P(z)$ lie in

$$|z| < \frac{1}{\rho^L}\left(1 + \frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^L)^{N(r)}\right). \tag{3.8}$$

Again let us consider $F(z) = (\rho^L)^{N(r)}z^{N(r)}P\left(\frac{1}{\rho^L z}\right)$. Now $F(z) = (\rho^L)^{N(r)}z^{N(r)}P\left(\frac{1}{\rho^L z}\right)$
$\approx (\rho^L)^{N(r)}z^{N(r)}\left\{a_0 + \frac{a_1}{\rho^L z} + ....... + \frac{a_{N(r)}}{(\rho^L z)^{N(r)}}\right\} = a_0(\rho^L)^{N(r)}z^{N(r)} + a_1(\rho^L)^{N(r)-1}z^{N(r)-1} + ... + a_{N(r)}$. Therefore $|F(z)| \geq |a_0|(\rho^L)^{N(r)}|z|^{N(r)} - \left|a_1(\rho^L)^{N(r)-1}z^{N(r)-1} + ... + a_{N(r)}\right|$. Again

$$\begin{aligned}&\left|a_1(\rho^L)^{N(r)-1}z^{N(r)-1} + ... + a_{N(r)}\right| \leq |a_1|(\rho^L)^{N(r)-1}|z|^{N(r)-1} + ... + \left|a_{N(r)}\right|\\ &\leq |a_1|(\rho^L)^{N(r)-1}\left(|z|^{N(r)-1} + ... + |z| + 1\right)\end{aligned}$$

provided $|z| \neq 0$. So $\left|a_1(\rho^L)^{N(r)-1}z^{N(r)-1} + ... + a_{N(r)}\right| \leq |a_1|(\rho^L)^{N(r)-1}|z|^{N(r)}\left(\frac{1}{|z|} + ...... + \frac{1}{|z|^{N(r)}}\right)$. So for $|z| \neq 0$, $|F(z)| \geq |a_0|(\rho^L)^{N(r)}|z|^{N(r)} - |a_1|(\rho^L)^{N(r)-1}|z|^{N(r)}\left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right)$
$= (\rho^L)^{N(r)-1}|z|^{N(r)}\left[|a_0|\rho^L - |a_1|\left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right)\right]$. Therefore for $|z| \neq 0$,

$$|F(z)| > (\rho^L)^{N(r)-1}|z|^{N(r)}\left[|a_0|\rho^L - |a_1|\sum_{k=1}^{\infty}\frac{1}{|z|^k}\right]. \tag{3.9}$$

The geometric series $\sum\limits_{k=1}^{\infty}\frac{1}{|z|^k}$ is convergent for $\frac{1}{|z|} < 1$ *i.e.*, for $|z| > 1$ and converges to

$$\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z|-1} \text{ if } |z| > 1. \tag{3.10}$$

Using (3.9) and (3.10) we have for $|z| > 1$, $|F(z)| > (\rho^L)^{N(r)-1} |z|^{N(r)} \left[|a_0| \rho^L - \frac{|a_1|}{|z|-1}\right]$. Hence for $|z| > 1$,

$$|F(z)| > 0 \text{ if } |z|^{N(r)} (\rho^L)^{N(r)-1} \left[|a_0| \rho^L - \frac{|a_1|}{|z|-1}\right] \geq 0$$

$$\text{i.e., if } |a_0| \rho^L - \frac{|a_1|}{|z|-1} \geq 0$$
$$\text{i.e., if } |a_0| \rho^L \geq \frac{|a_1|}{|z|-1}$$
$$\text{i.e., if } |z| \geq 1 + \frac{|a_1|}{|a_0| \rho^L} > 1.$$

Therefore $|F(z)| > 0$ for $|z| \geq 1 + \frac{|a_1|}{|a_0|\rho^L}$. So $F(z)$ does not vanish in $|z| \geq 1 + \frac{|a_1|}{|a_0|\rho^L}$. Equivalently all the zeros of $F(z)$ lie in $|z| < 1 + \frac{|a_1|}{|a_0|\rho^L}$. Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $a_0 \neq 0$ and $z_0 \neq 0$.
Now let us put $z = \frac{1}{\rho^L z_0}$ in $F(z)$. So we have $F\left(\frac{1}{\rho^L z_0}\right) = (\rho^L)^{N(r)} \left(\frac{1}{\rho^L z_0}\right)^{N(r)} .P(z_0) = \left(\frac{1}{z_0}\right)^{N(r)} .0 = 0$. Therefore $z = \frac{1}{\rho^L z_0}$ is a root of $F(z)$.
Hence

$$\left|\frac{1}{\rho^L z_0}\right| < 1 + \frac{|a_1|}{|a_0| \rho^L}$$

$$\text{i.e., } \frac{1}{|z_0|} < \rho^L \left(1 + \frac{|a_1|}{|a_0| \rho^L}\right)$$
$$\text{i.e., } |z_0| > \frac{1}{\rho^L \left(1 + \frac{|a_1|}{|a_0|\rho^L}\right)}.$$

As $z_0$ is an arbitrary zero of $P(z)$, all the zeros of $P(z)$ lie on

$$|z| > \frac{1}{\rho^L \left(1 + \frac{|a_1|}{|a_0|\rho^L}\right)}. \tag{3.11}$$

From (3.8) and (3.11) we get that all the zeros of $P(z)$ lie on the proper ring shaped region $\frac{1}{\rho^L\left(1+\frac{|a_1|}{|a_0|\rho^L}\right)} < |z| < \frac{1}{\rho^L} \left(1 + \frac{|a_0|}{|a_{N(r)}|} (\rho^L)^{N(r)}\right)$ where $|a_0| (\rho^L)^{N(r)} \geq |a_1| (\rho^L)^{N(r)-1} \geq ... \geq \left|a_{N(r)}\right|$ for $\rho^L > 0$.
This proves the theorem. □

In the line of Theorem 3.3, we may state the following theorem in view of Lemma 2.2 :

**Theorem 3.4.** *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

*with $L^*$-order $\rho^{L^*}$, $a_{N(r)} \neq 0$, $a_0 \neq 0$ and also $a_n \to 0$ for $n > N(r)$ for the disc $|z| \leq [re^{L(r)}]$ when $r$ is sufficiently large. Further, for $\rho^{L^*} > 0$,*

$$|a_0| (\rho^{L^*})^{N(r)} \geq |a_1| (\rho^{L^*})^{N(r)-1} \geq ..... \geq \left|a_{N(r)-1}\right| \rho^{L^*} \geq \left|a_{N(r)}\right|.$$

*Then all the zeros of $P(z)$ lie in the ring shaped region*

$$\frac{1}{\rho^{L^*}\left(1 + \frac{|a_1|}{|a_0|\rho^{L^*}}\right)} < |z| < \frac{1}{\rho^{L^*}}\left(1 + \frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^{L^*})^{N(r)}\right).$$

The proof is omitted.

**Corollary 3.1.** *From Theorem 3.3 we can easily conclude that all the zeros of*

$$P(z) = a_0 + a_1 z + ... + a_n z^n$$

*of degree $n$, $|a_n| \neq 0$ with the property $|a_0| \geq |a_1| \geq ... \geq |a_n|$ lie in the proper ring shaped region*

$$\frac{1}{\left(1 + \frac{|a_1|}{|a_0|}\right)} < |z| < \left(1 + \frac{|a_0|}{|a_n|}\right)$$

*just on putting $\rho^L = 1$.*

**Corollary 3.2.** *From Theorem 3.4 we can easily conclude that all the zeros of*

$$P(z) = a_0 + a_1 z + ... + a_n z^n$$

*of degree $n$, $|a_n| \neq 0$ with the property $|a_0| \geq |a_1| \geq ... \geq |a_n|$ lie in the proper ring shaped region*

$$\frac{1}{\left(1 + \frac{|a_1|}{|a_0|}\right)} < |z| < \left(1 + \frac{|a_0|}{|a_n|}\right)$$

*just on putting $\rho^{L^*} = 1$.*

**Theorem 3.5.** *Let $P(z)$ be an entire function with $L$-order $\rho^L$. For sufficiently large values of $r$ in the disk $|z| \leq [rL(r)]$, the Taylor's series expansion of $P(z)$*

$$P(z) = a_0 + a_{p_1} z^{p_1} + a_{p_2} z^{p_2} + ... + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}, a_0 \neq 0$$

*be such that $1 \leq p_1 < p_2 ... < p_m \leq N(r) - 1$, $p_i$'s are integers and for $\rho^L > 0$,*

$$|a_0| (\rho^L)^{N(r)} \geq \left|a_{p_1}\right| (\rho^L)^{N(r)-p_1} \geq ... \geq \left|a_{p_m}\right| (\rho^L)^{N(r)-p_m}.$$

*Then all the zeros of $P(z)$ lie in the proper ring shaped region*

$$\frac{1}{\rho^L t_0'} < |z| < \frac{1}{\rho^L} t_0$$

*where $t_0$ and $t_0'$ are the unique positive roots of the equations*

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)-p_m} - \left|a_{N(r)}\right| t^{N(r)-p_m-1} - |a_0| (\rho^L)^{N(r)} = 0 \text{ and}$$
$$f(t) \equiv |a_0| (\rho^L)^{p_1} t^{p_1} - |a_0| (\rho^L)^{p_1} t^{p_1-1} - \left|a_{p_1}\right| = 0$$

*respectively.*

*Proof.* Let

$$P(z) = a_0 + a_{p_1} z^{p_1} + ... + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}, \left|a_{N(r)}\right| \neq 0. \tag{3.12}$$

Also for $\rho^L > 0$, $|a_0| (\rho^L)^{N(r)} \geq \left|a_{p_1}\right| (\rho^L)^{N(r)-p_1} \geq ... \geq \left|a_{N(r)}\right|$. Let us consider

$$R(z) = (\rho^L)^{N(r)} P\left(\frac{z}{\rho^L}\right) = (\rho^L)^{N(r)} \left\{a_0 + a_{p_1} \frac{z^{p_1}}{(\rho^L)^{p_1}} + .... + a_{p_m} \frac{z^{p_m}}{(\rho^L)^{p_m}} + a_{N(r)} \frac{z^{N(r)}}{(\rho^L)^{N(r)}}\right\}$$
$$= a_0(\rho^L)^{N(r)} + a_{p_1}(\rho^L)^{N(r)-p_1} z^{p_1} + ... + a_{p_m}(\rho^L)^{N(r)-p_m} z^{p_m} + a_{N(r)} z^{N(r)}.$$

Therefore

$$|R(z)| \geq \left|a_{N(r)} z^{N(r)}\right| - \left|a_0(\rho^L)^{N(r)} + a_{p_1}(\rho^L)^{N(r)-p_1} z^{p_1} + ... + a_{p_m}(\rho^L)^{N(r)-p_m} z^{p_m}\right|. \tag{3.13}$$

Now for $|z| \neq 0$,

$$\begin{aligned}
&\left|a_0(\rho^L)^{N(r)} + a_{p_1}(\rho^L)^{N(r)-p_1} z^{p_1} + ... + a_{p_m}(\rho^L)^{N(r)-p_m} z^{p_m}\right| \\
&\leq |a_0| (\rho^L)^{N(r)} + \left|a_{p_1}\right| (\rho^L)^{N(r)-p_1} |z|^{p_1} + ... + \left|a_{p_m}\right| (\rho^L)^{N(r)-p_m} |z|^{p_m} \\
&\leq |a_0| (\rho^L)^{N(r)} (1 + |z|^{p_1} + ... + |z|^{p_m}) \\
&= |a_0| (\rho^L)^{N(r)} |z|^{p_m+1} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{p_m+1-p_2}} + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}}\right).
\end{aligned} \tag{3.14}$$

Using (3.13) and (3.14), we have for $|z| \neq 0$

$$\begin{aligned}
|R(z)| &\geq \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} |z|^{p_m+1} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}}\right) \\
&> \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} |z|^{p_m+1} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}} + ...\right) \\
&= \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} |z|^{p_m+1} \sum_{k=1}^{\infty} \frac{1}{|z|^k}.
\end{aligned} \tag{3.15}$$

The geometric series $\sum_{k=1}^{\infty}\frac{1}{|z|^k}$ is convergent for $\frac{1}{|z|} < 1$i.e.,for $|z| > 1$ and converges to $\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}$. Therefore $\sum_{k=1}^{\infty}\frac{1}{|z|^k} = \frac{1}{|z|-1}$ for $|z| > 1$. So on $|z| > 1$,

$$\begin{aligned}
&|R(z)| > 0 \text{ if } \left|a_{N(r)}\right| |z|^{N(r)} - \frac{|a_0| \left(\rho^L\right)^{N(r)} |z|^{p_m+1}}{|z|-1} \geq 0\\
&\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)} \geq \frac{|a_0| \left(\rho^L\right)^{N(r)} |z|^{p_m+1}}{|z|-1}\\
&\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)+1} - \left|a_{N(r)}\right| |z|^{N(r)} \geq |a_0| \left(\rho^L\right)^{N(r)} |z|^{p_m+1}\\
&\text{i.e., if } |z|^{p_m+1}\left(\left|a_{N(r)}\right| |z|^{N(r)-p_m} - \left|a_{N(r)}\right| |z|^{N(r)-p_m-1} - |a_0| \left(\rho^L\right)^{N(r)}\right) \geq 0.
\end{aligned}$$

Let us consider $g(t) \equiv \left|a_{N(r)}\right| |t|^{N(r)-p_m} - \left|a_{N(r)}\right| |t|^{N(r)-p_m-1} - |a_0| \left(\rho^L\right)^{N(r)} = 0$. Clearly $g(t) = 0$ has one positive root because the maximum number of changes in sign in $g(t)$ is one and $g(0) = -|a_0|\rho^{N(r)}$ is $-ve$, $g(\infty)$ is $+ve$.
Let $t_0$ be the positive root of $g(t) = 0$ and $t_0 > 1$. Clearly for $t > t_0$, $g(t) > 0$. If not for some $t_1 > t_0$, $g(t_1) < 0$.
Then $g(t_1) < 0$ and $g(\infty) > 0$. Therefore $g(t) = 0$ must have another positive root in $(t_1, \infty)$ which gives a contradiction .
Hence for $t \geq t_0$, $g(t) \geq 0$ and $t_0 > 1$. So $|R(z)| > 0$ for $|z| \geq t_0$.
Thus $R(z)$ does not vanish in $|z| \geq t_0$.

Hence all the zeros of $R(z)$ lie in $|z| < t_0$.

Let $z = z_0$ be any zero of $P(z)$. So $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \rho^L z_0$ in $R(z)$ we have $R(\rho^L z_0) = (\rho^L)^{N(r)}.P(z_0) = (\rho^L)^{N(r)}.0 = 0$. Therefore $R(\rho^L z_0) = 0$ and so $z = \rho^L z_0$ is a zero of $R(z)$ and consequently $\left|\rho^L z_0\right| < t_0$ which implies $|z_0| < \frac{t_0}{\rho^L}$. As $z_0$ is an arbitrary zero of $P(z)$,

$$\text{all the zeros of } P(z) \text{ lie in } |z| < \frac{t_0}{\rho^L}. \tag{3.16}$$

Again let us consider $F(z) = (\rho^L)^{N(r)} z^{N(r)} P\left(\frac{1}{\rho^L z}\right)$. Now

$$\begin{aligned}
F(z) &= (\rho^L)^{N(r)} z^{N(r)} . \left\{a_0 + a_{p_1}\frac{1}{(\rho^L)^{p_1} z^{p_1}} + ..... + a_{p_m}\frac{1}{(\rho^L)^{p_m} z^{p_m}} + a_{N(r)}\frac{1}{(\rho^L)^{N(r)} z^{N(r)}}\right\}\\
&= a_0(\rho^L)^{N(r)} z^{N(r)} + a_{p_1}(\rho^L)^{N(r)-p_1} z^{N(r)-p_1} + ... + a_{p_m}(\rho^L)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}.
\end{aligned}$$

Also

$$\begin{aligned}
&\left|a_{p_1}(\rho^L)^{N(r)-p_1} z^{N(r)-p_1} + ... + a_{p_m}(\rho^L)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}\right|\\
&\leq \left|a_{p_1}\right| (\rho^L)^{N(r)-p_1} |z|^{N(r)-p_1} + ... + \left|a_{p_m}\right| (\rho^L)^{N(r)-p_m} |z|^{N(r)-p_m} + \left|a_{N(r)}\right|\\
&\leq \left|a_{p_1}\right| (\rho^L)^{N(r)-p_1} \left(|z|^{N(r)-p_1} + |z|^{N(r)-p_2} + ... + |z|^{N(r)-p_m} + 1\right).
\end{aligned}$$

So for $|z| \neq 0$,

$$\begin{aligned}|F(z)| &\geq |a_0|\,(\rho^L)^{N(r)}\,|z|^{N(r)} - \left|a_{p_1}(\rho^L)^{N(r)-p_1}z^{N(r)-p_1} + \ldots + a_{p_m}(\rho^L)^{N(r)-p_m}z^{N(r)-p_m} + a_{N(r)}\right| \\ &\geq |a_0|\,(\rho^L)^{N(r)}\,|z|^{N(r)} - \left|a_{p_1}\right|(\rho^L)^{N(r)-p_1}\left(|z|^{N(r)-p_1} + |z|^{N(r)-p_2} + \ldots + |z|^{N(r)-p_m} + 1\right) \\ &= |a_0|\,(\rho^L)^{N(r)}\,|z|^{N(r)} - \left|a_{p_1}\right|(\rho^L)^{N(r)-p_1}\,|z|^{N(r)-p_1+1}\left(\frac{1}{|z|} + \frac{1}{|z|^{p_2-p_1+1}} + \ldots + \frac{1}{|z|^{N(r)-p_1+1}}\right)\end{aligned}$$

i.e., on $|z| \neq 0$, $|F(z)| > |a_0|\,(\rho^L)^{N(r)}\,|z|^{N(r)} - \left|a_{p_1}\right|(\rho^L)^{N(r)-p_1}\,|z|^{N(r)-p_1+1}\left(\sum_{k=1}^{\infty}\frac{1}{|z|^k}\right)$. The geometric series $\sum_{k=1}^{\infty}\frac{1}{|z|^k}$ is convergent for $\frac{1}{|z|} < 1$i.e., for $|z| > 1$ and converges to $\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}$. Therefore $\sum_{k=1}^{\infty}\frac{1}{|z|^k} = \frac{1}{|z|-1}$ for $|z| > 1$. Therefore for $|z| > 1$

$$\begin{aligned}|F(z)| &> |a_0|\,(\rho^L)^{N(r)}\,|z|^{N(r)} - \left|a_{p_1}\right|(\rho^L)^{N(r)-p_1}\,|z|^{N(r)-p_1+1}\left(\frac{1}{|z|-1}\right) \\ &= (\rho^L)^{N(r)-p_1}\left((\rho^L)^{p_1}\,|a_0|\,|z|^{N(r)} - \left|a_{p_1}\right|\frac{|z|^{N(r)-p_1+1}}{|z|-1}\right) \\ &= (\rho^L)^{N(r)-p_1}\,|z|^{N(r)-p_1+1}\left(|a_0|\left(\rho^L\right)^{p_1}|z|^{p_1-1} - \frac{\left|a_{p_1}\right|}{|z|-1}\right)\end{aligned}$$

For $|z| > 1$,

$$\begin{aligned}|F(z)| > 0 \text{ if } &|a_0|\,(\rho^L)^{p_1}\,|z|^{p_1-1} - \frac{\left|a_{p_1}\right|}{|z|-1} \geq 0 \\ \text{i.e., if } &|a_0|\,(\rho^L)^{p_1}\,|z|^{p_1-1} \geq \frac{\left|a_{p_1}\right|}{|z|-1} \\ \text{i.e., if } &|a_0|\,(\rho^L)^{p_1}\,|z|^{p_1} - |a_0|\,(\rho^L)^{p_1}\,|z|^{p_1-1} - \left|a_{p_1}\right| \geq 0.\end{aligned} \quad (3.17)$$

Therefore on $|z| > 1, |F(z)| > 0$ if (3.17) holds. Let us consider $f(t) = |a_0|\,(\rho^L)^{p_1}t^{p_1} - |a_0|\,(\rho^L)^{p_1}t^{p_1-1} - \left|a_{p_1}\right| = 0$. Clearly $f(t) = 0$ has exactly one positive root and is greater than one. Let $t_0'$ be the positive root of $f(t) = 0$. Therefore $t_0' > 1$. Obviously if $t \geq t_0'$ then $f(t) \geq 0$. So for $|F(z)| > 0$, $|z| \geq t_0'$. Therefore $F(z)$ does not vanish in $|z| \geq t_0'$.

Hence all the zeros of $F(z)$ lie in $|z| < t_0'$.

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.
Now putting $z = \frac{1}{\rho^L z_0}$ in $F(z)$ we obtain that $F\left(\frac{1}{\rho^L z_0}\right) = (\rho^L)^{N(r)}\left(\frac{1}{\rho^L z_0}\right)^{N(r)}.P(z_0) = \left(\frac{1}{z_0}\right)^{N(r)}.P(z_0)$ $= \left(\frac{1}{z_0}\right)^{N(r)}.0 = 0$. Therefore $z = \frac{1}{\rho^L z_0}$ is a zero of $F(z)$. Now $\left|\frac{1}{\rho^L z_0}\right| < t_0'$ *i.e.*, $\left|\frac{1}{z_0}\right| < \rho^L t_0'$ *i.e.*, $|\,z_0| > \frac{1}{\rho^L t_0'}$.
As $z_0$ is an arbitrary zero of $P(z)$ therefore we obtain that

$$\text{all the zeros of } P(z) \text{ lie in } |z| > \frac{1}{\rho^L t_0'}. \quad (3.18)$$

Using (3.16) and (3.18) we get that all the zeros of $P(z)$ lie in the ring shaped region $\frac{1}{\rho^{L} t_0'} < |z| < \frac{t_0}{\rho^{L}}$ where $t_0, t_0'$ are the unique positive roots of the equations $g(t) = 0$ and $f(t) = 0$ respectively whose forms are given in the statement of Theorem 3.3. This proves the theorem. □

In the line of Theorem 3.5, we may state the following theorem in view of Lemma 2.2 :

**Theorem 3.6.** *Let $P(z)$ be an entire function with $L^*$-order $\rho^{L^*}$. For sufficiently large values of $r$ in the disk $|z| \leq [re^{L(r)}]$, the Taylor's series expansion of $P(z)$*

$$P(z) = a_0 + a_{p_1} z^{p_1} + a_{p_2} z^{p_2} + ... + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}, a_0 \neq 0$$

*be such that $1 \leq p_1 < p_2 ... < p_m \leq N(r) - 1$, $p_i$'s are integers and for $\rho^{L^*} > 0$,*

$$|a_0| (\rho^{L^*})^{N(r)} \geq \left|a_{p_1}\right| (\rho^{L^*})^{N(r)-p_1} \geq ... \geq \left|a_{p_m}\right| (\rho^{L^*})^{N(r)-p_m}.$$

*Then all the zeros of $P(z)$ lie in the proper ring shaped region*

$$\frac{1}{\rho^{L^*} t_0'} < |z| < \frac{1}{\rho^{L^*}} t_0$$

*where $t_0$ and $t_0'$ are the unique positive roots of the equations*

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)-p_m} - \left|a_{N(r)}\right| t^{N(r)-p_m-1} - |a_0| (\rho^{L^*})^{N(r)} = 0 \text{ and}$$
$$f(t) \equiv |a_0| (\rho^{L^*})^{p_1} t^{p_1} - |a_0| (\rho^{L^*})^{p_1} t^{p_1-1} - \left|a_{p_1}\right| = 0$$

*respectively.*

The proof is omitted.

**Corollary 3.3.** *In view of Theorem 3.5 we may state that all the zeros of the polynomial $P(z) = a_0 + a_{p_1} z^{p_1} + ... + a_{p_m} z^{p_m} + a_n z^n$ of degree $n$ with $1 \leq p_1 < p_2 < ... < p_m \leq n - 1$, $p_i$'s are integers such that*

$$|a_0| \geq \left|a_{p_1}\right| \geq ........ \geq |a_n|$$

*lie in ring shaped region*

$$\frac{1}{t_0'} < |z| < t_0$$

*where $t_0, t_0'$ are the unique positive roots of the equations*

$$g(t) \equiv |a_n| t^{n-p_m} - |a_n| t^{n-p_m-1} - |a_0| = 0$$

*and*

$$f(t) \equiv |a_0| t^{p_1} - |a_0| t^{p_1-1} - \left|a_{p_1}\right| = 0$$

*respectively just substituting $\rho^{L} = 1$.*

**Corollary 3.4.** *In view of Theorem 3.6 we may state that all the zeros of the polynomial* $P(z) = a_0 + a_{p_1}z^{p_1} + ... + a_{p_m}z^{p_m} + a_n z^n$ *of degree n with* $1 \leq p_1 < p_2 < ... < p_m \leq n-1$, $p_i$*'s are integers such that*

$$|a_0| \geq \left|a_{p_1}\right| \geq ........ \geq |a_n|$$

*lie in ring shaped region*

$$\frac{1}{t_0'} < |z| < t_0$$

*where* $t_0, t_0'$ *are the unique positive roots of the equations*

$$g(t) \equiv |a_n|\, t^{n-p_m} - |a_n|\, t^{n-p_m-1} - |a_0| = 0$$

*and*

$$f(t) \equiv |a_0|\, t^{p_1} - |a_0|\, t^{p_1-1} - \left|a_{p_1}\right| = 0$$

*respectively just substituting* $\rho^{L^*} = 1$.

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# Primality Testing and Integer Factorization by using Fourier Transform of a Correlation Function Generated from the Riemann Zeta Function

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## Abstract

In this article, the author tries to make primarity testing and factorization of integers by using Fourier transform of a correlation function generated from the Riemann zeta function.

*Keywords:* Primarity testing, factorization, Fourier transform, Riemann zeta function.
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## 1. Introduction

In number theory, integer factorization or prime factorization is the decomposition of a composite number into smaller non-trivial divisors, which when multiplied together equal the original integer. When the numbers are very large, no efficient, non-quantum integer factorization algorithm is known; an effort by several researchers concluded in 2009, factoring a 232-digit number (RSA-768), utilizing hundreds of machines over a span of 2 years. The presumed difficulty of this problem is at the heart of widely used algorithms in cryptography such as RSA (Rivest *et al.*, 1978). Many areas of mathematics and computer science have been brought to bear on the problem, including elliptic curves, algebraic number theory, and quantum computing.

In this article, the author tries to make primarity testing and factorization of integers by using Fourier transform of a correlation function generated from the Riemann zeta function.

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## 2. Frequency Spectrum of a Correlation Function generated from the Riemann Zeta Function

Riemann zeta function is an analytic function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, which can also be given by (Hardy & Riesz, 2005).

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \quad (\mathrm{Re}[\,s\,] > 1)\,, \tag{2.1}$$

where $\Gamma(s)$ is a Gamma function.

We define the Fourier transform of $z_\sigma(t, \tau)$ shown as

$$Z_\sigma(t, \omega) = \lim_{T\to\infty} \int_{-T}^{+T} z_\sigma(t, \tau) e^{-i\omega\tau} d\tau\,, \tag{2.2}$$

where $z_\sigma(t, \tau)$ is a time-dependent autocorrelation function (Yen, 1987) defined by

$$z_\sigma(t, \tau) = \zeta(\sigma - i(t + \tau/2)) \cdot \zeta^*(\sigma - i(t - \tau/2))\,.$$

In this formula, $\zeta^*(s)$ is a conjugate of $\zeta(s)$ .

From the infinite sum of the Riemann zeta function given by $\zeta(\sigma - it) = \sum_{n=1}^{\infty} \frac{\exp(it\log n)}{n^\sigma}$ , we have

$$Z_\sigma(t, \omega) = \lim_{T\to\infty} \int_{-T}^{+T} \sum_{k=1}^{\infty} \frac{1}{k^\sigma} \exp\left[\, i\,(t + \tau/2)\log k\,\right] \cdot \sum_{l=1}^{\infty} \frac{1}{l^\sigma} \exp\left[\, -i\,(t - \tau/2)\log l\,\right] e^{-i\omega\tau} d\tau$$

$$= \lim_{T\to\infty} \int_{-T}^{+T} \sum_{k,l=1}^{\infty} \frac{1}{(kl)^\sigma} \exp\left[\, i\log(k/l)t\,\right] \exp\left[\, i\log(kl)\tau/2\,\right] e^{-i\omega\tau} d\tau.$$

For the integer $n$ , put $n = kl$ , then we can write

$$Z_\sigma(t, \omega) = \lim_{T\to\infty} \sum_{k,l=1}^{\infty} \frac{1}{n^\sigma} \exp\left[\, i\log(k/l)\,t\,\right] \int_{-T}^{+T} \exp\left(i\tau\log n/2\right) e^{-i\omega\tau} d\tau\,,$$

where $\int_{-T}^{+T} \exp\left(i\tau\log n/2\right) e^{-i\omega\tau} d\tau = \frac{2T\sin\left(\omega - \frac{1}{2}\log n\right)}{(\omega - \frac{1}{2}\log n)}$ .

When we let $a(n, t) = \sum_{n=kl} \exp\left[i\log(k/l)\,t\right]$, Eq.(2) can be rewritten as

$$Z_\sigma(t, \omega) = \lim_{T\to\infty} \sum_{n=1}^{\infty} \frac{a(n, t)}{n^\sigma} \frac{2T\sin\left(\omega - \frac{1}{2}\log n\right)}{\left(\omega - \frac{1}{2}\log n\right)} = \sum_{n=1}^{\infty} \frac{a(n, t)}{n^\sigma} 2\pi\delta\left(\omega - \frac{1}{2}\log n\right)\,,$$

where $a(n,t)$ is a real valued function given by

$$a(n,t) = \frac{1}{2}\sum_{n=kl}\left\{\exp\left[i\log\left(k/l\right)t\right] + \exp\left[i\log\left(l/k\right)t\right]\right\} = \sum_{n=kl}\cos\left[\,\log\left(k/l\right)t\,\right]$$

and $\delta(\omega)$ is a Dirac's delta function.

**Lemma 2.1.** *$a(n,t)$ is a multiplicative on $n$ .*

*Proof.* As we can write $a(n,t) = \sum_{n=kl}\exp[i\log(k/l)t]$ , the multiplicative property of which can be shown from

$$a(n,t) = \sum_{k|n}\exp\left(it\log(k^2/n)\right) = \frac{1}{n^{it}}\sum_{k|n}k^{2it},$$

where the subscript $k|n$ indicates integers $k$ which divide $n$ .

If $f(n)$ is multiplicative, then $F(n) = \sum_{d|n} f(d)$ is multiplicative. From which, we have $a(mn,t) = a(m,t)a(n,t)$ for the case when satisfying $(m,n) = 1$ , because $k^{2it}$ is multiplicative. □

From the definition of $a(n,t)$ , we can obtain the following recurrence formula given by (Musha, 2012).

$$a(p^r,t) = a(p^{r-1},t)\cos(t\log p) + \cos(rt\log p) \quad (r = 1,2,3,\cdots). \tag{2.3}$$

From which, it can be proved that $a(p^r,t) = \dfrac{\sin[(r+1)t\log p]}{\sin(t\log p)}$. (2.4)

From Eq.(3), we have $Z_\sigma\left(t,\frac{1}{2}\log n\right) = \dfrac{2\pi\delta(0)}{n^\sigma}a(n,t)$ .

For the integer $n$ given by $n = p^a q^b r^c\cdots$ , we have

$$Z_\sigma\left(t,\tfrac{1}{2}\log n\right) = \frac{2\pi\delta(0)}{n^\sigma}\frac{\sin[(a+1)t\log p]}{\sin(t\log p)}\frac{\sin[(b+1)t\log q]}{\sin(t\log q)}\frac{\sin[(c+1)t\log r]}{\sin(t\log r)}\cdots,$$

from Lemma.1 and Eq.(5).

From the Fourier transform of $Z_a\left(t,\frac{1}{2}\log n\right)$ given by $F_n(\omega) = \int_{-\infty}^{+\infty} Z_\sigma\left(t,\frac{1}{2}\log n\right)e^{-i\omega t}dt$ , we can obtain the following Lemma.

**Lemma 2.2.** *If $n = p_1p_2p_3\cdots p_k$ , where $p_1,p_2,p_3,\cdots,p_k$ are different primes, $F_n(\omega)$ is consisted of $2^{k-1}$ discrete spectrum.*

*Proof.* From Eq. (4), we have

$$a(n,t) = 2\cos(t\log p_1)\cdot 2\cos(t\log p_2)\cdot 2\cos(t\log p_3)\cdots 2\cos(t\log p_k).$$

By the trigonometrical formula shown as $\cos\alpha\cdot\cos\beta=\frac{1}{2}\{\cos(\alpha-\beta)+\cos(\alpha+\beta)\}$, we have

$$a(n,t)=2^2\times\frac{1}{2}\left\{\cos[t(\log p_1-\log p_2)]+\cos[t(\log p_1+\log p_2)]\right\}\cdot 2\cos(t\log p_3)\cdots 2\cos(t\log p_k)$$
$$=2^2\left\{\cos[t(\log p_1-\log p_2)]\cos(t\log p_3)\quad+\cos[t(\log p_1+\log p_2)]\cos(t\log p_3)\right\}\cdots 2\cos(t\log p_k)$$
$$=2^2\times\frac{1}{2}\left\{\cos[t(\log p_1-\log p_2-\log p_3)]\ +\cos[t(\log p_1-\log p_2+\log p_3)]\right.$$
$$\left.+\cos[t(\log p_1+\log p_2-\log p_3)]+\cos[t(\log p_1+\log p_2+\log p_3)]\right\}2\cos(t\log p_4)\cdots 2\cos(t\log p_k)$$

By repeating the above computations, we have

$$a(n,t)=2\sum_{i=1}^{2^{k-1}}\cos[\,t(\lambda_{i1}\log p_1+\lambda_{i2}\log p_2+\cdots+\lambda_{ik}\log p_k)],$$

where $\lambda_{i1}=+1$ and $\lambda_{ij}=+1$ or $-1$ for $j>1$.

As $\log p_1$, $\log p_2$, $\log p_3$, $\cdots$, $\log p_k$ are linearly independent over $\mathbf{Z}$ (Kac, 1959), thus $F_n(\omega)$ is consisted of $2^{k-1}$ different spectrum. □

Then we obtain following Theorems.

**Theorem 2.1.** *If and only $F_n(\omega)$ is consisted of a single spectra for $\omega\geq 0$, then n is a prime.*

*Proof.* The Fourier transform of $\cos(t\log p)$ can be given by $\pi[\delta(\omega-\log p)+\delta(\omega+\log p)]$, and thus it is clear from Lemma 2.2. □

**Theorem 2.2.** *If and only $F_n(\omega)$ is consisted of two spectrum for $\omega\geq 0$, then n has either form of $n=p\cdot q\ (p\neq q)$, $n=p^2$ or $n=p^3$.*

*Proof.* From Theorem I, there is only a case for the integer $n=p_1p_2\cdots p_k$, when $F_n(\omega)$ is consisted of two spectrum, that is $n=p\cdot q\quad(p\neq q)$.

From Eq.(4), we have following equations for $a(p^r,t)$;

$$r=1,\ a(p,t)\ =2\cos(t\log p)$$
$$r=2,\ a(p^2,t)=1+2\cos(2t\log p)$$
$$r=3,\ a(p^3,t)=2\cos(t\log p)+2\cos(3t\log p)$$
$$r=4,\ a(p^4,t)=1+2\cos(2t\log p)+2\cos(4t\log p)$$
$$r=5,\ a(p^5,t)=2\cos(t\log p)+2\cos(3t\log p)+2\cos(5t\log p)$$
$$r=6,\ a(p^6,t)=1+2\cos(2t\log p)+2\cos(4t\log p)+2\cos(6t\log p)$$
$$r=7,\ a(p^7,t)=2\cos(t\log p)+2\cos(3t\log p)+2\cos(5t\log p)+2\cos(7t\log p)$$
$$\vdots$$

Including the spectra at $\omega=0$, there are cases for $r=2$ and $r=3$ when $a(n,t)$ has two spectrum. □

**Theorem 2.3.** *If $F_n(\omega)$ is consisted of two spectrums at frequencies $\omega_1$ and $\omega_2$ and $n = p \cdot q$, we can obtain factors of an integer $n$ given by $p = \exp\left(\frac{\omega_2 - \omega_1}{2}\right)$ and $q = \exp\left(\frac{\omega_1 + \omega_2}{2}\right)$.*

*Proof.* If $n = p \cdot q$, then we obtain $Z_\sigma\left(t, \frac{1}{2}\log n\right) = \frac{4\pi\delta(0)}{n^\sigma} \times \cos(t \log p) \cdot \cos(t \log q) = \frac{2\pi\delta(0)}{n^\sigma}\left\{\cos[(\log q - \log p)t] + \cos[(\log q + \log p)t]\right\}$.

When we let $\omega_1 = \log q - \log p$, $\omega_2 = \log q + \log p$, we have $p = \exp\left(\frac{\omega_2 - \omega_1}{2}\right)$, $q = \exp\left(\frac{\omega_1 + \omega_2}{2}\right)$. □

## 3. Primarity Testing and Factorization from Fourier spectrum

From Theorems 2.1, 2.2 and 2.3, we can make primality testing and factorization of the integer $n$ consisted of two primes from the Fourier spectrum $F_n(\omega)$ $(\omega \geq 0)$ by following procedures;

At first, compute the Fourier transform $Z_\sigma(t, \omega) = \int_{-\infty}^{+\infty} z_\sigma(t, \tau) e^{-i\omega\tau} d\tau$, where $z_\sigma(t, \tau) = \zeta(\sigma - i(t + \tau/2)) \cdot \zeta^*(\sigma - i(t - \tau/2))$, from which we can obtain the Fourier spectrum by $F_n(\omega) = \int_{-\infty}^{+\infty} Z_\sigma\left(t, \frac{1}{2}\log n\right) e^{-i\omega t} dt$. Then we can make primality testing and integer factorization of an integer $n$, the process of which is shown in Figure 1.

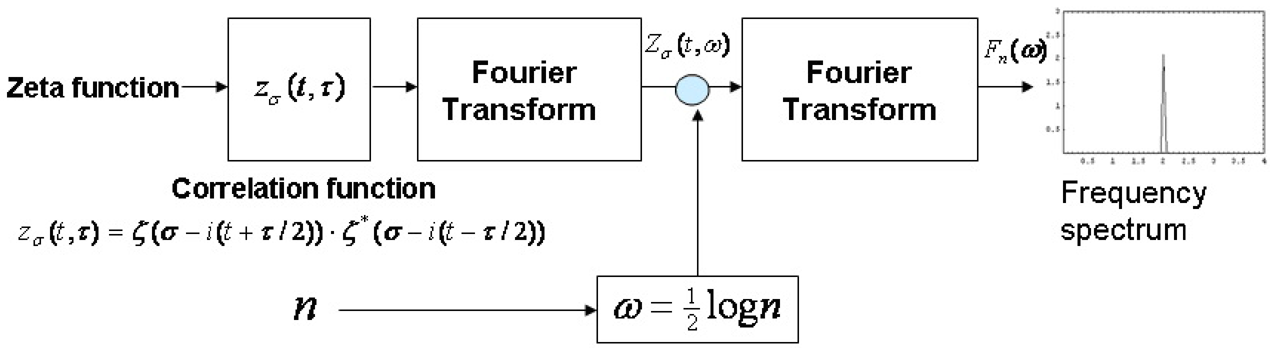


**Figure 1.** Process to conduct primality testing for the integer $n$.

From this process, we can recognize the prime as a single spectra from the frequency analysis result. If there are two spectrum observed from the calculation result, $n$ has either form of $n = p \cdot q$ $(p \neq q)$, $n = p^2$ or $n = p^3$.

In this case, we can obtain factors of an integer $n$ from Theorem 2.3.

As the Fourier transform $Z_\sigma(t, \omega) = \int_{-\infty}^{+\infty} z_\sigma(t, \tau) e^{-i\omega\tau} d\tau$ can be computed by using discrete FFT (fast Fourier transform) algorithm for the calculation of Wigner distribution function (Boashash & Black, 1987), (Dellomo & Jacyna, 1991) because $Z_\sigma(t, \omega)$ can be regarded as a Wigner distribution of the Riemann's zeta function, we can obtain the Fourier spectrum of $F_n(\omega)$ by conducting FFT calculations.

By using this method, we can propose some possible applications which use the theory presented in this paper.

- Primary testing of large numbers such as Mersenne numbers $2^m - 1$ can be conducted by using the algorithm shown in Figure 1 from the approximation, $\omega = \frac{1}{2}\log(2^m - 1) = \frac{m}{2}\log 2 - 1/2^{m+1} - 1/2^{2m+2} - \cdots$ .

- Factorization of an integer $n$ consisted of two primes can be conducted by using this method. By using FFT algorithm, there is a possibility to complete the computation within a polynomial time, whereas there is no known efficient algorithm that runs in polynomial time (Ribenboim, 1991).

- Breaking the public-key crypto system, which is considered to be hard by using the conventional computer systems, because the RSA crypto-system depends on the factorization of an integer composed of two large primes.

It is also known that Fourier transform can be conducted by the quantum computer, the schematic diagram for the quantum Fourier transform is shown in Figure 2 (Nielsen & Chuang, 2000).

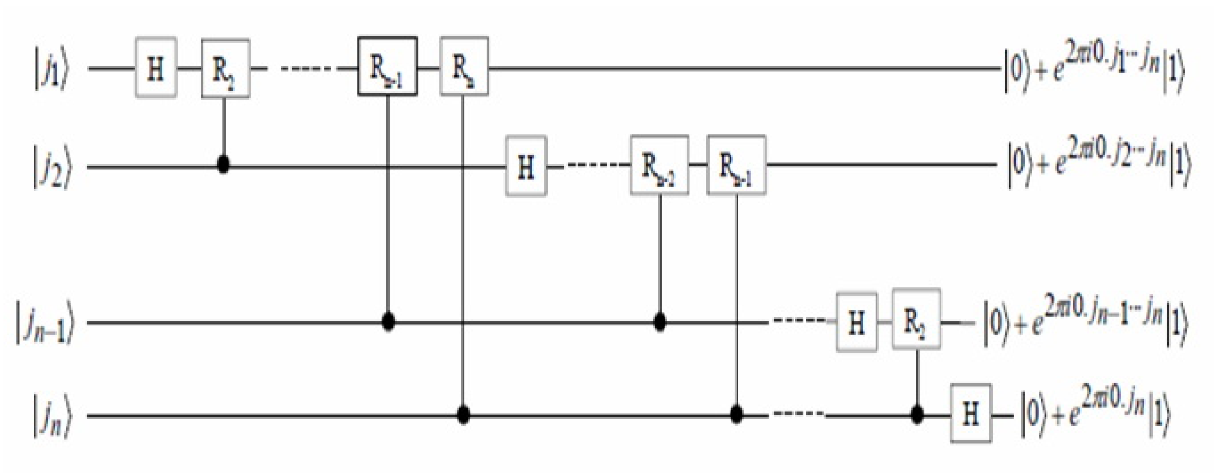

**Figure 2.** Schematic diagram for the quantum Fourier transform.

In this figure, $H$ is a Hadamard gate and $R_k$ is a unitary transformation given by

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{bmatrix}.$$

Hence it can be seen that primality testing and integer factorization of an integer $n$ consisted of two primes can be conducted efficiently by using quantum computation besides the notably Shor's integer factorization algorithm (Yang, 2002), which gives us the possibility to break the RSA cryptosystem.

## 4. Conclusion

From the spectrum obtained by the Fourier transform of a correlation function generated from the Riemann zeta function given by $F_n(\omega) = \int_{-\infty}^{+\infty} Z_\sigma\left(t, \frac{1}{2}\log n\right) e^{-i\omega t} dt$, we can see the primarity of an integer $n$ if and only the $F_n(\omega)$ has a single spectra. Moreover we can factorize the integer $n$ consisted of two primes by using this method.

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# Sixth Order Multiple Coarse Grid Computation for Solving 1D Partial Differential Equation

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## Abstract

We present a new method using multiple coarse grid computation technique to solve one dimensional (1D) partial differential equation (PDE). Our method is based on a fourth order discretization scheme on two scale grids and the Richardson extrapolation. For a particular implementation, we use multiple coarse grid computation to compute the fourth order solutions on the fine grid and all the coarse grids. Since every fine grid point has a corresponding coarse grid point with fourth order solution, the Richardson extrapolation procedure is applied for every fine grid point to increase the order of solution accuracy from fourth order to sixth order. We compare the maximum absolute error and the order of solution accuracy for our new method, the standard fourth order compact (FOC) scheme and Wang-Zhang's sixth order multiscale multigrid method. Two convection-diffusion problems are solved numerically to validate our proposed method.

*Keywords:* partial differential equation, multiple coarse grid computation, multigrid method.
*2010 MSC:* 65N06, 65N55, 65F10.

## 1. Introduction

Numerical solutions of partial differential equations (PDEs) play a crucial role in many simulation and engineering modeling applications, such as airplane manufacturing (Gamet *et al.*, 1999), auto manufacturing (Gerlinger *et al.*, 1998), medical imaging (Kang *et al.*, 2004), oil exploration and production (Li *et al.*, 2005), semiconductor (Carey *et al.*, 1996), communications (Kim & Kim, 2004), etc. Over the past several decades, computational mathematicians and engineers have developed many efficient fast algorithms to reduce the computation time. However, the increasing

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demand for higher resolution simulations in less computer time has continuously challenged the computational scientists to come up with more efficient, scalable numerical algorithms to solve PDEs.

In many scientific and engineering applications, such as the global ocean modeling and wide area weather forecasting, the computational domains are huge and the grid spaces are not small. In the context of the finite difference methods, the standard second order discretization scheme or the first order upwind difference scheme yield unsatisfactory results because they may need fine mesh griddings to compute approximate solutions of acceptable accuracy. In addition, the second order scheme may also produce numerical solutions with nonphysical oscillations for the convection dominated problems (Spotz, 1995).

Higher order (more than two) discretization methods are considered to be useful to reduce computational cost in very large scale modelings and simulations, which use relatively coarser mesh griddings to yield approximate solutions of comparable accuracy, compared with lower order discretization. Generally, higher order discretization schemes need more complicated procedures and more preprocessing costs to construct the coefficient matrix. However, they usually yield linear systems of much smaller size, compared with those from the lower order methods.

For the development of fourth order compact difference schemes, Gupta *et al*. proposed a fourth order nine-point compact (FOC) scheme to discretize the two dimensional (2D) convection-diffusion equation with variable coefficients (Gupta *et al.*, 1984). There are also some other similar fourth order compact schemes that have been developed for the convection-diffusion equations. Readers are referred to (Li *et al.*, 1995; Spotz, 1995; Spotz & Carey, 1995) and the references therein for more details.

For the sixth order schemes, Chu and Fan (Chu & Fan, 1998, 1999) proposed a three point combined compact difference (CCD) scheme for solving 2D Stommel Ocean model, which is a convection-diffusion equation. Their scheme can achieve sixth order accuracy for the inner grid points and fifth order accuracy for the boundary grid points. CCD scheme is considered as an *implicit* scheme because it does not compute the solution of the variables of interest directly. It also has a stability problem that for certain problems, if a large meshsize is used, the computed solution may be oscillatory (Zhang & Zhao, 2005).

In contrary, the *explicit* compact schemes compute the solutions of the variables directly. In addition, the explicit schemes have an additional advantage that they can avoid the oscillations in computed solutions. However, the higher order explicit compact schemes are more complicated to develop in higher dimensions, compared with the implicit schemes. As far as we know, there is no existing explicit compact scheme on a single scale grid that is higher than the fourth order.

By using the idea of multiscale computation, Sun and Zhang (Sun & Zhang, 2004) first proposed a sixth order explicit finite difference discretization strategy, which is based on the Richardson extrapolation technique and an operator interpolation scheme. Recently, Wang and Zhang developed an efficient and scalable sixth order explicit compact scheme for 2D/3D Poisson and convection-diffusion equations by using multiscale mutigrid method and an operator based interpolation combined with extrapolation technique (Wang & Zhang, 2009, 2011, 2010). The However, for the operator based interpolation, if the coefficient matrix $A$ is not diagonally dominant like the convection-diffusion equation with very large cell Reynolds number, it may take a large number of iterations to converge. In this paper, we present another technique called the multiple coarse grid computation technique. This approach can be used to compute the fourth order solutions on the fine grid and every coarse grid, which means that we can directly apply Richardson extrapolation for every grid point on the find grid and no operator based interpolation is needed.

An outline of the paper is as follows. In Section 2, we illustrate our sixth order strategy by using multiple coarse grid computation technique. Numerical results will be provided in Section 3. Section 4 contains the concluding remarks.

## 2. Sixth Order Multiple Coarse Grid Computation

Our motivation is to build an efficient and scalable method for solving PDEs like the convection-diffusion equations with high order of solution accuracy. In addition, we want the new method to have good potential to be modified to work on parallel computers. In (Wang & Zhang, 2009), Wang and Zhang successfully increase the order of solution accuracy from fourth order to sixth order by using multiscale multigrid method, Richardson extrapolation and an operator based interpolation. Important properties of the Richardson extrapolation has been studied by Zlatev *et al*. Readers are referred to (Zlatev *et al.*, 2010) and the references therein for more details. The interpolation strategy is an mesh-refinement type of iterative method and it is very efficient for some PDEs like the Poisson equation. Since their discretization scheme is based on the standard explicit fourth order compact scheme, so their is no nonphysical oscillation in the computed solutions. The proof and numerical analysis of this property can be found in (Spotz, 1995). However, it is not efficient and scalable for some problems like the convection-diffusion equation with high Reynolds numbers (Wang & Zhang, 2011). For some cases, the interpolation procedure may take thousands of iterations to converge. In addition, this method does not have a good potential for parallel implementation.

The idea of using multiple coarse grid computation is from the parallel superconvergent multigrid method. In addition to splitting the original grid and filtering residual vector to exploit parallelism, one can use the concurrent relaxation method on multiple grids (Zhu, 1993). The multigrid superconvergent method uses multiple coarse grids to generate better correction for the fine grid solution than that from a single coarse grid. The reason is that for standard multigrid method of 1D problem as in figure 1, the residual of the fine grid is projected to only *even* coarse grid. But we can also project the residual to *odd* coarse grids. Therefore, a combination of error correction from all the coarse grids may make the fine grid converge faster than that from a single coarse grid. In general, for a $d$ dimensional problem, the fine grid can be easily coarsened into $2^d$ coarse grids. If the computation work for each coarse grid can be loaded to a separate processor and computed simultaneously, we can develop an parallel solver for solving PDEs.

### *2.1. 1D multiple coarse grid computation*

Let's consider the multiple coarse grid computation technique for the one dimensional (1D) convection diffusion equation, which can be written as

$$u_x + b(x)u_x + c(x)u = f(x), \qquad 0 \le x \le l, \tag{2.1}$$

where the known functions $b(x)$, $c(x)$ and $f(x)$ are assumed to have the necessary derivatives up to certain orders. Eq. (2.1) can be discretized by some finite difference scheme to result in a system of linear equations

$$A^h u^h = f^h, \tag{2.2}$$

where $h$ is the uniform grid spacing of the discretized domain $\Omega^h$.

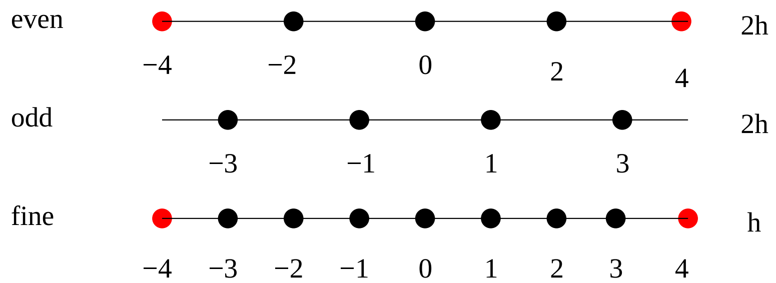


**Figure 1.** Illustration of the multiple coarse grid for 1D problem.

In order to achieve sixth order solution accuracy, we need to compute the fourth order solutions for the fine grid and two coarse grids like figure 1. Then we can apply the Richardson extrapolation. The fourth order compact (FOC) scheme we use is from (Wang & Zhang, 2011).

From figure 1, we can find out that two coarse grids are generated in such a way that all the even-numbered grid points from $\Omega_h$ belong to coarse grid $\Omega_{even}$ and all the odd-numbered grid points belong to coarse grid $\Omega_{odd}$. So we have

$$\Omega_{even} = \{x_j | x_j \in \Omega_h \; and \; (j = even)\},$$
$$\Omega_{odd} = \{x_j | x_j \in \Omega_h \; and \; (j = odd)\}.$$

We note that the even indexed coarse grid is easy to be solved by double the mesh size from $h$ to $2h$. However, the coarse grid $\Omega_{odd}$ only contains the *black* color grid points from fine grid but no *red* color boundary grid points. It is very difficult to develop the finite difference schemes for coarse grid $\Omega_{odd}$ if we only have the inner grid points. One possible approach is to add these red color boundaries to $\Omega_{odd}$ and develop special computational stencil for grid point $u_{-3}$ and $u_3$ as shown in figure 2. For the 1D problem in figure 2, the computational stencil for the grid points near the boundaries are different with other inner grid points. For the inner grid points like $u_{-1}$ and $u_1$, their finite difference schemes are based on $2h$ meshsize. However, if we take grid point $u_{-3}$ in $\Omega_{odd}$ as an example, its compact finite difference scheme needs the boundary grid point $u_{-4}$ and inner grid point $u_{-1}$. The meshsize between $u_{-4}$ and $u_{-1}$ are $h$ and $2h$.

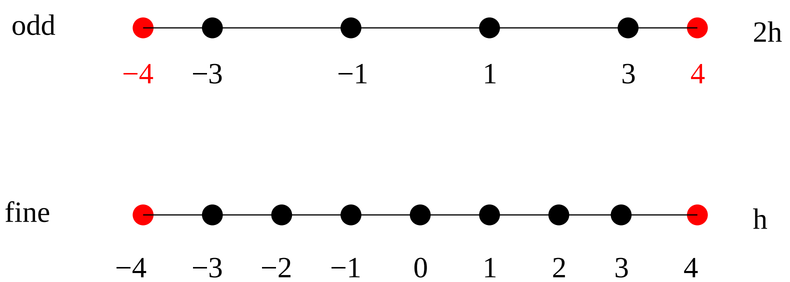


**Figure 2.** $\Omega_{odd}$ with two added red color boundary grid points.

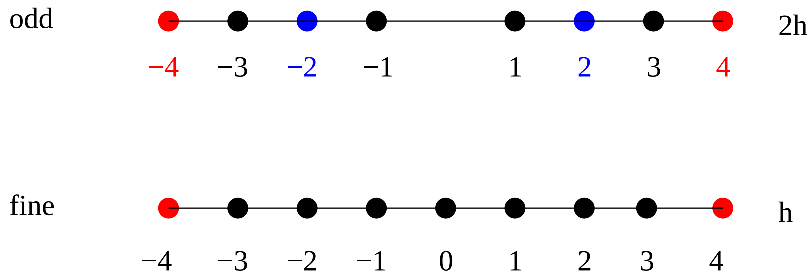


**Figure 3.** $\Omega_{odd}$ with two red color boundary and two blue color inner grid points .

**Lemma 2.1.** *For coarse grid as shown in figure 2, the solution accuracy for the central difference operator becomes first order.*

**Proof.** It can be easily verified by using Taylor series expansion.

□

Since the second order central difference operator is degraded to the first order, the FOC scheme which is based on the approximation for the second order terms will be degraded to the second order for these near boundary grid points. In order to compute fourth order solution for every coarse grid point, we add two more grid points to the $\Omega_{odd}$ like the *blue* color grid points in figure 3.

By adding these four grid points, now we can discretize every grid point in $\Omega_{odd}$ with fourth order accuracy using FOC scheme. Let's assume the $\Omega_{odd}$ contains $Nx$ grid points

$$u_{odd}(0), u_{odd}(1), \ \ldots, u_{odd}(Nx)$$

as in figure 4. Then the $\Omega_{even}$ will contains $Nx-3$ grid points and fine grid will contains $2Nx-7$ grid points. The grid points on $\Omega_{odd}$ are approximated as follows:

- For $j \in \{1, 2, Nx-2, Nx-1\}$, $u_{odd}(j)$ is approximated by three-point computational stencil from FOC scheme using grid points $u_{odd}(j-1)$ and $u_{odd}(j+1)$ with meshsize $h$. The truncation error is $O(h^4)$.

- For $j = 3$, $u_{odd}(j)$ is approximated by three-point computational stencil from FOC scheme using grid points $u_{odd}(j-2)$ and $u_{odd}(j+1)$ with meshsize $2h$. The truncation error is $O((2h)^4)$.

- For $j \in [4, Nx-4]$, $u_{odd}(j)$ is approximated by three-point computational stencil from FOC scheme using grid points $u_{odd}(j-1)$ and $u_{odd}(j+1)$ with meshsize $2h$. The truncation error is $O((2h)^4)$.

- For $j = Nx-3$, $u_{odd}(j)$ is approximated by three-point computational stencil from FOC scheme using grid points $u_{odd}(j-1)$ and $u_{odd}(j+2)$ with meshsize $2h$. The truncation error is $O((2h)^4)$.


**Figure 4.** Representation of modified $\Omega_{odd}$ for 1D problem.

By using above discretization strategy, we can approximate the fourth order solution for every grid point on $\Omega_{odd}$. After we get fourth order solutions for the fine grid and two coarse grids, each grid point on the fine grid will have a corresponding grid point on either $\Omega_{even}$ or $\Omega_{odd}$. Then we apply Richardson extrapolation (Cheney & Kincard, 1999) for every fine grid point to approximate the sixth order solution like

$$\tilde{u}_j^h = \frac{16u_j^h - u_j^{2h}}{15}, \tag{2.3}$$

where $u_j^h$ is $jth$ grid point from fine grid and $u_j^{2h}$ is the corresponding coarse grid point.

## 3. Numerical Results

Two 1D convection-diffusion equations are solved using the multiple coarse grid computation strategy discussed in the previous sections. We compared the truncated error and the order of accuracy by using our multiple coarse grid computation technique (MCG), standard fourth order scheme (FOC), and the sixth order operator based interpolation scheme (SOC) in (Wang & Zhang, 2011).

The codes are written in Fortran 77 programming language and run on one node of the Lipscomb HPC Cluster at the University of Kentucky. Each node has 36GB of local memory and runs at 2.66GHz. The initial guess for our test cases is the zero vector. The stopping criteria for the iterative methods we tested and the operator based interpolation procedure is $10^{-10}$. The errors reported are the maximum absolute errors over the discrete grid of the finest level.

For the order of solution accuracy, we denote $E(h)$ and $E(H)$ to be the solution error with meshsize $h$ and $H$, respectively. The order of accuracy $m$ is calculated from the formula

$$\frac{E(h)}{E(H)} = \frac{h^m}{H^m}$$
$$\Longrightarrow m = \log_{(h/H)}(E(h)/E(H)).$$

The order of accuracy is formally defined when the meshsize approaches zero. Therefore, when the meshsize is relatively large, the discretization scheme may not achieve its formal order of accuracy.

**Problem 1**. Let's consider the examples from Sun's previous work (Sun & Zhang, 2004), which is a 1D convection-diffusion equation like

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} - u = -\cos x - 2\sin x, \qquad 0 \le x \le \pi. \tag{3.1}$$

Eq. (3.1) has the Dirichlet boundary conditions as $u(0) = u(\pi) = 0$. The analytic solution for this problem is $u(x) = \sin x$.

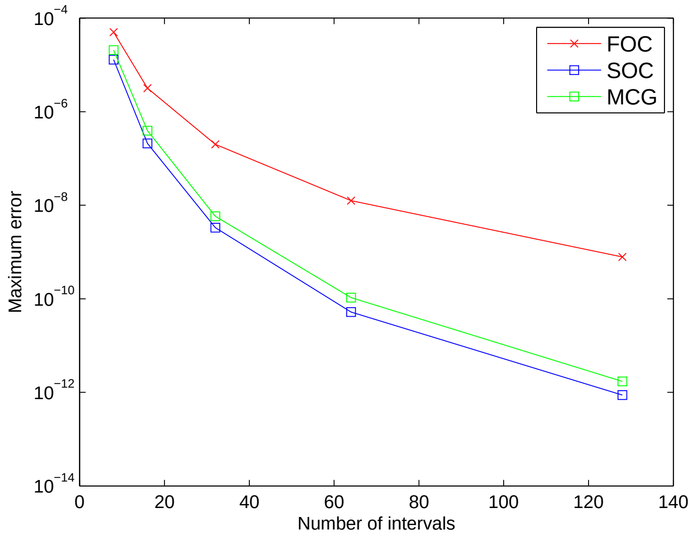


**Figure 5.** Comparison of maximum errors of FOC, SOC and MCG methods for Problem 1.

The computational results are listed in Table 1 and figure 5. From Table 1, we can see that the multiple coarse grid method (MCG) is more accurate than the fourth order scheme (FOC). Although the MCG method is not as accurate as the SOC but it can achieve the sixth order solution accuracy when the number of intervals is bigger than 8. The reason why MCG is less accurate than SOC is that there are two near boundary grid point using meshsize $h$ to approximate instead of $2h$ in $\Omega_{odd}$.

**Table 1.** Comparison of maximum errors and the order of accuracy by using FOC, SOC, and MCG methods for Eq. (3.1).

| | FOC | | SOC | | MCG | |
|---|---|---|---|---|---|---|
| $h$ | Error | Order | Error | Order | Error | Order |
| $\pi/8$ | 5.02e-5 | 4.0 | 1.30e-5 | 5.9 | 2.08e-5 | 5.7 |
| $\pi/16$ | 3.18e-6 | 4.0 | 2.10e-7 | 6.0 | 3.94e-7 | 6.1 |
| $\pi/32$ | 2.00e-7 | 4.0 | 3.32e-9 | 6.0 | 5.81e-9 | 5.8 |
| $\pi/64$ | 1.25e-8 | 4.1 | 5.20e-11 | 6.0 | 1.06e-10 | 6.0 |
| $\pi/128$ | 7.83e-10 | 4.1 | 8.73e-13 | 6.0 | 1.71e-12 | 6.0 |

**Problem 2**. We solve another classical 1D convection-diffusion equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} = 0, \qquad 0 \le x \le 1. \tag{3.2}$$

The boundary condition for Eq. (3.2) is $u_0 = 0$ and $u_1 = 1$. The analytic solution is $u(x) = (e^x - 1)/(e - 1)$.

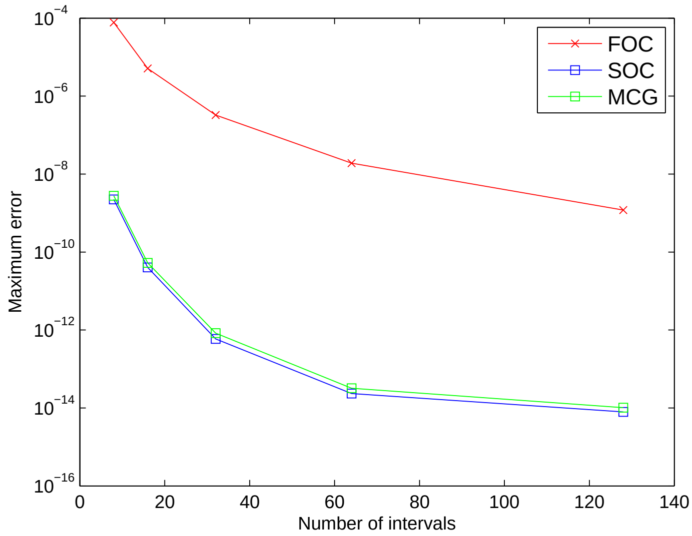


**Figure 6.** Comparison of maximum errors of FOC, SOC and MCG methods for Problem 2.

The numerical results of Problem 2 are listed in Table 2 and figure 6. We note that when $n > 32$, the order of solution accuracy is not high enough as we hope. The reason is that the computed solutions with $h = 1/64$ and $h = 1/128$ are not as accurate as they should be, due to the stopping criteria we set. Once again, the solutions from our MCG method are more accurate than the FOC method and can achieve the sixth order when $n < 64$.

**Table 2.** Comparison of maximum errors and the order of accuracy by using FOC, SOC, and MCG methods for Eq. (3.2).

| | FOC | | SOC | | MCG | |
|---|---|---|---|---|---|---|
| $h$ | Error | Order | Error | Order | Error | Order |
| 1/8 | 7.76 e-5 | 3.9 | 2.24e-9 | 5.9 | 2.78-9 | 5.7 |
| 1/16 | 5.12e-6 | 4.0 | 4.01e-11 | 6.0 | 5.29-11 | 6.0 |
| 1/32 | 3.27e-7 | 4.0 | 5.91e-13 | 6.0 | 8.27-13 | 5.9 |
| 1/64 | 1.91e-8 | 4.0 | 2.34e-14 | 4.9 | 3.21-14 | 4.8 |
| 1/128 | 1.19e-9 | 4.0 | 7.93e-15 | 1.6 | 1.02-14 | 1.6 |

We want to mention here that the SOC method for both test cases is slightly more accurate than the MCG method, but the MCG method has a very good potential for parallel implementation. The computing tasks for MCG procedure can be divided to three independent processors (one for find grid and two for coarse grids). In addition, since the MCG method does not need the operator based interpolation procedure to approximate the sixth order fine grid solution, it will save a large amount of CPU costs for some high Reynolds number problems (Wang & Zhang, 2011).

## 4. Concluding Remarks and Future Work

We presented a new sixth order solution method based on the fourth order discretization and multiple coarse grid computation for solving 1D convection-diffusion equation. Our numerical experiments show that the new sixth order strategy is more accurate than the standard fourth order scheme and can achieve the sixth order solution accuracy.

It is worth pointing out that our solution strategy can be applied to solve many other types of PDEs, because it does not require the additional work to redesign the discretization schemes. The advantage of using multiple coarse grids is that we can use it to increase the order of accuracy without using operator based interpolation scheme. However, there is still a lot of work that needs to be done to develop a useful multiple coarse grid computation method that can be applied to real-world problems. In this paper, we just use the standard Gauss-Seidel iterative method for MCG strategy, because our goal is to test whether the MCG method can achieve the sixth order accuracy or not. For some real applications, we should use multigrid method and implement the multiple coarse grid computation in the multigrid cycle.

For the future research work, we will extend our 1D multiple coarse grid computation method to higher dimensional problems. For 2D problems, we will generate four course grids by the index of $x$ and $y$ direction as $(even, even)$, $(even, odd)$, $(odd, even)$ and $(odd, odd)$. Here, $(even, even)$ is the course grid in standard multigrid method. Like 1D strategy, only the $(even, even)$ course grid has the full boundary conditions. We need to find a way to add artificial boundary grid points for other three course grids. Another possible solution is to use algebraic multigrid method instead of geometric multigrid method, this is also one of our research interest in the future.

For the parallelization, the parallel multiscale multigrid (MCG) method has been discussed in (Xiao, 1994; Zhu, 1993). However, these parallel MCG methods are only used to speed up the convergence. As we mentioned in previous section, the computation of each course grid and the fine grid is independent. If we want to solve a 3D problem, we can use nine processors to solve the fourth order solutions on the fine grid and eight coarse grids. Then an Richardson extrapolation, which can also been parallelized, can increase the order of accuracy to sixth order.

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# On Nonuniform Polynomial Stability for Evolution Operators on the Half-line

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## Abstract

The main aim of this paper is to study a concept of nonuniform polynomial stability for evolution operators on the half-line. The obtained results are variants for nonuniform polynomial stability of some well-known theorems due to Barbashin, Datko, Rolewicz and Zabczyk in the case of uniform exponential stability. This paper generalizes well-known results for the nonuniform exponential stability (Lupa & Megan, 2012) and the uniform polynomial stability (Megan & Ceausu, 2012).

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## 1. Introduction and preliminaries

The notion of exponential stability plays an important role in the theory of differential equations in Banach spaces, particularly in the study of asymptotical behaviors. It has gained prominence since appearance of two fundamental monographs of J. L. Massera, J. J. Schäffer (Massera & Schäffer, 1966) and J. L. Daleckii, M. G. Krein (Daleckii & Krein, 1974). These were followed by the important books of C. Chicone and Yu. Latushkin (Chicone & Latushkin, 1999) and L. Barreira and C. Valls (Barreira & Valls, 2008).

The most important stability concept used in the qualitative theory of differential equations is the uniform exponential stability. In some situations, particulary in the nonautonomous setting, the concept of uniform exponential stability is too restrictive and it is important to look for a more general behavior.

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Two different perspectives can be identified to generalize the concept of uniform exponential stability, on the one hand one can define exponential stabilities that depends on the initial time (and therefore are nonuniform) and, on the other hand, one can consider grow rates that are not necessarily exponential.

The first approach leads to the concepts of nonuniform exponential stabilities and can be found in the works (Barreira & Valls, 2008), (Lupa & Megan, 2012), (Megan, 1995), (Minda & Megan, 2011), (Pinto, 1988) and the second approach is presented in the papers (Barreira & Valls, 2009), (Bento & Silva, 2009), (Bento & Silva, 2012), (Megan & Ramneantu, 2011), (Megan & Minda, 2011).

A natural generalization is to consider stability concepts that are both nonuniform and not necessarily exponential. This was the approach followed by Barreira and Valls in (Barreira & Valls, 2009) and A. Bento and C. Silva in (Bento & Silva, 2009), (Bento & Silva, 2012), who studied a nonuniform polynomial dichotomy concept. A principal motivation for weakening the assumption of uniform exponential behavior is that from the point of view of ergodic theory, almost all variational equations in a finite dimensional space admit a nonuniform exponential dichotomy.

In this paper we consider a concept of nonuniform polynomial stability for evolution operators in Banach spaces. This concept has been considered in the case of invertible evolution operators in the papers (Barreira & Valls, 2009) due to L. Barreira and C. Valls, respectively in (Bento & Silva, 2009), (Bento & Silva, 2012) due to A. Bento and C. Silva.

Some results concerning polynomial stability for evolution operators were published in our papers (Megan & Ramneantu, 2011), (Megan & Ceausu, 2012), (Megan & Minda, 2011). We remark that the results obtained in (Megan & Ramneantu, 2011) are for the case of evolution operators with uniform exponential growth. In this paper we consider the case of evolution operators with nonuniform polynomial growth.

The obtained results in this paper can be considered as variants for nonuniform polynomial stability of some well-known theorems due to Barbashin ((Barbashin, 1967)), Datko ((Datko, 1972)) and Rolewicz ((Rolewicz, 1986)) in the case of uniform exponential stability. We remark that our proofs are not adaptations for polynomial stability of the proofs presented in (Barbashin, 1967), (Datko, 1972) and (Rolewicz, 1986). The case of nonuniform exponential stability has been studied in (Lupa & Megan, 2012), (Minda & Megan, 2011), respectively (Megan & Ramneantu, 2011), (Megan & Ceausu, 2012).

Moreover, we note that we consider evolution operators which are not supposed to be invertible and the polynomial stability concept studied in this paper uses the evolution operators in forward time. Thus the stability results obtained in this paper hold for a much larger class of differential equations than in the classical theory of uniform exponential stability.

Let $X$ be a real or complex Banach space and let $I$ be the identity operator on $X$. The norm on $X$ and on $\mathcal{B}(X)$, the algebra of all bounded linear operators acting on $X$, will be denoted by $\|\cdot\|$.

Let

$$\Delta = \{(t, s) \in \mathbb{R}_+^2 : \ t \geq s\}.$$

We recall that a mapping $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is called an *evolution operator* on $X$ if

$(e_1)$ $\Phi(t, t) = I$, for all $t \geqslant 0$;

$(e_2)$ $\Phi(t,s)\Phi(s,r) = \Phi(t,r)$, for all $(t,s),(s,r) \in \Delta$.

**Definition 1.1.** An evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is said to be

$(i)$ *with polynomial growth* (and denote p.g) if there exist $M \geq 1$, $\omega > 0$ and $\varepsilon \geq 0$ such that

$$(s+1)^{\omega} \parallel \Phi(t,s) \parallel \leq M(t+1)^{\omega}(s+1)^{\varepsilon}, \quad for \ \ all \ \ (t,s) \in \Delta;$$

$(ii)$ *polynomially stable* (and denote p.s) if there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$(t+1)^{\alpha} \parallel \Phi(t,s) \parallel \leq N(s+1)^{\alpha+\beta}, \quad for \ \ all \ \ (t,s) \in \Delta;$$

$(iii)$ *exponentially stable* (and denote e.s) if there exist $N_1 \geq 1$, $\alpha_1 > 0$ and $\beta_1 \geq 0$ such that

$$e^{\alpha_1 t} \parallel \Phi(t,s) \parallel \leq N_1 e^{(\alpha_1+\beta_1)s}, \quad for \ \ all \ \ (t,s) \in \Delta.$$

**Definition 1.2.** An evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is said to be

$(i)$ *measurable*, if for all $(s,x) \in \mathbb{R}_+ \times X$ the mapping $t \longmapsto \parallel \Phi(t,s)x \parallel$ is measurable on $[s,\infty)$.

$(ii)$ $*$*-measurable*, if for all $(s,x^*) \in \mathbb{R}_+ \times X^*$ the mapping $s \longmapsto \parallel \Phi(t,s)^* x^* \parallel$ is measurable on $[0,t]$.

## 2. Results

**Theorem 2.1.** *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a measurable evolution operator. If $\Phi$ is p.s then there exist $D \geq 1$, $d > 0$ and $c \geq 0$ such that*

$$\int_s^{\infty} (\tau+1)^{d-1} \|\Phi(\tau,s)x\| d\tau \leq D(s+1)^{d+c} \|x\|, \tag{2.1}$$

*for all $s \geq 0$ and $x \in X$.*

*Proof.* If $\Phi$ is p.s, then according to Definition 1.1 (ii) there exist the constants $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that, for all $d \in (0,\alpha)$ and $c = \beta$ we have

$$\int_s^{\infty} (\tau+1)^{d-1} \|\Phi(\tau,s)x\| d\tau \leq N(s+1)^{\alpha+\beta} \|x\| \int_s^{\infty} (\tau+1)^{d-\alpha-1} d\tau \leq D(s+1)^{d+c} \|x\|,$$

for all $(s,x) \in \mathbb{R}_+ \times X$, where $D = \frac{N+\alpha-d}{\alpha-d}$.

□

**Theorem 2.2.** *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a measurable evolution operator with p.g and with the property that there exist $D \geq 1$, $c \geq 0$ and $d > \varepsilon$ such that (2.1) holds, where $\varepsilon$ is given by Definition 1.1(i). Then $\Phi$ is p.s.*

*Proof.* Let $x \in X$ and $t \geq 2s + 1$. Because

$$\int_{\frac{t-1}{2}}^{t} (\tau + 1)^{a-1} d\tau = (t+1)^a \frac{2^a - 1}{a 2^a},$$

for all $t \geq 0$ and $a > 0$ we have

$$(t+1)^{d-\varepsilon} \|\Phi(t,s)x\| = N \int_{\frac{t-1}{2}}^{t} (\tau+1)^{d-\varepsilon-1} \|\Phi(t,s)x\| d\tau$$

$$= N \int_{\frac{t-1}{2}}^{t} (\tau+1)^{d-\varepsilon-1} \|\Phi(\tau,s)x\| M \left(\frac{t+1}{\tau+1}\right)^{\omega} (\tau+1)^{\varepsilon} d\tau$$

$$\leq 2^{\omega} NM \int_{s}^{\infty} (\tau+1)^{d-1} \|\Phi(\tau,s)x\| d\tau \leq 2^{\omega} NMD (s+1)^{d+c} \|x\|.$$

Hence, we have that

$$(t+1)^{d-\varepsilon} \|\Phi(t,s)x\| \leq 2^{\omega} NMD (s+1)^{d-\varepsilon+c+\varepsilon} \|x\|,$$

for all $t \geq 2s + 1$ and $x \in X$, where $N = \frac{(d-\varepsilon)2^{d-\varepsilon}}{2^{d-\varepsilon}-1}$.

For $t \in [s, 2s+1)$ we have

$$(t+1)^{d-\varepsilon} \|\Phi(t,s)x\| \leq 2^{d+\omega-\varepsilon} M (s+1)^d \|x\|$$

and hence,

$$(t+1)^{d-\varepsilon} \|\Phi(t,s)x\| \leq K (s+1)^{d-\varepsilon+c+\varepsilon} \|x\|,$$

for all $(t, s, x) \in \Delta \times X$, where $K = \max\{2^{\omega} NMD, 2^{d-\varepsilon+\omega} M\}$.
Finally, we obtain that $\Phi$ is p.s. □

A discrete variant of the Theorem 2.2 is

**Theorem 2.3.** *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be an evolution operator with p.g and with the property that there exist the constants $D \geq 1$, $d > 0$ and $c \geq 0$ such that*

$$\sum_{k=n}^{\infty} (k+1)^d \|\Phi(k,n)x\| \leq D(n+1)^{d+c} \|x\|,$$

*for all $n \in \mathbb{N}$ and $x \in X$. Then $\Phi$ is p.s.*

*Proof.* According the hypothesis, if we consider $k = m$ then we have

$$(m+1)^d \|\Phi(m,n)x\| \leq D(n+1)^{d+c} \|x\|,$$

for all $m, n \in \mathbb{N}$, $m \geq n$ and $x \in X$, which proves that $\Phi$ is p.s. □

*Remark.* Theorem 2.3 can be considered a Zabczyk's (Zabczyk, 1974) type theorem for polynomial stability.

**Theorem 2.4.** *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be an evolution operator. Then $\Phi$ is p.s if and only if there exist the constants $B \geq 1$ and $b > c \geq 0$ such that*

$$\sum_{k=0}^{n}(k+1)^{-b-1}\|\Phi(n,k)x\| \leq B(n+1)^{c-b}\|x\|,$$

*for all $n \in \mathbb{N}$ and $x \in X$.*

*Proof.* Necessity. If we consider $b \in (\beta, \alpha+\beta)$, $c = \beta$ and $B = \frac{N+\alpha+\beta-b}{\alpha+\beta-b}$ we have

$$\sum_{k=0}^{n}(k+1)^{-b-1}\|\Phi(n,k)x\| \leq N(n+1)^{-\alpha}\|x\|\sum_{k=0}^{n}(k+1)^{\alpha+\beta-b-1}$$

$$\leq N(n+1)^{-\alpha}\|x\|\left(1+\int_0^n(\tau+1)^{\alpha+\beta-b-1}d\tau\right) \leq B(n+1)^{c-b}\|x\|,$$

for all $n \in \mathbb{N}$ and $x \in X$.
Sufficiency. Let $n \geq k \geq 0$ with $n, k \in \mathbb{N}$. According to the hypothesis we have that

$$(k+1)^{-b-1}\|\Phi(n,k)x\| \leq B(n+1)^{c-b}\|x\|$$

which implies

$$(n+1)^{b-c}\|\Phi(n,k)x\| \leq B(k+1)^{b-c+1+c}\|x\|,$$

for all $x \in X$. Hence, $\Phi$ is p.s. □

*Remark.* Theorem 2.4 can be considered a Barbashin's type theorem for polynomial stability (see (Barbashin, 1967)).

We consider the set

$$\mathcal{R} = \{R : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ | R \ \ nondecreasing, \ \ R(t) > 0, \ \ \forall \ \ t > 0\}.$$

**Theorem 2.5.** *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a $*$-measurable evolution operator with p.g. Then $\Phi$ is p.s if and only if there exist $B \geq 1$, $b > c \geq 0$ and a function $R \in \mathcal{R}$ such that*

$$\int_0^t R\left((\tau+1)^{-b-1}\|\Phi(t,\tau)^* x^*\|\right)d\tau \leq BR\left((t+1)^{c-b}\|x^*\|\right),$$

*for all $(t, s, x^*) \in \Delta \times X^*$.*

*Proof.* Necessity. Let us consider $R(t) = t$, $t \geq 0$. If $\Phi$ is p.s, then there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that for all $b \in (\beta, \alpha+\beta)$ and $c = \beta$ we have

$$\int_0^t(\tau+1)^{-b-1}\|\Phi(t,\tau)^* x^*\|d\tau \leq N(t+1)^{-\alpha}\|x^*\|\int_0^t(\tau+1)^{\alpha+\beta-b-1}d\tau = B(t+1)^{c-b}\|x^*\|,$$

where $B = \frac{N+\alpha+\beta-b}{\alpha+\beta-b}$.
Sufficiency. Let $x \in X$ with $\|x\| \le 1$ and $a-1 > B$. For $t \ge as + a - 1$ we have

$$(a-1)R\left(M^{-1}a^{-b-\omega-1}(s+1)^{-b-c-1}|\langle x^*, \Phi(t,s)x\rangle|\right)$$

$$= \int_s^{s+a-1} R\left(M^{-1}a^{-b-\omega-1}(s+1)^{-b-c-1}|\langle \Phi(t,\tau)^*x^*, \Phi(\tau,s)x\rangle|\right)d\tau$$
$$\le \int_s^{as+a-1} R\left((\tau+1)^{-b-1}\|\Phi(t,\tau)^*x^*\|a^{-b-\omega-1}\left(\frac{\tau+1}{s+1}\right)^{b+\omega+1}\right)d\tau$$
$$\le \int_0^t R\left((\tau+1)^{-b-1}\|\Phi(t,\tau)^*x^*\|\right)d\tau < (a-1)R\left((t+1)^{c-b}\|x^*\|\right).$$

Since $R$ is nondecreasing, we obtain that

$$M^{-1}a^{-b-\omega-1}(s+1)^{-b-c-1}|\langle x^*, \Phi(t,s)x\rangle| \le (t+1)^{c-b}\|x^*\|\|x\|.$$

By taking supremum relative to $\|x^*\| \le 1$, we have that

$$(t+1)^{b-c}\|\Phi(t,s)\| \le Ma^{b+\omega+1}(s+1)^{b+c+1}.$$

If $t \in [s, as+a-1)$ we have

$$(t+1)^{b-c}\|\Phi(t,s)\| \le M\left(\frac{t+1}{s+1}\right)^{b-c+\omega}(s+1)^b \le Ma^{b+\omega+1}(s+1)^b,$$

and, further,

$$(t+1)^{b-c}\|\Phi(t,s)\| \le Ma^{b+\omega+1}(s+1)^{b+c+1},$$

for all $(t,s) \in \Delta$, which proves that $\Phi$ is p.s. □

*Remark.* Theorem 2.5 can be considered a Rolewicz's type theorem for polynomial stability (see (Rolewicz, 1986)).

**Corollary 2.6.** Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a $*$-measurable evolution operator with p.g. Then $\Phi$ is p.s if and only if there exist $B \ge 1$ and $b > c \ge 0$ such that

$$\int_0^t (\tau+1)^{-b-1}\|\Phi(t,\tau)^*x^*\|d\tau \le B(t+1)^{c-b}\|x^*\|,$$

for all $(t,s,x^*) \in \Delta \times X^*$.
Proof. It follows from Theorem 2.5 for $R(t) = t$.

*Remark.* A similar result was obtained by N. Lupa and M. Megan in (Lupa & Megan, 2012) for the case of nonuniform exponential stability.

## 3. Examples

In this section we will give some examples that illustrate the connection between the exponential stability and the polynomial stability, as well as the connection between polynomial growth and polynomial stability. Furthermore, we will present some examples of evolution operators which are not p.s and the integral from (2.1) is convergent, respectively divergent.

In contrast with uniform case (where uniform exponential stability implies uniform polynomial stability, see (Megan & Ramneantu, 2011)) in the nonuniform case there is no connection between the concepts of exponential stability and polynomial stability, as shown in the following examples.

**Example 3.1.** We consider the function

$$u : [1, \infty) \longrightarrow \mathbb{R}_+^*, \quad u(t) = (t+1)^3 + 1$$

and the evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(X), \quad \Phi(t,s) = \frac{(s+1)^2 u(s)}{(t+1)^2 u(t)} I.$$

We have that

$$(t+1)^2 \|\Phi(t,s)\| \leq 2(s+1)^5, \quad for \ all \ (t,s) \in \Delta.$$

It results that $\Phi$ is p.s. If we suppose that $\Phi$ is e.s, then there exist $N_1 \geq 1$, $\alpha_1 > 0$ and $\beta_1 \geq 0$ such that

$$e^{\alpha_1 t}(s+1)^2[(s+1)^3 + 1] \leq N_1 e^{(\alpha_1+\beta_1)s}(t+1)^2[(t+1)^3 + 1], \quad for \ all \ (t,s) \in \Delta.$$

For $s = 0$ and $t \longrightarrow \infty$, we obtain a contradiction and hence $\Phi$ is not e.s.

**Example 3.2.** The evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(X), \quad \Phi(t,s) = \frac{e^{(2-\cos s)s}}{e^{(2-\cos t)t}} I$$

satisfies the condition

$$e^t \|\Phi(t,s)\| \leq e^{3s}, \quad for \ all \ (t,s) \in \Delta.$$

Hence $\Phi$ is e.s. If we suppose that $\Phi$ is p.s then there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$(t+1)^\alpha e^{(2-\cos s)s} \leq N(s+1)^{\alpha+\beta} e^{(2-\cos t)t}, \quad for \ all \ (t,s) \in \Delta.$$

From here, for $t = 2(n+1)\pi$ and $s = (2n+1)\pi$ we obtain

$$(2n\pi + 2\pi + 1)^\alpha \, e^{4n\pi+\pi} \leq N \, (2n\pi + \pi)^{\alpha+\beta},$$

which for $n \longrightarrow \infty$ yields a contradiction.

It is obvious that if an evolution operator is p.s then it has p.g. The next example presents an evolution operator with p.g, which is not p.s and the integral from (2.1) is divergent.

**Example 3.3.** The evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(X), \quad \Phi(t,s) = \frac{(s+1)^{1-\cos(s+1)}}{(t+1)^{1-\cos(t+1)}} I$$

satisfies the relation

$$(s+1)^{\omega} \|\Phi(t,s)\| \leq (t+1)^{\omega}(s+1)^{\varepsilon}, \quad for \;\; all \;\; (t,s) \in \Delta.$$

It results that $\Phi$ has p.g for all $\omega > 0$ and $\varepsilon = 2$.
If we suppose that $\Phi$ is p.s then there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$(t+1)^{\alpha} \frac{(s+1)^{1-\cos(s+1)}}{(t+1)^{1-\cos(t+1)}} \leq N(s+1)^{\alpha+\beta},$$

for all $(t,s) \in \Delta$. For $s = \frac{\pi}{2} - 1$ and $t = 2\pi + 2n\pi - 1$, we obtain

$$(2\pi + 2n\pi)^{\alpha} \frac{\pi}{2} \leq N \left(\frac{\pi}{2}\right)^{\alpha+\beta},$$

which if $n \longrightarrow \infty$, leads to a contradiction. We obtain thus that $\Phi$ is not p.s.
Let $d \geq 2$ and $s \geq 0$. Then we have

$$\int_s^{\infty} (\tau+1)^{d-1} \|\Phi(\tau,s)x\| d\tau \geq (s+1)^{1-\cos(s+1)} \|x\| \int_s^{\infty} (\tau+1)^{d-3} d\tau = \infty.$$

The next evolution operator does is not p.s and the integral from (2.1) is divergent.

**Example 3.4.** We consider the set

$$A = \{n + \frac{1}{n+1} : \; n \in \mathbb{N}\}$$

and a function $u : [0,\infty) \longrightarrow [1,\infty)$

$$u(t) = \begin{cases} e^{t+1}, & t \notin A \\ e^2, & t \in A. \end{cases}$$

and the evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(X), \quad \Phi(t,s) = \frac{u(s)}{u(t)} I$$

Let $d > 0$ and $s \geq 0$. Then we have

$$\int_s^{\infty} (\tau+1)^{d-1} \|\Phi(\tau,s)x\| d\tau = u(s)\|x\| \int_s^{\infty} (\tau+1)^{d-1} e^{-(\tau+1)} d\tau$$

$$\leq u(s)\|x\| \int_{s+1}^{\infty} y^{d-1} e^{-y} dy = \|x\| u(s) \Gamma(d) < \infty.$$

If we suppose that $\Phi$ has p.g then there exist $M \geq 1$, $\omega > 0$ and $\varepsilon \geq 0$ such that

$$(s+1)^{\omega} u(s) \leq M(t+1)^{\omega}(s+1)^{\varepsilon} u(t), \quad for \;\; all \;\; (t,s) \in \Delta.$$

For $s = n$ and $t = n + \frac{1}{n+1}$ we obtain

$$e^{n+1}(n+1)^{2\omega} \leq Me^2(n+1)^{\varepsilon}(n^2+2n+2)^{\omega},$$

which for $n \longrightarrow \infty$ yields a contradiction. Hence, $\Phi$ does not have p.g and so $\Phi$ is not p.s.

## 4. Open Problem

Finally, we put the following open problems:

1 There exist evolution operators which are not p.s and the relation (2.1) is satisfied?

2 There are evolution operators with p.g with $\varepsilon > 0$, which are not p.s and the relation (2.1) is satisfied for $d \in (0, \varepsilon)$?

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# Zweier I-convergent Sequence Spaces Defined by a Sequence of Moduli

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## Abstract

In this article we introduce the sequence spaces $\mathcal{Z}^I(F)$, $\mathcal{Z}_0^I(F)$ and $\mathcal{Z}_\infty^I(F)$ for a sequence of moduli $F = (f_k)$ and study some of the topological and algebraic properties on these spaces.

*Keywords:* Ideal, filter, sequence of moduli, Lipschitz function, I-convergence field, I-convergent, monotone and solid spaces.
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## 1. Introduction

Let $\mathbb{R}$, and $\mathbb{C}$ be the sets of all real and complex numbers respectively. We write

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \ or \ \mathbb{C}\}$$

the space of all real or complex sequences. Let $\ell_\infty$, $c$ and $c_0$ denote the Banach spaces of bounded, convergent and null sequences respectively normed by $\|x\|_\infty = \sup_k |x_k|$. Each linear subspace of $\omega$, for example $\lambda, \mu \subset \omega$ is called a sequence space. A sequence space $\lambda$ with linear topology is called a K-space provided each of maps $p_i \longrightarrow \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space $\lambda$ is called an FK-space provided $\lambda$ is a complete linear metric space. An FK-space whose topology is normable is called a BK-space. Let $\lambda$ and $\mu$ be two sequence spaces and $A = (a_{nk})$ is an infinite matrix of real or complex numbers $a_{nk}$, where $n, k \in \mathbb{N}$. Then we say that $A$ defines

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a matrix mapping from $\lambda$ to $\mu$ and we denote it by writing $A : \lambda \longrightarrow \mu$. If for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the $A$ transform of $x$ is in $\mu$, where

$$(Ax)_n = \sum_k a_{nk} x_k, \qquad (n \in \mathbb{N}). \tag{1.1}$$

By $(\lambda : \mu)$, we denote the class of matrices $A$ such that $A : \lambda \longrightarrow \mu$. Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of (1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$. The approach of constructing the new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Başar and Mursaleen (Altay *et al.*, 2006), Başar and Altay (Altay & Başar, 2003), Malkowsky (Malkowsky, 1997), Ng and Lee (Ng & Lee, 1978) and Wang (Wang, 1978). Şengönül (Şengönül, 2007) defined the sequence $y = (y_i)$ which is frequently used as the $Z^p$ transform of the sequence $x = (x_i)$ i.e, $y_i = px_i + (1-p)x_{i-1}$ where $x_{-1} = 0, p \neq 0, 1 < p < \infty$ and $Z^p$ denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, (i = k), \\ 1 - p, (i - 1 = k); (i, k \in \mathbb{N}), \\ 0, \text{otherwise}. \end{cases}$$

Following Basar and Altay (Altay & Başar, 2003), Şengönül (Şengönül, 2007) introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows $\mathcal{Z} = \{x = (x_k) \in \omega : Z^p x \in c\}$, $\mathcal{Z}_0 = \{x = (x_k) \in \omega : Z^p x \in c_0\}$. Here we quote below some of the results due to Şengönül (Şengönül, 2007) which we will need in order to establish the results of this article.

**Theorem 1.1** ((Şengönül, 2007), Theorem 2.1)**.** *The sets $\mathcal{Z}$ and $\mathcal{Z}_0$ are the linear spaces with the co-ordinate wise addition and scalar multiplication which are the BK-spaces with the norm*

$$\|x\|_{\mathcal{Z}} = \|x\|_{\mathcal{Z}_0} = \|Z^p x\|_c.$$

**Theorem 1.2** ((Şengönül, 2007), Theorem 2.2)**.** *The sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ are linearly isomorphic to the spaces $c$ and $c_0$ respectively, i.e $\mathcal{Z} \cong c$ and $\mathcal{Z}_0 \cong c_0$.*

**Theorem 1.3** ((Şengönül, 2007), Theorem 2.3)**.** *The inclusions $\mathcal{Z}_0 \subset \mathcal{Z}$ strictly hold for $p \neq 1$.*

**Theorem 1.4** ((Şengönül, 2007), Theorem 2.6)**.** *$\mathcal{Z}_0$ is solid.*

**Theorem 1.5** ((Şengönül, 2007), Theorem 3.6)**.** *$\mathcal{Z}$ is not a solid sequence space.*

The concept of statistical convergence was first introduced by Fast (Fast, 1951) and also independently by Buck (Buck, 1953) and Schoenberg (Schoenberg, 1959) for real and complex sequences. Further this concept was studied by Connor (Connor, 1988, 1989; Connor & Kline, 1996), Connor, Fridy and Kline (Fridy & Kline, 1994) and many others. Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. A sequence $x = (x_k)$ is said to be Statistically convergent to $L$ if for a given $\epsilon > 0$

$$\lim_k \frac{1}{k} |\{i : |x_i - L| \geq \epsilon, i \leq k\}| = 0.$$

Later on it was studied by Fridy (Fridy, 1985, 1993) from the sequence space point of view and linked it with the summability theory. The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Šalát, Wilczyński (Kostyrko *et al.*, 2000). Later on it was studied by Šalát, Tripathy, Ziman (Šalát *et al.*, 2004; Šalát *et al.*, 2005) and Demirci (Connor *et al.*, 2001). Here we give some preliminaries about the notion of I-convergence.

Let X be a non empty set. A set I$\subseteq 2^X$ ($2^X$ denoting the power set of X) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e $A \in I$, $B \subseteq A \Rightarrow B \in I$. A non empty family of sets $\pounds(I) \subseteq 2^X$ is said to be filter on X if and only if $\phi \notin \pounds$(I), for $A, B \in \pounds(I)$ we have $A \cap B \in \pounds(I)$ and for each $A \in \pounds(I)$ and $A \subseteq B$ implies $B \in \pounds(I)$. An Ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. For each ideal I, there is a filter $\pounds(I)$ corresponding to I. i.e $\pounds(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} - K$.

**Definition 1.1.** A sequence space $E$ is said to be solid or normal if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequence of scalars $(\alpha_k)$ with $|\alpha_k| < 1$ for all $k \in \mathbb{N}$.

**Definition 1.2.** A sequence space $E$ is said to be monotone if it contains the canonical preimages of all its stepspaces.

**Definition 1.3.** A sequence space $E$ is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

**Definition 1.4.** A sequence space $E$ is said to be a sequence algebra if $(x_k y_k) \in E$ whenever $(x_k), (y_k) \in E$.

**Definition 1.5.** A sequence space $E$ is said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $(x_k) \in E$ where $\pi(k)$ is a permutation on $\mathbb{N}$.

**Definition 1.6.** A sequence $(x_k) \in \omega$ is said to be I-convergent to a number $L$ if for every $\epsilon > 0$. $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I$. In this case we write I-lim $x_k = L$.
The space $c^I$ of all I-convergent sequences to $L$ is given by

$$c^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I, \text{ for some L} \in \mathbb{C}\}.$$

**Definition 1.7.** A sequence $(x_k) \in \omega$ is said to be I-null if $L = 0$. In this case we write I-lim $x_k = 0$.

**Definition 1.8.** A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in I$.

**Definition 1.9.** A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that $\{k \in \mathbb{N} : |x_k| \geq M\} \in I$.

**Definition 1.10.** A modulus function $f$ is said to satisfy $\triangle_2$-condition if for all values of u there exists a constant $K > 0$ such that $f(Lu) \leq KLf(u)$ for all values of $L > 1$.

**Definition 1.11.** Take for I the class $I_f$ of all finite subsets of $\mathbb{N}$. Then $I_f$ is a non-trivial admissible ideal and $I_f$ convergence coincides with the usual convergence with respect to the metric in $X$ (see (Khan & Ebadullah, 2011; Kostyrko *et al.*, 2000)).

**Definition 1.12.** For I= $I_\delta$ and $A \subset \mathbb{N}$ with $\delta(A) = 0$ respectively. $I_\delta$ is a non-trivial admissible ideal, $I_\delta$-convergence is said to be logarithmic statistical covergence (see (Khan & Ebadullah, 2011; Kostyrko *et al.*, 2000)).

**Definition 1.13.** A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \rightarrow \mathbb{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where $K$is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$ (see (Khan & Ebadullah, 2011)).

**Definition 1.14.** A convergence field of I-covergence is a set

$$F(I) = \{x = (x_k) \in \ell_\infty : \text{there exists } I - \lim x \in \mathbb{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of $\ell_\infty$ with respect to the supremum norm, $F(I) = \ell_\infty \cap c^I$ (see (Khan & Ebadullah, 2011; Tripathy & Hazarika, 2011)).

Define a function $\hbar : F(I) \rightarrow \mathbb{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \rightarrow \mathbb{R}$ is a Lipschitz function (see (Khan & Ebadullah, 2011)). The following Lemmas will be used for establishing some results of this article.

**Lemma 1.1.** *Let E be a sequence space. If E is solid then E is monotone (see (Kamthan & Gupta, 1981), page 53).*

**Lemma 1.2.** *Let K∈ £(I) and M⊆N. If M∉I, then M∩N ∉I (see (Tripathy & Hazarika, 2009, 2011)).*

**Lemma 1.3.** *If $I \subset 2^N$ and M⊆N. If M ∉I, then M∩N ∉I (see (Tripathy & Hazarika, 2009, 2011)).*

The idea of modulus was structured in 1953 by Nakano (See (Nakano, 1953)). A function $f : [0, \infty) \longrightarrow [0, \infty)$ is called a modulus if (1) $f(t) = 0$ if and only if $t = 0$,
(2) $f(t + u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
(3) $f$ is nondecreasing, and
(4) $f$ is continuous from the right at zero.

Ruckle (Ruckle, 1968, 1967, 1973) used the idea of a modulus function $f$ to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space, and Ruckle (Ruckle, 1973) proved that the intersection of all such $X(f)$ spaces is $\phi$, the space of all finite sequences. The space $X(f)$ is closely related to the space $\ell_1$ which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle (Ruckle, 1968, 1967, 1973) proved that, for any modulus $f$, $X(f) \subset \ell_1$ and $X(f)^\alpha = \ell_\infty$, where $X(f)^\alpha = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}$. The space $X(f)$ is a Banach space with respect to the norm $\|x\| = \sum_{k=1}^{\infty} f(|x_k|) < \infty$.(See[31]).

Spaces of the type $X(f)$ are a special case of the spaces structured by Gramsch in (Gramsch, 1971). From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by Garling (Garling, 1966, 1968), Köthe (Köthe, 1970) and Ruckle (Ruckle, 1968, 1967, 1973). After then Kolk (Kolk, 1993, 1994) gave an extension of $X(f)$ by considering a sequence of modulii $F = (f_k)$ and defined the sequence space $X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}$.(See[22-23]).

(c.f (Dems, 2005; Gurdal, 2004; Khan *et al.*, 2012*b*,*a*, 2013; Šalát, 1980; Tripathy & Hazarika, 2009, 2011)).

Recently Khan and Ebadullah in (Khan *et al.*, 2013) introduced the following classes of sequences $\mathcal{Z}^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k - L|) \geq \varepsilon, \text{ for some L} \in \mathbb{C}\ \} \in I\}$, $\mathcal{Z}_0^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq \varepsilon\} \in I\}$, $\mathcal{Z}_\infty^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq M, \text{ for each fixed M>0}\} \in I\}$.

We also denote by $m_{\mathcal{Z}}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}^I(f)$ and $m_{\mathcal{Z}_0}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}_0^I(f)$.

**In this article we introduce the following class of sequence spaces:**

$$\mathcal{Z}^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k - L|) \geq \varepsilon, \text{ for some L} \in \mathbb{C}\ \} \in I\},$$
$$\mathcal{Z}_0^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k|) \geq \varepsilon\} \in I\},$$
$$\mathcal{Z}_\infty^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k|) \geq M, \text{ for each fixed M>0}\} \in I\}.$$

We also denote by $m_{\mathcal{Z}}^I(F) = \mathcal{Z}_\infty^I(F) \cap \mathcal{Z}^I(F)$ and $m_{\mathcal{Z}_0}^I(F) = \mathcal{Z}_\infty^I(F) \cap \mathcal{Z}_0^I(F)$.

## 2. Main Results

**Theorem 2.1.** *For a sequence of moduli $F = (f_k)$, the classes of sequences $\mathcal{Z}^I(F), \mathcal{Z}_0^I(F), m_{\mathcal{Z}}^I(F)$ and $m_{\mathcal{Z}_0}^I(F)$ are linear spaces.*

*Proof.* We shall prove the result for the space $\mathcal{Z}^I(F)$. The proof for the other spaces will follow similarly.
Let $(x_k), (y_k) \in \mathcal{Z}^I(F)$ and let $\alpha, \beta$ be scalars. Then

$$I - \lim f_k(|x_k - L_1|) = 0, \text{ for some} L_1 \in \mathbb{C};$$

$$I - \lim f_k(|y_k - L_2|) = 0, \text{ for some} L_2 \in \mathbb{C};$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in N : f_k(|x_k - L_1|) > \frac{\epsilon}{2}\} \in I, \tag{2.1}$$

$$A_2 = \{k \in N : f_k(|y_k - L_2|) > \frac{\epsilon}{2}\} \in I. \tag{2.2}$$

Since $f_k$ is a modulus function, we have

$$f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2) \leq f_k(|\alpha||x_k - L_1|) + f_k(|\beta||y_k - L_2|) \leq f_k(|x_k - L_1|) + f_k(|y_k - L_2|).$$

Now, by (2.1) and (2.2), $\{k \in \text{N}: f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) > \epsilon\} \subset A_1 \cup A_2$. Therefore $(\alpha x_k + \beta y_k) \in \mathcal{Z}^I(F)$ Hence $\mathcal{Z}^I(F)$ is a linear space.

□

We state the following result without proof in view of Theorem 2.1.

**Theorem 2.2.** *The spaces* $m^I_{\mathcal{Z}}(F)$ *and* $m^I_{\mathcal{Z}_0}(F)$ *are normed linear spaces, normed by*

$$\|x_k\|_* = \sup_k f_k(|x_k|). \tag{2.3}$$

**Theorem 2.3.** *A sequence* $x = (x_k) \in m^I_{\mathcal{Z}}(F)$ *I-converges if and only if for every* $\epsilon > 0$ *there exists* $N_\epsilon \in \mathbb{N}$ *such that*

$$\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F) \tag{2.4}$$

*Proof.* Suppose that $L = I - \lim x$. Then $B_\epsilon = \{k \in \mathbb{N} : |x_k - L| < \frac{\epsilon}{2}\} \in m^I_{\mathcal{Z}}(F)$. For all $\epsilon > 0$. Fix an $N_\epsilon \in B_\epsilon$.Then we have $|x_{N_\epsilon} - x_k| \le |x_{N_\epsilon} - L| + |L - x_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ which holds for all $k \in B_\epsilon$. Hence $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F)$.

Conversely, suppose that $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F)$. That is $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F)$ for all $\epsilon > 0$. Then the set $C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m^I_{\mathcal{Z}}(F)$ for all $\epsilon > 0$. Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I_{\mathcal{Z}}(F)$ as well as $C_{\frac{\epsilon}{2}} \in m^I_{\mathcal{Z}}(F)$.

Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I_{\mathcal{Z}}(F)$. This implies that $J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$ that is $\{k \in \mathbb{N} : x_k \in J\} \in m^I_{\mathcal{Z}}(F)$ that is $diamJ \le diamJ_\epsilon$ where the diam of J denotes the length of interval J.
In this way, by induction we get the sequence of closed intervals $J_\epsilon = I_0 \supseteq I_1 \supseteq ... \supseteq I_k \supseteq ...$ with the property that $diamI_k \le \frac{1}{2} diamI_{k-1}$ for (k=2,3,4.....) and $\{k \in \mathbb{N} : x_k \in I_k\} \in m^I_{\mathcal{Z}}(F)$ for $(k = 1, 2, 3, ...)$. Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f_k(\xi) = I - \lim f_k(x)$, that is $L = I - \lim f_k(x)$. □

**Theorem 2.4.** *Let* $(f_k)$ *and* $(g_k)$ *be modulus functions for some fixed k that satisfy the* $\triangle_2$*-condition. If X is any of the spaces* $\mathcal{Z}^I, \mathcal{Z}^I_0, m^I_{\mathcal{Z}}$ *and* $m^I_{\mathcal{Z}_0}$ *etc, then the following assertions hold.*
*(a)* $X(g_k) \subseteq X(f_k.g_k)$,
*(b)* $X(f_k) \cap X(g_k) \subseteq X(f_k + g_k)$.

*Proof.* (a) Let $(x_n) \in \mathcal{Z}^I_0(g_k)$. Then

$$I - \lim_n g_k(|x_n|) = 0. \tag{2.5}$$

Let $\epsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $f_k(t) < \epsilon$ for $0 < t < \delta$.
Write $y_n = g_k(|x_n|)$ and consider $\lim_n f_k(y_n) = \lim_n f_k(y_n)_{y_n<\delta} + \lim_n f_k(y_n)_{y_n>\delta}$. We have

$$\lim_n f_k(y_n) \le f_k(2) \lim_n (y_n). \tag{2.6}$$

For $y_n > \delta$, we have $y_n < \frac{y_n}{\delta} < 1 + \frac{y_n}{\delta}$. Since $f_k$ is non-decreasing, it follows that $f_k(y_n) < f_k(1 + \frac{y_n}{\delta}) < \frac{1}{2} f_k(2) + \frac{1}{2} f_k(\frac{2y_n}{\delta})$ Since $f_k$ satisfies the $\triangle_2$-condition, we have $f_k(y_n) < \frac{1}{2} K \frac{y_n}{\delta} f_k(2) + \frac{1}{2} K \frac{y_n}{\delta} f_k(2) = K \frac{y_n}{\delta} f_k(2)$.

Hence

$$\lim_n f_k(y_n) \le \max(1,\mathrm{K}) \delta^{-1} f_k(2) \lim_n (y_n). \tag{2.7}$$

From (2.5), (2.6) and (2.7), we have $(x_n) \in \mathcal{Z}_0^I(f_k.g_k)$. Thus $\mathcal{Z}_0^I(g_k) \subseteq \mathcal{Z}_0^I(f_k.g_k)$. The other cases can be proved similarly.

(b) Let $(x_n) \in \mathcal{Z}_0^I(f_k) \cap \mathcal{Z}_0^I(g_k)$. Then $I - \lim_n f_k(|x_n|) = 0$ and $I - \lim_n g_k(|x_n|) = 0$.

The rest of the proof follows from the following equality $\lim_n (f_k + g_k)(|x_n|) = \lim_n f_k(|x_n|) + \lim_n g_k(|x_n|)$. □

**Corollary 2.1.** *$X \subseteq X(f_k)$ for some fixed k and $X = \mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$.*

**Theorem 2.5.** *The spaces $\mathcal{Z}_0^I(F)$ and $m_{\mathcal{Z}_0}^I(F)$ are solid and monotone.*

*Proof.* We shall prove the result for $\mathcal{Z}_0^I(F)$. Let $(x_k) \in \mathcal{Z}_0^I(F)$. Then

$$I - \lim_k f_k(|x_k|) = 0. \tag{2.8}$$

Let $(\alpha_k)$ be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.
Then the result follows from [9] and the following inequality $f_k(|\alpha_k x_k|) \leq |\alpha_k| f_k(|x_k|) \leq f_k(|x_k|)$ for all $k \in \mathbb{N}$. That the space $\mathcal{Z}_0^I(F)$ is monotone follows from the Lemma 1.20. For $m_{\mathcal{Z}_0}^I(F)$ the result can be proved similarly. □

**Theorem 2.6.** *The spaces $\mathcal{Z}^I(F)$ and $m_{\mathcal{Z}}^I(F)$ are neither solid nor monotone in general.*

*Proof.* Here we give a counter example. Let $I = I_\delta$ and $f_k(x) = x^2$ for some fixed k and for all $x \in [0, \infty)$. Consider the K-step space $X_K(f_k)$ of $X$ defined as follows. Let $(x_n) \in X$ and let $(y_n) \in X_K$ be such that

$$(y_n) = \begin{cases} (x_n), \text{if n is even,} \\ 0, \textit{otherwise}. \end{cases}$$

Consider the sequence $(x_n)$ defined by $(x_n) = 1$ for all $n \in \mathbb{N}$. Then $(x_n) \in \mathcal{Z}^I(F)$ but its K-stepspace preimage does not belong to $\mathcal{Z}^I(F)$. Thus $\mathcal{Z}^I(F)$ is not monotone. Hence $\mathcal{Z}^I(F)$ is not solid. □

**Theorem 2.7.** *The spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are sequence algebras.*

*Proof.* We prove that $\mathcal{Z}_0^I(F)$ is a sequence algebra. Let $(x_k), (y_k) \in \mathcal{Z}_0^I(F)$. Then $I - \lim f_k(|x_k|) = 0$ and $I - \lim f_k(|y_k|) = 0$. Then we have $I - \lim f_k(|(x_k.y_k)|) = 0$. Thus $(x_k.y_k) \in \mathcal{Z}_0^I(F)$ is a sequence algebra. For the space $\mathcal{Z}^I(F)$, the result can be proved similarly. □

**Theorem 2.8.** *The spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not convergence free in general.*

*Proof.* Here we give a counter example. Let $I = I_f$ and $f_k(x) = x^3$ for some fixed k and for all $x \in [0, \infty)$. Consider the sequence $(x_n)$ and $(y_n)$ defined by $x_n = \frac{1}{n}$ *and* $y_n = n$ for all $n \in \mathbb{N}$. Then $(x_n) \in \mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$, but $(y_n) \notin \mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$. Hence the spaces $\mathcal{Z}_0^I(F)$ and $\mathcal{Z}_0^I(F)$ are not convergence free. □

**Theorem 2.9.** *If $I$ is not maximal and $I \neq I_f$, then the spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not symmetric.*

*Proof.* Let $A \in I$ be infinite and $f_k(x) = x$ for some fixed k and for all $x \in [0, \infty)$. If

$$x_n = \begin{cases} 1, \text{for } n \in A, \\ 0, otherwise, \end{cases}$$

then by lemma 1.22 $(x_n) \in \mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F)$.
Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(n) = \begin{cases} \phi(n), \text{for } n \in K, \\ \psi(n), otherwise, \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(n)} \notin \mathcal{Z}^I(F)$ and $x_{\pi(n)} \notin \mathcal{Z}_0^I(F)$. Hence $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not symmetric. □

**Theorem 2.10.** $\mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F) \subset \mathcal{Z}_\infty^I(F)$.

*Proof.* Let $(x_k) \in \mathcal{Z}^I(F)$. Then there exists $L \in \mathbb{C}$ such that $I - \lim f_k(|x_k - L|) = 0$. We have $f_k(|x_k|) \leq \frac{1}{2} f_k(|x_k - L|) + f_k \frac{1}{2}(|L|)$. Taking the supremum over $k$ on both sides we get $(x_k) \in \mathcal{Z}_\infty^I(F)$. The inclusion $\mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F)$ is obvious. □

**Theorem 2.11.** *The function $\hbar : m_{\mathcal{Z}}^I(F) \to \mathbb{R}$ is the Lipschitz function, where $m_{\mathcal{Z}}^I(F) = \mathcal{Z}_\infty^I(F) \cap \mathcal{Z}^I(F)$, and hence uniformly continuous.*

*Proof.* Let $x, y \in m_{\mathcal{Z}}^I(F)$, $x \neq y$. Then the sets

$$A_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| \geq \|x - y\|_*\} \in I,$$
$$A_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| \geq \|x - y\|_*\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \|x - y\|_*\} \in m_{\mathcal{Z}}^I(F),$$
$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \|x - y\|_*\} \in m_{\mathcal{Z}}^I(F).$$

Hence also $B = B_x \cap B_y \in m_{\mathcal{Z}}^I(F)$, so that $B \neq \phi$. Now taking k in $B$,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_k| + |x_k - y_k| + |y_k - \hbar(y)| \leq 3\|x - y\|_*.$$

Thus $\hbar$ is a Lipschitz function. For the space $m_{\mathcal{Z}_0}^I(F)$ the result can be proved similarity. □

**Theorem 2.12.** *If $x, y \in m_{\mathcal{Z}}^I(F)$, then $(x.y) \in m_{\mathcal{Z}}^I(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.*

*Proof.* For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \epsilon\} \in m_{\mathcal{Z}}^I(F),$$
$$B_x = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \epsilon\} \in m_{\mathcal{Z}}^I(F).$$

Now,

$$|x_k y_k - \hbar(x)\hbar(y)| = |x_k y_k - x_k \hbar(y) + x_k \hbar(y) - \hbar(x)\hbar(y)| \leq |x_k||y_k - \hbar(y)| + |\hbar(y)||x_k - \hbar(x)|. \quad (2.9)$$

As $m_{\mathcal{Z}}^I(F) \subseteq \mathcal{Z}_\infty^I(F)$, there exists an $M \in \mathbb{R}$ such that $|x_k| < M$ and $|\hbar(y)| < M$.

Using eqn[10] we get $|x_k y_k - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$ For all $k \in B_x \cap B_y \in m^I(F)$. Hence $(x.y) \in m_{\mathcal{Z}}^I(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For the space $m_{\mathcal{Z}_0}^I(F)$ the result can be proved similarity. □

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# Exponential Stability versus Polynomial Stability for Skew-Evolution Semiflows in Infinite Dimensional Spaces

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## Abstract

As the dynamical systems that model processes issued from engineering, economics or physics are extremely complex, of great interest is to study the solutions of differential equations by means of associated evolution families. In this paper we emphasize some notions of asymptotic stability for skew-evolution semiflows on Banach spaces, such as exponential and polynomial stability, in a nonuniform setting. Examples for every concept and connections between them are also presented, as well as some characterizations.

*Keywords:* Skew-evolution semiflow, exponential stability, Barreira-Valls exponential stability, polynomial stability.
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## 1. Preliminaries

The theory of asymptotic properties for evolution equations has witnessed lately an explosive development. We intend to emphasize in our paper a framework which enables us to obtain characterizations in a unitary approach for the asymptotic stability on Banach spaces. The notion of skew-evolution semiflow, introduced in (Megan & Stoica, 2008), is more appropriate for the study in the nonuniform case. They depend on three variables, making thus possible the generalization for skew-product semiflows and evolution operators, which depend only on two. Hence, the study of asymptotic behaviors for skew-evolution semiflows in the nonuniform setting arises as natural, relative to the third variable. The notion has proved itself of interest in the development of the stability theory, in a uniform as well as in a nonuniform setting, being already adopted by some researchers, as, for example, A.J.G. Bento and C.M. Silva (see (Bento & Silva, 2012)), P. Viet Hai (see (Hai, 2010) and (Hai, 2011)) and T. Yue, X.Q. Song and D.Q. Li (see (Yue *et al.*, 2014)), which have contributed to the expansion of the concept of skew-evolution semiflows and deepened the study of their asymptotic behaviors and applications. Some properties for skew-evolution semiflows are defined and characterized in (Stoica, 2010).

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The definitions of various types of stability are illustrated by examples and the connections between them are emphasized. Our aim is also to give some integral characterizations for them. We present a concept of nonuniform exponential stability, given and studied by L. Barreira and C. Valls in (Barreira & Valls, 2008), which we call *"Barreira-Valls exponential stability"*. In this paper, some generalizations for the results obtained in the uniform setting in (Stoica & Megan, 2010) are proved in the nonuniform case.

## 2. Skew-evolution semiflows

This section gives the notion of skew-evolution semiflow on a Banach space, defined by means of an evolution semiflow and of an evolution cocycle.

Let $(X, d)$ be a metric space, $V$ a Banach space and $V^*$ its topological dual. Let $\mathcal{B}(V)$ be the space of all $V$-valued bounded operators defined on $V$. The norm of vectors on $V$ and on $V^*$ and of operators on $\mathcal{B}(V)$ is denoted by $\|\cdot\|$. $I$ is the identity operator. Let us denote $Y = X \times V$ and $T = \left\{(t, t_0) \in \mathbb{R}_+^2 : t \geq t_0\right\}$.

**Definition 2.1.** A mapping $\varphi : T \times X \to X$ is said to be *evolution semiflow* on $X$ if following properties are satisfied:

$(es_1)$ $\varphi(t, t, x) = x,\ \forall (t, x) \in \mathbb{R}_+ \times X;$

$(es_2)$ $\varphi(t, s, \varphi(s, t_0, x)) = \varphi(t, t_0, x),\ \forall (t, s), (s, t_0) \in T,\ \forall x \in X.$

**Definition 2.2.** A mapping $\Phi : T \times X \to \mathcal{B}(V)$ is called *evolution cocycle* over an evolution semiflow $\varphi$ if it satisfies following properties:

$(ec_1)$ $\Phi(t, t, x) = I,\ \forall t \geq 0,\ \forall x \in X;$

$(ec_2)$ $\Phi(t, s, \varphi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x), \forall (t, s), (s, t_0) \in T, \forall x \in X.$

Let $\Phi$ be an evolution cocycle over an evolution semiflow $\varphi$. The mapping

$$C : T \times Y \to Y,\ C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v) \tag{2.1}$$

is called *skew-evolution semiflow* on $Y$.

**Example 2.1.** Let us denote $C = C(\mathbb{R}, \mathbb{R})$ the set of all continuous functions $x : \mathbb{R} \to \mathbb{R}$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}$. For every $x, y \in C$, we define

$$d_n(x, y) = \sup_{t \in [-n, n]} |x(t) - y(t)|.$$

The set $C$ is metrizable with respect to the metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)}.$$

We consider for every $n \in \mathbb{N}^*$ a decreasing function

$$x_n : \mathbb{R}_+ \to \left(\frac{1}{2n+1}, \frac{1}{2n}\right), \text{ with the property } \lim_{t \to \infty} x_n(t) = \frac{1}{2n+1}.$$

We denote

$$x_n^s(t) = x_n(t + s),\ \forall t, s \geq 0.$$

Let $X$ be the closure in $C$ of the set $\{x_n^s, n \in \mathbb{N}^*, s \in \mathbb{R}_+\}$. The mapping

$$\varphi : T \times X \to X,\ \varphi(t, s, x) = x_{t-s},\ \text{where } x_{t-s}(\tau) = x(t - s + \tau),\ \forall \tau \geq 0,$$

is an evolution semiflow on $X$. Let us consider the Banach space $V = \mathbb{R}^p$, $p \geq 1$, with the norm $\left\|(v_1, ..., v_p)\right\| = |v_1| + ... + |v_p|$. Then the mapping

$$\Phi : T \times X \to \mathcal{B}(V),\ \Phi(t, s, x)v = \left(e^{\alpha_1 \int_s^t x(\tau - s)d\tau} v_1, ..., e^{\alpha_p \int_s^t x(\tau - s)d\tau} v_p\right),$$

where $(\alpha_1, ..., \alpha_p) \in \mathbb{R}^p$ is fixed, is an evolution cocycle over the evolution semiflow $\varphi$ and $C = (\varphi, \Phi)$ is a skew-evolution semiflow on $Y$.

**Example 2.2.** For $X = \mathbb{R}_+$, the mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t, s, x) = x$ is an evolution semiflow. For every evolution cocycle $\Phi$ over $\varphi$, we obtain that the mapping $E_\Phi : T \to \mathcal{B}(V)$, $E_\Phi(t, s) = \Phi(t, s, 0)$ is an evolution operator on $V$.

**Example 2.3.** If $C = (\varphi, \Phi)$ denotes a skew-evolution semiflow and $\alpha \in \mathbb{R}$ a parameter, then $C_\alpha = (\varphi, \Phi_\alpha)$, where

$$\Phi_\alpha : T \times X \to \mathcal{B}(V),\ \Phi_\alpha(t, t_0, x) = e^{\alpha(t-t_0)}\Phi(t, t_0, x), \tag{2.2}$$

is also a skew-evolution semiflow, called the *α-shifted skew-evolution semiflow*.

## 3. Exponential stability

In this section we consider several concepts of exponential stability for skew-evolution semiflows. Some connections between these concepts are established. We will emphasize that they are not equivalent.

The nonuniform exponential stability is given by

**Definition 3.1.** A skew-evolution semiflow $C = (\varphi, \Phi)$ is *exponentially stable* (*e.s.*) if there exist a mapping $N : \mathbb{R}_+ \to [1, \infty)$ and a constant $\alpha > 0$ such that, for all $(t, s) \in T$, following relation takes place:

$$\|\Phi(t, t_0, x)v\| \leq N(s)e^{-\alpha t} \|\Phi(s, t_0, x)v\|,\ \forall (x, v) \in Y. \tag{3.1}$$

A concept of nonuniform exponential stability, which we will name *"Barreira-Valls exponential stability"*, is given by L. Barreira and C. Valls in (Barreira & Valls, 2008) for evolution equations.

**Definition 3.2.** A skew-evolution semiflow $C = (\varphi, \Phi)$ is *Barreira-Valls exponentially stable* (*BV.e.s.*) if there exist some constants $N \geq 1$, $\alpha > 0$ and $\beta$ such that, for all $(t, s), (s, t_0) \in T$, the relation holds:

$$\|\Phi(t, t_0, x)v\| \leq Ne^{-\alpha t}e^{\beta s} \|\Phi(s, t_0, x)v\|,\ \forall\ (x, v) \in Y. \tag{3.2}$$

The asymptotic property of nonuniform stability is considered in

**Definition 3.3.** A skew-evolution semiflow $C = (\varphi, \Phi)$ is *stable* (*s.*) if there exists a mapping $N : \mathbb{R}_+ \to [1, \infty)$ such that, for all $(t, s), (s, t_0) \in T$, the relation is true:

$$\|\Phi(t, t_0, x)v\| \leq N(s) \|\Phi(s, t_0, x)v\|,\ \forall\ (x, v) \in Y. \tag{3.3}$$

Let us remind the property of exponential growth for skew-evolution semiflows, given by

**Definition 3.4.** A skew-evolution semiflow $C = (\varphi, \Phi)$ has *exponential growth* (*e.g.*) if there exist two mappings $M, \omega : \mathbb{R}_+ \to [1, \infty)$, $\omega$ nondecreasing, such that, for all $(t, s), (s, t_0) \in T$, we have:

$$||\Phi(t, t_0, x)v|| \leq M(s)e^{\omega(s)(t-s)} \, ||\Phi(s, t_0, x)v|| \, , \ \forall \ (x, v) \in Y. \tag{3.4}$$

*Remark.* The relations concerning the previously defined asymptotic properties for skew-evolution semiflows are given by

$$(BV.e.s.) \Longrightarrow (e.s.) \Longrightarrow (s.) \Longrightarrow (e.g.) \tag{3.5}$$

The reciprocal statements are not true, as shown in what follows.

The following example presents a skew-evolution semiflow which is exponentially stable but not Barreira-Valls exponentially stable.

**Example 3.1.** Let $X = \mathbb{R}_+$. The mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t, s, x) = x$ is an evolution semiflow on $\mathbb{R}_+$.

Let us consider a continuous function $u : \mathbb{R}_+ \to [1, \infty)$ with

$$u(n) = e^{n \cdot 2^{2n}} \text{ and } u\left(n + \frac{1}{2^{2n}}\right) = e^4.$$

We define

$$\Phi_u(t, s, x)v = \frac{u(s)e^s}{u(t)e^t} v, \ \text{where } (t, s) \in T, \ (x, v) \in Y.$$

As following relation

$$||\Phi_u(t, s, x)v|| \leq u(s)e^s e^{-t} \, ||v||$$

holds for all $(t, s, x, v) \in T \times Y$, it results that the skew-evolution semiflow $C_u = (\varphi, \Phi_u)$ is exponentially stable.

Let us now suppose that the skew-evolution semiflow $C_u = (\varphi, \Phi_u)$ is Barreira-Valls exponentially stable. Then, according to Definition 3.2, there exist $N \geq 1, \alpha > 0, \beta > 0$ and $t_1 > 0$ such that

$$\frac{u(s)e^s}{u(t)e^t} \leq N e^{-\alpha t} e^{\beta s}, \ \forall t \geq s \geq t_1.$$

For $t = n + \frac{1}{2^{2n}}$ and $s = n$ it follows that

$$e^{n\left(2^{2n}+1\right)} \leq N e^{n + \frac{1}{2^{2n}} + 4} e^{-\alpha\left(n + \frac{1}{2^{2n}}\right)} e^{\beta n},$$

which is equivalent with

$$e^{n\left(2^{2n}-\beta\right)} \leq N e^{\frac{1}{2^{2n}} + 4 - \alpha\left(n + \frac{1}{2^{2n}}\right)}.$$

For $n \to \infty$, a contradiction is obtained, which proves that $C_u$ is not Barreira-Valls exponentially stable.

There exist skew-evolution semiflows that are stable but not exponentially stable, as results from the following

**Example 3.2.** Let us consider $X = \mathbb{R}_+$, $V = \mathbb{R}$ and

$$u : \mathbb{R}_+ \to [1, \infty) \text{ with the property } \lim_{t \to \infty} \frac{u(t)}{e^t} = 0.$$

The mapping

$$\Phi_u : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}),\ \Phi_u(t, s, x)v = \frac{u(s)}{u(t)}v$$

is an evolution cocycle. As $|\Phi_u(t, s, x)v| \leq u(s)|v|$, $\forall (t, s, x, v) \in T \times Y$, it follows that $C_u = (\varphi, \Phi_u)$ is a stable skew-evolution semiflow, for every evolution semiflow $\varphi$.

On the other hand, if we suppose that $C_u$ is exponentially stable, according to Definition 3.1, there exist a mapping $N : \mathbb{R}_+ \to [1, \infty)$ and a constant $\alpha > 0$ such that, for all $(t, s), (s, t_0) \in T$, we have

$$\|\Phi_u(t, t_0, x)v\| \leq N(s)e^{-\alpha t} \|\Phi_u(s, t_0, x)v\|,\ \forall\ (x, v) \in Y.$$

It follows that

$$\frac{u(s)}{N(s)} \leq \frac{u(t)}{e^{\alpha t}}.$$

For $t \to \infty$ we obtain a contradiction, and, hence, $C_u$ is not exponentially stable.

Following example gives a skew-evolution semiflow that has exponential growth but is not stable.

**Example 3.3.** We consider $X = \mathbb{R}_+$, $V = \mathbb{R}$ and

$$u : \mathbb{R}_+ \to [1, \infty) \text{ with the property } \lim_{t\to\infty} \frac{e^t}{u(t)} = \infty.$$

The mapping

$$\Phi_u : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}),\ \Phi_u(t, s, x)v = \frac{u(s)e^t}{u(t)e^s}v$$

is an evolution cocycle. We have $|\Phi(t, s, x)v| \leq u(s)e^{t-s}|v|$, $\forall (t, s, x, v) \in T \times Y$. Hence, $C_u = (\varphi, \Phi_u)$ is a skew-evolution semiflow, over every evolution semiflow $\varphi$, and has exponential growth.

Let us suppose that $C_u$ is stable. According to Definition 3.3, there exists a mapping $N : \mathbb{R}_+ \to [1, \infty)$ such that $u(s)e^t \leq N(s)u(t)e^s$, for all $(t, s) \in T$. If $t \to \infty$, a contradiction is obtained. Hence, $C_u$ is not stable.

## 4. Polynomial stability

In this section, we introduce a new concept of nonuniform stability for skew-evolution semiflows, given by the next

**Definition 4.1.** A skew-evolution semiflow $C = (\varphi, \Phi)$ is called *polynomially stable* (*p.s.*) if there exist a mapping $N : \mathbb{R}_+ \to [1, \infty)$ and a constant $\gamma > 0$ such that:

$$\|\Phi(t, s, x)v\|\, ds \leq N(s)(t - s)^{-\gamma}\, \|v\|, \tag{4.1}$$

for all $t > s \geq 0$ and all $(x, v) \in Y$.

*Remark.* If a skew-evolution semiflow $C$ is exponentially stable, then it is polynomially stable.

$$(e.s.) \Longrightarrow (p.s.)$$

The reciprocal statement is not true, as shown in

**Example 4.1.** Let $X = \mathbb{R}_+$, $V = \mathbb{R}$ and the mapping $u : \mathbb{R}_+ \to \mathbb{R}$ given by $u(t) = t + 1$. The mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$, where $\varphi(t, s, x) = x$ is an evolution semiflow on $\mathbb{R}_+$. We consider

$$\Phi_u : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}),\ \Phi_u(t, s, x)v = \frac{u(s)}{u(t)}v.$$

Then, as we have

$$|\Phi_u(t, s, x)v| \leq \frac{s^2}{t}|v| = s\frac{s}{t}|v|,\ \forall t \geq s \geq 1 = t_0,\ \forall (x, v) \in Y,$$

it follows that $C = (\varphi, \Phi)$ is a Barreira-Valls polynomially stable skew-evolution semiflow.

If we suppose that $C$ is exponentially stable, according to Definition 3.1, there exist $N : \mathbb{R}_+ \to [1, \infty)$ and $\alpha > 0$ such that

$$\frac{s + 1}{t + 1} \leq N(s)e^{-\alpha t},\ \forall t \geq s \geq t_0,$$

which is equivalent with

$$\frac{e^{\alpha t}}{t + 1} \leq \frac{N(t_0)}{t_0 + 1},\ \forall t \geq t_0,$$

and which, for $t \to \infty$, leads to a contradiction. Hence, $C$ is not exponentially stable.

*Remark.* For $\alpha \geq \beta$ in Definition 3.2, a Barreira-Valls exponentially stable skew-evolution semiflow $C$ is polynomially stable.

$$(B.V.e.s.) \Longrightarrow (p.s.)$$

**Example 4.2.** Let us consider $X = \mathbb{R}_+$, $V = \mathbb{R}$ and the mapping $u : \mathbb{R}_+ \to \mathbb{R}$ given by $u(t) = t^2 + 1$. The mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$, where $\varphi(t, s, x) = t - s + x$ is an evolution semiflow on $\mathbb{R}_+$. We define

$$\Phi_u : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}),\ \Phi_u(t, s, x)v = \frac{u(s)}{u(t)}v.$$

Then, as the relation

$$|\Phi_u(t, s, x)v| \leq (s^2 + 1)(t - s)^{-2}|v|,\ \forall t > s \geq 0,\ \forall (x, v) \in Y$$

holds, it follows that $C = (\varphi, \Phi)$ is a polynomially stable skew-evolution semiflow. On the other hand, $C$ is not Barreira-Valls exponentially stable.

A similar concept to the nonuniform exponential growth can be considered the following nonuniform asymptotic property, given by

**Definition 4.2.** A skew-evolution semiflow $C = (\varphi, \Phi)$ has *polynomial growth* (*p.g.*) if there exist two mappings $M, \gamma : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that:

$$\|\Phi(t, s, x)v\| \leq M(s)(t - s)^{\gamma(s)} \|v\|, \tag{4.2}$$

for all $t > s \geq 0$ and all $(x, v) \in Y$.

*Remark.* If a skew-evolution semiflow $C$ has polynomial growth, then it has exponential growth.

$$(p.g.) \Longrightarrow (e.g.)$$

In order to obtain an integral characterization for the property of nonuniform polynomial stability for skew-evolution semiflows, we introduce the following concept, given by

**Definition 4.3.** A skew-evolution semiflow $C = (\varphi, \Phi)$ is said to be $*$-*strongly measurable* ($* - s.m.$) if for every $(t, t_0, x, v^*) \in T \times X \times V^*$ the mapping defined by $s \mapsto \|\Phi(t, s, \varphi(s, t_0, x))^* v^*\|$ is measurable on $[t_0, t]$.

A particular class of $*$-strongly measurable skew-evolution semiflows is given by the next

**Definition 4.4.** A $*$-strongly measurable skew-evolution semiflow $C = (\varphi, \Phi)$ is called $*$-*integrally stable* ($* - i.s.$) if there exists a nondecreasing mapping $B : \mathbb{R}_+ \to [1, \infty)$ such that:

$$\int_s^t \left\|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\right\| d\tau \le B(s) \left\|v^*\right\|, \tag{4.3}$$

for all $(t, s) \in T$, all $x \in X$ and all $v^* \in V^*$ with $\|v^*\| \le 1$.

**Theorem 4.3.** *Let $C = (\varphi, \Phi)$ be a $*$-strongly measurable skew-evolution semiflow with polynomial growth. If $C$ is $*$-integrally stable, then $C$ is stable.*

*Proof.* Let us consider the function

$$\gamma_1 : \mathbb{R}_+ \to \mathbb{R}_+,\ \gamma_1(t) = \frac{1}{1 + \gamma(t)},$$

where the mapping $\gamma$ is given by Definition 4.2. We remark that for $t \ge s + 1$ we have

$$\int_s^t (\tau - s)^{-\gamma(s)} d\tau = \int_0^{t-s} u^{-\gamma(s)} du \ge \int_0^1 u^{-\gamma(s)} du = \gamma_1(s).$$

Hence, it follows that

$$\gamma_1(s) | < v^*, \Phi(t, s, x)v > | \le$$
$$\le \int_s^t (\tau - s)^{-\gamma(s)} \left\|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\right\| \|\Phi(\tau, s, x)v\| d\tau \le$$
$$\le M(s) \|v\| \int_s^t \left\|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\right\| d\tau \le M(s)B(s) \|v\| \left\|v^*\right\|,$$

where the existence of function $M$ is assured by Definition 4.2. We obtain

$$\|\Phi(t, s, x)v\| \le M_1(s) \|v\|,\ \forall t \ge s + 1 > s \ge 0,\ \forall (x, v) \in Y,$$

where we have denoted

$$M_1(s) = \frac{M(s)B(s)}{\gamma(s)},\ s \ge 0.$$

On the other hand, for $t \in [s, s + 1)$, we have

$$\|\Phi(t, s, x)v\| \le M(s)(t - s)^{\gamma(s)} \|v\| \le M(s) \|v\|,$$

and, hence, it follows that

$$\|\Phi(t, s, x)v\| \le [M(s) + M_1(s)] \|v\|,\ \forall (t, s) \in T,\ \forall (x, v) \in Y,$$

which proves that the skew-evolution semiflow $C$ is stable. □

The main result of this section is the following

**Theorem 4.4.** *Let $C = (\varphi, \Phi)$ be a $*$-strongly measurable skew-evolution semiflow with polynomial growth. If $C$ is $*$-integrally stable, then $C$ is polynomially stable.*

*Proof.* As the skew-evolution semiflow $C = (\varphi, \Phi)$ is $*$-integrally stable, according to Theorem 4.3, it follows that there exists a mapping $M_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$| < v^*, \Phi(t,s,x)v > | = | < \Phi(t,\tau,\varphi(\tau,s,x))^* v^*, \Phi(\tau,s,x)v > | \leq$$

$$\leq \|\Phi(\tau,s,x)v\| \left\|\Phi(t,\tau,\varphi(\tau,s,x))^* v^*\right\| \leq M_2(s)\|v\| \left\|\Phi(t,\tau,\varphi(\tau,s,x))^* v^*\right\|.$$

By integrating on $[s,t]$ we obtain for $(x,v) \in Y$ and $v^* \in V^*$ with $\|v^*\| \leq 1$

$$(t-s)| < v^*, \Phi(t,s,x)v > | \leq M_2(s)\|v\| \int_s^t \left\|\Phi(t,\tau,\varphi(\tau,s,x))^* v^*\right\| d\tau \leq$$

$$\leq M_2(s)B(s)\|v\| \left\|v^*\right\|,$$

which implies

$$(t-s)\|\Phi(t,s,x)v\| \leq M_2(s)B(s)\|v\|.$$

Hence, following relation

$$\|\Phi(t,s,x)v\| \leq M_2(s)B(s)(t-s)^{-1}\|v\|$$

holds for all $(t,s) \in T$ and all $(x,v) \in Y$.

Finally, it results that the skew-evolution semiflow $C = (\varphi, \Phi)$ is polynomially stable. □

*Remark.* In (Stoica & Megan, 2010), a variant of Theorem 4.4 for the case of uniform exponential stability is proved, as a generalization of a well known theorem of E.A. Barbashin, given in (Barbashin, 1967) for differential systems and of a result obtained in (Buşe *et al.*, 2007) by C. Buşe, M. Megan, M. Prajea and P. Preda for evolution operators. We remark that, in the nonuniform setting, the property of $*$-integral stability only implies the polynomial stability.

*Remark.* The reciprocal of Theorem 4.4 is not true. The skew-evolution semiflow given in Example 4.2 is polynomially stable but not $*$-strongly measurable. If we suppose that $C$ is $*$-strongly measurable, we have

$$\int_s^t \frac{\tau^2+1}{t^2+1} d\tau = \frac{t-s}{t^2+1}\left(1 + \frac{t^2+ts+s^2}{3}\right) \leq N(s).$$

For $t \to \infty$, a contradiction is obtained.

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# Common Fixed Points of Fuzzy Mappings in Quasi-Pseudo Metric and Quasi-Metric Spaces

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## Abstract

In this paper, we prove common fixed point theorems for fuzzy mappings satisfying a new inequality initiated by Constantin (1991) in complete quasi-pseudo metric space and we also obtain some new common fixed point theorems for a pair of fuzzy mappings on complete quasi-metric space under a generalized contractive condition. Our results generalized many recent fixed point theorems.

*Keywords:* fuzzy sets, fuzzy mappings, common fixed points, quasi-pseudo metric space, quasi-metric space, fuzzy contraction mappings.
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## 1. Introduction

It is a well known fact that the results of fixed points are very useful for determining the existence and uniqueness of solutions to various mathematical models. Over the period of last forty years the theory of fixed points has been developed regarding the results which are related to finding the fixed points of self and non-self nonlinear mappings. In 1922, Banach proved a contraction principle which states that for a complete metric space $(X, d)$, the mapping $T: X \to X$ satisfying the following contraction condition

$$d(Tx, Ty) \leq \alpha d(x, y) \quad \text{for all} \quad x, y \in X, \quad \text{where} \quad 0 < \alpha < 1$$

has a unique fixed point in $X$. Banach contraction principle plays a fundamental role in the emergence of modern fixed point theory and it gains more attention because it is based on iteration, so it can be easily applied using computer. Initially Zadeh (1965) introduced the concept of Fuzzy Sets in 1965, has been an attempt to develop a mathematical framework in which two system or phenomena which due to intrinsic indefiniteness-as distinguished from mere statistical variation can't themselves be characterized precisely. The classical work of Zadeh (1965) stimulated a great interest among mathematicians, engineers, biologists, economists, psychologists and experts in other areas who use mathematical method in their research.

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The notion of fixed points for fuzzy mappings was introduced by Weiss (1975) and Butnariu (1982). Fixed point theorems for fuzzy set valued mappings have been studied by Heilpern (1981) who introduced the concept of fuzzy contraction mappings and established Banach contraction principle for fuzzy mappings in complete metric linear spaces which is a fuzzy extension of Banach fixed point theorem and Nadler (1969) theorem for multi-valued mappings. Park & Jeong (1997) proved some common fixed point theorems for fuzzy mappings satisfying in complete metric space which are fuzzy extensions of some theorems in Beg & A. (1992); Park & Jeong (1997).

Motivated and inspired by the works of Arora & V. (2000), Constantin (1991) and Park & Jeong (1997) the purpose of this paper is to prove some common fixed point theorems for fuzzy mappings satisfying new contractive-type condition of Constantin (1991) in complete quasi-pseudo metric space. Our results are the fuzzy extensions of some theorems in Beg & A. (1992); Iseki (1995); Popa (1985); Singh & Whitfield (1982) . Also, our results generalize the results of Arora & V. (2000), Heilpern (1981), and Park & Jeong (1997).

Recently Chen (2011, 2012) considered a new type contraction $\psi$ contractive mapping in complete quasi metric space. The aim of this paper is to introduced a new class of fuzzy contraction mappings, which will be call fuzzy $\psi$ contractive mappings in complete quasi metric space and to prove the existence of common fixed point for these contractions.

## 2. Basic concepts

For this purpose we need the following definitions and Lemmas.

**Definition 2.1.** Sahin *et al.* (2005) A quasi-pseudo metric on a non-empty set $X$ is a non-negative real valued function $d$ on $X \times X$ such that, for all $x, y, z \in X$:

(i) $d(x, x) = 0$, and

(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A pair $(X, d)$ is called a quasi-pseudo metric space, if $d$ is a quasi-pseudo metric on $X$. A quasi-pseudo metric $d$ such that $x = y$ whenever $d(x, y) = 0$ is a quasi metric so that a quasi pseudo metric space we do not assume that $d(x, y) = d(y, x)$ for every $x$ and $y$. Each quasi-pseudo metric $d$ on $X$ induces a topology $\tau(d)$ which has base the family of all $d$ balls $B_\varepsilon(x)$, where $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$ If $d$ is a quasi-pseudo metric on $X$, then the function $d^{-1}$ defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ is also quasi-pseudo metric on $X$. By $d \wedge d^{-1}$ and $d \vee d^{-1}$ we denote $\min\{d, d^{-1}\}$ and $\max\{d, d^{-1}\}$ respectively.

**Definition 2.2.** Gregori. & Pastor (1999) Let $(X, d)$ be a quasi-pseudo metric space and let $A$ and $B$ be non-empty subsets of $X$. Then the Hausdroff distance between subsets of $A$ and $B$ is defined by

$$H(A, B) = max\{sup_{a\in A} d(a, B), sup_{b\in B} d(b, A)\}$$

where $d(a, B) = inf\{d(a, x) : x \in B\}$.

Note that: $H(A, B) \geq 0$ with $H(A, B) = 0$ if and only if closure of $A$ is equal to closure of $B$, $H(A, B) = H(B, A)$ and $H(A, B) \leq H(A, C) + H(C, B)$ for any non-empty subset $A, B$ and $C$ of $X$ when $d$ is a metric on $X$, clearly $H$ is the usual Hausdroff distance.

**Definition 2.3.** Gregori. & Pastor (1999) Let $(X, d)$ be a quasi-pseudo metric space. The families $W^*(X)$ and $W'(X)$ of fuzzy sets on $(X, d)$ are defined by

$$W^*(X) = \{A \text{ in } I^X : A_1 \text{ is non-empty, } d\text{ - closed and } d^{-1}\text{-}compact\},$$

$$W'(X) = \{A \text{ in } I^X : A_1 \text{ is non-empty, } d\text{ - closed and } d\text{-compact}\}.$$

As per Heilpern (1981), the family W(X) of fuzzy sets on metric linear space $(X, d)$ is defined as follows: $A \in W(X)$ if and only if $A_\alpha$ is compact and convex in $X$ for each $\alpha \in [0, 1]$ and $supA(x) = 1$ for $x \in X$.
If $(X, d)$ is a metric linear space, then we have

$$W(X) \subset W^*(X) = W'(X) = \{A \in I^X : A_1 \text{ is non-empty and } d\text{-compact } \} \subset I^X.$$

**Definition 2.4.** Gregori. & Pastor (1999) Let $(X, d)$ be a quasi-pseudo metric space and let $A, B \in W^*(X)$ or $A, B \in W'(X)$ and $\alpha \in [0, 1]$. Then we define

$$p_\alpha(A, B) = Inf\{d(x, y) : x \in A_\alpha, y \in B_\alpha\},$$

$$\delta_\alpha(A, B) = sup\{d(x, y) : x \in A_\alpha, y \in B_\alpha\},$$

$$D_\alpha(A, B) = H(A_\alpha, B_\alpha),$$

where $H$ is the Hausdroff distance deduced from the quasi-pseudo metric $d$ on $X$, $p(A, B) = Sup\{p_\alpha(A, B) : \alpha \in [0, 1]\}$, $\delta(A, B) = Sup\{\delta_\alpha(A, B) : \alpha \in [0, 1]\}$, $D(A, B) = Sup\{ D_\alpha(A, B) : \alpha \in [0, 1]\}$. It is noted that $p_\alpha$ is non-decreasing function of $\alpha$.

**Definition 2.5.** Gregori. & Pastor (1999) Let $X$ be an arbitrary set and $Y$ be any quasi-pseudo metric space. $G$ is said to be a fuzzy mapping if $G$ is a mapping from the set $X$ into $W^*(Y)$ or $W'(Y)$. This definition is more general than the one given in Heilpern (1981). A fuzzy mapping $G$ is a fuzzy subset on $X \times Y$ with membership function $G(x)(y)$. The function $G(x)(y)$ is the grade of membership of $y$ in $G(x)$.

**Definition 2.6.** Sahin *et al.* (2005) A point $x$ is a fixed point of the mapping $G : X \to I^X$, if $\{x\} \subseteq G(x)$.

Note that : If $A, B \in I^X$, then $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$.
The following Lemmas were proved by Gregori. & Pastor (1999).

**Lemma 2.1.** *Let $(X, d)$ be a quasi-pseudo metric space and let $x \in X$ and $A \in W^*(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ iff $p_\alpha(x, A) = 0$, for each $\alpha \in [0, 1]$.*

**Lemma 2.2.** *Let $(X, d)$ be a quasi-pseudo metric space and let $A \in W^*(X)$. Then $p_\alpha(x, A) \leq d(x, y)+p_\alpha(y, A)$ for any $x, y \in X$ and $\alpha \in [0, 1]$.*

**Lemma 2.3.** *Let $(X, d)$ be a quasi-pseudo metric space and let $\{x_0\} \subset A$. Then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $A, B \in W^*(X)$ and $\alpha \in [0, 1]$.*

Above Lemmas were proved by Heilpern (1981) for the family $W(X)$ in a metric linear space.

**Proposition 1.** *Let $(X, d)$ be a complete quasi-pseudo metric space and $G : X \to W^*(X)$ be a fuzzy mapping and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

**Proposition 2.** *Let $(X, d)$ be a quasi-pseudo metric space and $A, B \in CP(X)$ and $a \in A$, then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Now we shall use the notations as in Isufati & Hoxha (2010).
In the following, the letter $\Gamma$ denotes the set of positive integers.
If $A$ is a subset of a topological space $(X, \tau)$, we will denote by $cl_\tau A$ the closure of $A$ in $(X, \tau)$.
A quasi-metric on a non-empty set $X$ is a non-negative real-valued function d on $X \times X$ such that for all $x, y, z \in X$ :

(i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$,
(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A pair (X,d) is called a quasi-metric space, if d is a quasi-metric on X.
Each quasi-metric $d$ on $X$ induces a $T_0$ topology $\mathcal{T}(d)$ on $X$, which has a base, the family of all $d-$ balls $\{B_d(x, r) : x \in X, r > 0\}$, where, $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

If $d$ is a quasi-metric on $X$, then the function $d^{-1}$ defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ is also quasi metric on X. By $d \wedge d^{-1}$ we denote min $\{d, d^{-1}\}$ and also we denote $d^s$ the metric on $X$ by $d^s(x, y) = max\{d(x, y), d(y, x)\}$ for all $x, y \in X$.

A sequence $(x_n)_{n\in\Gamma}$ in a quasi metric space $(X, d)$ is called left $k-$ Cauchy Reilly *et al.* (1982) if for each $\varepsilon > 0$ there is a $n_\varepsilon \in \Gamma$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in \Gamma$ with $m \geq n \geq n_\varepsilon$. Let $(X, d)$ be a quasi-metric space and let $\mathcal{K}_0^s(X)$ be the collection of all non-empty compact subset of the metric space $(X, d^s)$. Then the Hausdroff distance $H_d$ on $\mathcal{K}_0^s(X)$ is defined by

$$H_d(A, B) = \max\{\ \sup d(a, B) : a \in A,\ \ \sup d(A, b) : b \in B\}\ \text{whenever } A, B \in \mathcal{K}_0^s(X).$$

A fuzzy set on X is an element of $I^X$ where $I = [0, 1]$. If A is a fuzzy set in $X$, then the number $A(x)$ is called the grade of membership of $x$ in A. The $\alpha$- level set of $A$, denoted by $A_\alpha$, and defined by $A_\alpha = \{x \in X : A(x) \geq \alpha\}$ for each $\alpha \in (0, 1]$ and $A_0 = \overline{\{x : A(x) > 0\}}$ where the closure is taken in $(X, d^s)$.

**Definition 2.7.** Gregori & Romaguera (2000) Let $(X, d)$ be a quasi-metric space. A fuzzy set A in quasi-metric space $(X, d)$ will be called an approximate quantity. The family $\mathcal{A}(X)$ of all fuzzy sets on $(X, d)$ is defined by $\mathcal{A}(X) = \{A \in I^X : A_\alpha$ is $d^s$ -compact for each $\alpha \in [0, 1]$ and $supA(x) = 1 : x \in X\}$.

**Definition 2.8.** Gregori & Romaguera (2000) Let $A, B \in \mathcal{A}(X)$ then $A$ is said to be more accurate than B, denoted by $A \subset B$ if and only if $A(x) \leq B(x)$ for all $x \in X$.

**Definition 2.9.** Gregori & Romaguera (2000) Let $(X, d)$ be a quasi-metric space and let $A, B \in \mathcal{A}(X)$ and $\alpha$ in $[0, 1]$. Then we define $p_\alpha(A, B) = Inf\{d(x, y) : x \in A_\alpha, y \in B_\alpha\} = d(A_\alpha, B_\alpha)$, $D_\alpha(A, B) = H_d(A_\alpha, B_\alpha)$, $p(A, B) = sup\{P_\alpha(A, B) : \alpha \in [0, 1]\}$, $D(A, B) = sup\{D_\alpha(A, B) : \alpha \in [0, 1]\}$, for $x \in X$, we write $p_\alpha(x, A)$ instead of $p_\alpha(\{x\}, A)$. We denote that $p_\alpha$ is a non-decreasing function of $\alpha$ and $D$ is metric on $\mathcal{A}(x)$.

**Definition 2.10.** Gregori & Romaguera (2000) A fuzzy mapping on a quasi-metric space $(X, d)$ is a function $F$ defined on $X$, which satisfies the following two conditions

(i) $F(x) \in \mathcal{A}(X)$ for all $x \in X$,
(ii) If $a, z \in X$ such that $(F(z))(a) = 1$ and $p(a, F(a)) = 0$ then $(F(a))(a) = 1$.

We need the following lemmas for our main result which was given by Gregori & Romaguera (2000).

**Lemma 2.4.** *Gregori & Romaguera (2000) Let $(X, d)$ be a quasi-metric space and let $A, B \in \mathcal{A}(X)$ and $x \in A_1$. There exist $y \in B_1$ such that $d(x, y) \leq D_1(A, B)$.*

**Lemma 2.5.** *Gregori & Romaguera (2000) Let $(X, d)$ be a quasi-metric space and let $A \in \mathcal{A}(X)$ and $y \in A$. Then $p(x, A) \leq d(x, y)$ for each $x \in X$.*

**Lemma 2.6.** *Gregori & Romaguera (2000) Let $x \in X$, $A \in \mathcal{A}(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$, then $\{x\} \subset A$ if and only if $p_\alpha(x, A) = 0$ for each $\alpha \in [0, 1]$.*

**Lemma 2.7.** *Gregori & Romaguera (2000) Let $(X, d)$ be a quasi-metric space and $A \in \mathcal{A}(X)$. Then $p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$.*

**Lemma 2.8.** *Gregori & Romaguera (2000) Let $(X, d)$ be a quasi-metric space and let $A \in \mathcal{A}(X)$ and $x \in A$. Then $p_\alpha(x, B) \leq D_\alpha(A, B)$ for each $B \in \mathcal{A}(X)$ and each $\alpha \in [0, 1]$.*

**Lemma 2.9.** *Gregori & Romaguera (2000) Let A and B be non-empty compact subset of a quasi-metric space $(X, d)$ if $a \in A$, then there exists $b \in B$, such that $d(a, b) \leq H(A, B)$.*

**Lemma 2.10.** *Gregori & Romaguera (2000) Let $(X, d)$ be a complete quasi metric space and let F be a fuzzy mapping from X into $\mathcal{A}(X)$ and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

We consider the set of function $\Psi = \{\psi : R^{+5} \to R^+\}$ satisfying the following conditions

(i) $\psi$ strictly increasing, continuous function in each coordinate and
(ii) for all $g \in R^+$ such that $\psi(g, g, g, 0, 2g) < g$, $\psi(g, g, g, 2g, 0) < g$, $\psi(0, 0, g, g, 0) < g$ and $\psi(g, 0, 0, g, g) < g$.

**Example 2.11.** *Let $\psi : R^{+5} \to R^{+5}$ denote by $\psi(g_1, g_2, g_3, g_4, g_5) = k\ max(g_1, g_2, g_3, \frac{g_4}{2}, \frac{g_5}{2})$ for $k \in (0, 1)$ then $\psi$ satisfies above conditions (i) and (ii).*

## 3. Main Result

Following Constantin (1991) we consider the set $\mathcal{G}$ of all continuous functions $g : [0, \infty)^5 \to [0, \infty)$ with the following properties:

(1) $g$ is non-decreasing in the $2^{nd}$, $3^{th}$, $4^{th}$ and $5^{th}$ variable,
(2) if $u, v \in [0, \infty)$ are such that $u \leq g(v, v, u, u + v, 0)$ or $u \leq g(v, u, v, 0, u + v)$ then $u \leq qv$ where $0 < q < 1$ is a given constant,
(3) if $u \in [0, \infty)$ is such that $u \leq g(u, 0, 0, u, u)$ then $u = 0$.

Now we are ready to prove our main theorems.

**Theorem 3.1.** *Let X be a complete quasi-pseudo metric space and let $G_1$ and $G_2$ be fuzzy mappings from X into $W^*(X)$. If there is a $g \in \mathcal{G}$ such that for $x, y \in X$*

$$D(G_1(x), G_2(y)) \leq g(d(x, y), p(x, G_1(x)), p(y, G_2(y)), p(x, G_2(y)), p(y, G_1(x)))$$

*then there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.*

*Proof.* Let $x_0 \in X$. Then by Proposition 2.1 there exists an $x_1 \in X$ such that $\{x_1\} \subset G_1(X_0)$. From Proposition 2.1 there exists $x_2 \in (G_2(x_1))_1$. Since $(G_1(x_0))_1, (G_2(x_1))_1 \in CP(X)$ then by Proposition 2.2 we obtain,

$$\begin{aligned} d(x_1, x_2) \leq\ & D_1(G_1(x_0), G_2(x_1)) \leq D(G_1(x_0), G_2(x_1)) \leq g(d(x_0, x_1), p(x_0, G_1(x_0)), p(x_1, G_2(x_1)), \\ & p(x_0, G_2(x_1)), p(x_1, G_1(x_0))) \leq g(d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_1) + d(x_1, x_2), 0) \end{aligned}$$

therefore, $d(x_1, x_2) \leq qd(x_0, x_1)$. Following similar process we obtain, $d(x_2, x_3) \leq qd(x_1, x_2)$. By induction, we produce a sequence $(x_n)$ of points of $X$ such that for each $k \geq 0$ $\{x_{2k+1}\} \subset G_1(x_{2k})$, and $\{x_{2k+2}\} \subset G_2(x_{2k+1})$, $d(x_n, x_{n+1}) \leq qd(x_{n-1}, x_n) \leq \cdots \leq q^n d(x_0, x_1)$. Furthermore, for $m > n$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \{q^n + q^{n+1} + \cdots + q^{m-1}\} d(x_0, x_1) \leq \frac{q^n}{(1 - q)} d(x_0, x_1). \end{aligned}$$

It follows that $(x_n)$ is a Cauchy sequence in $X$. Since $X$ is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Next, we show that $\{z\} \subset G_i(z), i = 1, 2$. Now by Lemma 2.2 $p_0(z, G_2(z)) \leq d(z, x_{2n+1}) + p_0(x_{2n+1}, G_2(z))$. Then by Lemma 2.3,

$$\begin{aligned} p(z, G_2(z)) &\leq d(z, x_{2n+1}) + D(G_1(x_{2n}), G_2(z)) \leq d(z, x_{2n+1}) + f(d(x_{2n}, z), p(x_{2n}, G_1(x_{2n})), \\ &\quad p(z, G_2(z)), p(x_{2n}, G_2(z)), p(z, G_1(x_{2n}))) \\ &\leq d(z, x_{2n+1}) + g(d(x_{2n}, z), d(x_{2n}, x_{2n+1}), p(z, G_2(z)), p(x_{2n}, G_2(z)), d(z, x_{2n+1})). \end{aligned}$$

As $n \to \infty$, we obtain from above inequality that $p(z, G_2(z)) \leq g(0, 0, p(z, G_2(z)), p(z, G_2(z)), 0)$, so by properties of $g$ we have $p(z, G_2(z)) = 0$. by (2). So by Lemma 2.1, we get $\{z\} \subset G_2(z)$. Similarly, it can be shown that $\{z\} \subset G_1(z)$. □

As corollaries of Theorem 3.1, we have the following:

**Corollary 3.2** (Park & Jeong (1997); Theorem 3.1 ). *Let $X$ be a complete quasi-pseudo metric space and let $G_1$ and $G_2$ be fuzzy mappings from $X$ into $W^*(X)$. If there exists a constant $\alpha, 0 \leq \alpha < 1$, such that for each $x, y \in X$, $D(G_1(x), G_2(y)) \leq \alpha \cdot \max\{d(x, y), p(x, G_1(x)), p(y, G_2(y)), \frac{[p(x,G_2(y))+p(y,G_1(x))]}{2}\}$ then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

*Proof.* We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = \alpha{\cdot}\max\{x_1, x_2, x_3, \frac{(x_4+x_5)}{2}\}$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain Corollary 3.1. □

**Corollary 3.3** (Park & Jeong (1997); Theorem 3.2). *Let $X$ be a complete quasi-pseudo metric space and let $G_1$ and $G_2$ be fuzzy mappings from $X$ into $W^*(X)$. satisfying $D(G_1(x), G_2(y)) \leq k[p(x, G_1(x)) \cdot p(y, G_2(y))]^{\frac{1}{2}}$, for all $x, y \in X$ and $0 < k < 1$. Then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

*Proof.* We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = k[x_2 \cdot x_3]^{\frac{1}{2}}$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain Corollary 3.2. □

**Corollary 3.4** (Park & Jeong (1997); Theorem 3.4). *Let $X$ be a complete quasi-pseudo metric space and let $G_1$ and $G_2$ be fuzzy mappings from $X$ into $W^*(X)$, such that*

$$D(G_1(x), G_2(y)) \leq \alpha \cdot \frac{p(y, G_1(y))[1 + p(x, G_2(x))]}{1 + d(x, y)} + \beta d(x, y)$$

*for all $x \neq y, \alpha, \beta > 0$ and $\alpha + \beta < 1$. Then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

*Proof.* We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = \alpha \cdot \frac{x_3(1+x_2)}{(1+x_1)} + \beta x_1$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain Corollary 3.3. □

**Corollary 3.5** (Arora & V. (2000); Theorem 3.2). *Let $X$ be a complete quasi-pseudo metric space and let $G_1$ and $G_2$ be fuzzy mappings from $X$ into $W^*(X)$. If there exists a constant $r, 0 \leq r < 1$, such that for each $x, y \in X$, $D(G_1(x), G_2(y)) \leq r{\cdot}\max\{d(x, y), p(x, G_1(x)), p(y, G_2(y)), p(x, G_2(y)), p(y, G_1(x))\}$ then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

*Proof.* We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = r{\cdot}\max\{x_1, x_2, x_3, x_4, x_5\}$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain corollary 3.4. □

The following Corollary is a fuzzy version of the fixed point theorem for multi-valued mappings of Iseki (1995).

**Corollary 3.6.** *Let $X$ be a complete quasi-pseudo metric space and let $G_1$ and $G_2$ be fuzzy mappings from $X$ into $W^*(X)$. If for each $x, y \in X$ , such that $D(G_1(x), G_2(y)) \leq \alpha[p(x, G_1(x)) + p(y, G_2(y))] + \beta[p(x, G_2(y)) + p(y, G_1(x))] + \gamma d(x, y)$ where $\alpha, \beta, \gamma$ are non-negative and $2\alpha + 2\beta + \gamma < 1$. Then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

*Proof.* We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = \alpha[x_2 + x_3] + \beta[x_4 + x_5] + \gamma x_1$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain corollary 3.5. □

The following Corollary is a fuzzy version of the fixed point theorem for multi-valued mappings of Singh & Whitfield (1982).

**Corollary 3.7.** *Let $X$ be a complete quasi-pseudo metric space and let $G_1$ and $G_2$ be fuzzy mappings from $X$ into $W^*(X)$. If there exists a constant $\alpha, 0 \leq \alpha < 1$, such that for each $x, y \in X$, $D(G_1(x), G_2(y)) \leq \alpha \cdot \max\{d(x, y), \frac{[p(x,G_1(x))+p(y,G_2(y))]}{2}, \frac{[p(x,G_2(y))+p(y,G_1(x))]}{2}\}$ then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

*Proof.* We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = \alpha \cdot \max\{x_1, \frac{[x_2+x_3]}{2}, \frac{[x_4+x_5]}{2}\}$ Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain Corollary 3.6. □

*Remark.* If there exists a function $g \in \mathcal{G}$ such that for all $x, y \in X$

$$\delta(G_1(x), G_2(y)) \leq g(d(x, y), p(x, G_1(x)), p(y, G_2(y)), p(x, G_2(y)), p(y, G_1(x))),$$

then the conclusion of Theorem 3.1 remains valid. This result is considered as special case of Theorem 3.1 because ( see, Hicks (1997); page 414) $D(G_1(x), G_2(y)) \leq \delta(G_1(x), G_2(y))$. Moreover, this result generalize Theorem 3.3 of Park & Jeong (1997).

The following theorem extends Theorem 3.1 to a sequence of fuzzy mappings:

**Theorem 3.8.** *Let $X$ be a complete quasi-pseudo metric space and let $\{G_n : n \in Z^+\}$ be fuzzy mappings from $X$ into $W^*(X)$. If there is a $g \in \mathcal{G}$ such that for all $x, y \in X$*

$$D(G_0(x), G_n(y)) \leq g(d(x, y), p(x, G_0(x)), p(y, G_n(y)), p(x, G_n(y)), p(y, G_0(x)))$$

*then there exists a common fixed point of the family $\{G_n : n \in Z^+\}$.*

*Proof.* From Theorem 3.1, we get a common fixed point $x_i, i = 1, 2, \ldots,$ for each pair $(G_0, G_i), i = 1, 2, \ldots$. Applying Lemma 2.2, one can have that $p_\alpha(x_i, G_0 x_i) = P_\alpha(x_i, G_i(x_i)) = 0$, for all $i = 1, 2, \ldots$. Thus one can deduce form Lemma 2.3, for $i \neq j$, that

$$\begin{aligned} d(x_i, x_j) &= p_\alpha(x_i, G_j(x_j)) \leq D_\alpha(G_i(x_i), G_j(x_j)) \leq D(G_i(x_i), G_j(x_j)) \\ &\leq g(d(x_i, x_j), p(x_i, G_i(x_i)), p(x_j, G_j(x_j)), p(x_i, G_j(x_j)), p(x_j, G_i(x_i))) \\ &= g(d(x_i, x_j), 0, 0, d(x_i, x_j), d(x_i, x_j)). \end{aligned}$$

Therefore $d(x_i, x_j) = 0$, i.e., $x_i = x_j$ for all $i, j \in \mathbb{N}$. □

**Corollary 3.9.** *(Arora & V. (2000);Theorem (3.4)) Let $X$ be a complete quasi-pseudo metric space and let $\{G_n : n \in \mathbb{N}^+\}$ be fuzzy mappings from $X$ into $W^*(X)$. If for each $x, y \in X$, and $r \in (0, \frac{1}{2}), n = 1, 2, , ...,$ such that $D(G_0(x), G_i(y)) \leq r\max\{d(x, y), p(x, G_o(x)), p(y, G_i(y)), p(x, G_i(y)), p(y, G_0(x))\}$. Then there exists a common fixed point of the family $\{G_n : n \in \mathbb{N}^+\}$.*

**Theorem 3.10.** *Let $(X, d)$ be a complete quasi-metric space, let $T_1, T_2 \colon X \to \mathcal{A}(X)$ be fuzzy $\psi$ contractive mappings satisfies $D(T_1x, T_2y) \leq \psi\{(d(x,y), p(x, T_1x), p(y, T_2y), p(x, T_2y), p(y, T_1x)\}$ then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.*

*Proof.* Let $x_0 \in X$ then by Lemma 2.10 there exists an element $x_1 \in X$ such that $\{x_1\} \subset T_1(x_0)$ for $x_1 \in T_2(x_1))_1$ is non-empty compact subset of X. Since $(T_1(x_0))_1, (T_2(x_1))_1 \in CP(X)$ and $x_1 \in (T_1(x_0)_1)$, then by lemma 2.9 asserts that there exists $x_2 \in (T_2(x_1))_1$ such that $d(x_1, x_2) \leq D_1(T_1(x_0), T_2(x_1))$ so, from Lemma 2.6 and properties of $\psi$ function, we have

$$\begin{aligned} d(x_1, x_2) &\leq D_1(T_1(x_0), T_2(x_1)) \leq D(T_1(x_0), T_2(x_1)) \\ &\leq \psi(d(x_0, x_1), p(x_0, T_1x_0), p(x_1, T_2x_1), p(x_0, T_2x_1), p(x_1, T_1x_0)) \\ &\leq \psi(d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_1) + d(x_1, x_2), 0) \end{aligned}$$

and

$$\begin{aligned} d(x_2, x_1) &\leq D_1(T_2(x_1), T_1(x_0)) \leq D(T_2(x_1), T_1(x_0)) \\ &\leq \psi(d(x_1, x_0), p(x_1, T_2x_1), p(x_0, T_1x_0), p(x_1, T_1x_0), p(x_0, T_2x_1)) \\ &\leq \psi(d(x_1, x_0), d(x_1, x_2), d(x_0, x_1), 0, d(x_0, x_1) + d(x_1, x_2)) \end{aligned}$$

by induction, we have a sequence $(x_n)$ of points such that for all $n \in R^+ \cup \{0\}$ we have $\{x_{2n+1}\} \subset T_1(x_{2n})$ and $\{x_{2n+2}\} \subset T_2(x_{2n+1})$ then

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0) \tag{3.1}$$

and

$$d(x_{n+1}, x_n) \leq \psi(d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), 0, d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \tag{3.2}$$

so, by the properties of the $\psi$ function we have that for each $n \in R^+$ $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ and $d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})$. The sequence $(b_m)_{m \in R^+}$, such that $b_m = d(x_m, x_{m+1})$ is a non-increasing sequence and bounded below. Thus it must converges to some $b \geq 0$. By the inequality 3.1 and 3.2 we have

$$b \leq b_m \leq \psi(b_{m-1}, b_{m-1}, b_m, b_{m-1} + b_m, 0) < b \tag{3.3}$$

passing to the limit, as $m \longrightarrow \infty$, and by properties of the $\psi$ function we have $b \leq b \leq \psi(b, b, b, 2b, 0) < b$ which is contradiction. Hence $b = 0$. Thus, the sequence $(x_n)_{n \in R^+}$ must be a Cauchy sequence.

Similarly, the sequence $(c_n)_{n \in R^+}$ such that $c_n = d(x_{n+1}, x_n)$ is a non-increasing sequence and bounded below. Thus, it must converges to some $c \geq 0$.
By the inequality 3.1 and 3.2 we have

$$c \leq c_n \leq \psi(c_{n-1}, c_{n-1}, c_n, c_{n-1} + c_n, 0) < b \tag{3.4}$$

passing to the limit, as $n \longrightarrow \infty$, and by properties of the $\psi$ function we have $c \leq c \leq \psi(c, c, c, 2c, 0) < c$ which is possible if and only if $c = 0$.

We next claim that to prove that for each $\varepsilon > 0$, there exists $n_0(\varepsilon) \in R^+$, such that for all $m > n > n_0(\varepsilon)$

$$d(x_m, x_n) < \varepsilon. \tag{3.5}$$

Suppose that 3.5 is false then, there exists some $\varepsilon > 0$ such that for all $k \in R^+$, there exists the smallest number $m_k$, such that $m_k, n_k \in R^+$ with $m_k > n_k \leq k$ satisfying $d(x_{m_k}, x_{n_k}) \geq \varepsilon$ so,

$$\begin{aligned} \varepsilon &\leq d(x_{m_k}, x_{n_k}) \leq D(Tx_{m_k-1}, Tx_{n_k-1}) \\ &\leq \psi(d(x_{m_k-1}, x_{n_k-1}), p(x_{m_k-1}, Tx_{m_k-1}), p(x_{n_k-1}, Tx_{n_k-1}), p(x_{m_k-1}, Tx_{n_k-1}), p(x_{n_k-1}, Tx_{m_k-1}) \\ &\leq \psi(d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}), d(x_{m_k-1}, x_{n_k}), d(x_{n_k-1}, x_{m_k}) \\ &\leq \psi(c_{m_k-1} + d(x_{m_k}, x_{n_k}) + c_{n_k-1}, c_{m_k-1}, c_{n_k-1}, c_{m_k-1} + d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{n_k}) + c_{n_k-1}). \end{aligned}$$

Letting $k \to \infty$ we have $\varepsilon \leq \psi(\varepsilon, 0, 0, \varepsilon, \varepsilon) < \varepsilon$ which is a contradiction. It follows from 3.5 that $(x_n)$ is a Cauchy sequence since $(X, d)$ is a complete quasi-metric space,then there exists $z \in X$ such that $lim_{n\to\infty} x_n = z$.

Next we show that $\{z\} \subset T_2(z)$.

By Lemmas 2.7 and 2.8 we get $p_\alpha(z, T_2z) \leq d(z, x_{2n+1}) + p_\alpha(x_{2n+1}, T_2z) \leq d(z, x_{2n+1}) + D_\alpha(T_1x_n, T_2z)$ for each $\alpha \in [0, 1]$. Taking supremum on $\alpha$ in the last inequality, we obtain from the properties of $\psi$ that

$$\begin{aligned} p_\alpha(z, T_2z) &\leq d(z, x_{2n+1}) + p_\alpha(x_{2n+1}, T_2z) \leq d(z, x_{2n+1}) + D_\alpha(T_1x_{2n}, T_2z) \\ &\leq d(z, x_{2n+1}) + \psi(d(x_{2n}, z), p(x_{2n}, T_1x_{2n}), p(z, T_2z), p(x_{2n}, T_2z), d(z, x_{2n+1})) \\ &\leq d(z, x_{2n+1}) + \psi(d(x_{2n}, z), d(x_{2n}, x_{2n+1}), p(z, T_2z), p(x_{2n}, T_2z), d(z, x_{2n+1})). \end{aligned}$$

As $n \to \infty$, we have $p(z, T_2z) \leq \psi(0, 0, p(z, T_2z), p(z, T_2z), 0) < p(z, T_2z)$. It yields that $p(z, T_2z) = 0$. So, we get from Lemma 2.10 that $\{z\} \subset T_2z$. Similarly we prove that $\{z\} \subset T_1z$. □

**Corollary 3.11.** *Let $(X, d)$ be a complete quasi metric space and let $T : X \to \mathcal{A}(X)$ be a fuzzy $\psi$ contraction mapping then there exists $z \in X$ such that $\{z\} \subset T(z)$.*

*Proof.* If put $T_1 = T_2 = T$ in theorem 3.3 we get the conclusion of corollary 3.8. □

**Corollary 3.12.** *Let $(X, d)$ be a complete quasi metric space and let $T : X \to \mathcal{A}(X)$ be a fuzzy $\psi$ contraction mapping, such that for all $x, y \in X$ $D(T_1x, T_2y) \leq \psi(d(x, y), p(x, T_1x), p(y, T_2y), \frac{p(x,T_2y)}{2}, \frac{p(y,T_1x)}{2})$ then there exists $z \in X$ such that $\{z\} \subset T_1z$ and $\{z\} \subset T_2z$.*

*Proof.* We consider the function $\psi : R^{+5} \to R^{+5}$ denoted by $\psi(t_1, t_2, t_3, t_4, t_5) = k \max\{t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}\}$ for $k \in (o, 1)$. Since $\psi \in \Psi$ we can apply theorem 3.3 and obtain Corollary 3.9. □

*Remark.* As examples of the main results we can taking theorems in which the contractions conditions are compatible with the condition $(i)$ and $(ii)$.

*Remark.* If there is a $\psi \in \Psi$ such that for all $x, y \in X$

$$\delta(T_1x, T_2y) \leq \psi(d(x, y), p(x, T_1x), p(y, T_2y), p(x, T_2y), p(y, T_1x))$$

then the conclusion of Theorem 3.3 remains valid. This result is considered as a special case of Theorem 3.3 because $D_1(T_1x, T_2y) \leq \delta(T_1x, T_2y)$ for all $x, y \in X$. The following theorem generalizes Theorem 3.3 to a sequence of fuzzy contractive mappings.

**Theorem 3.13.** *Let $(T_n : n \in (0, \infty) \cup \{0\})$ be a sequence of fuzzy mappings from a complete quasi metric space $X$ into $\mathcal{A}(X)$ . If there is a $\psi \in \Psi$ such that for all $x, y \in X$*

$$D(T_0x, T_ny) \leq \psi(d(x, y), p(x, T_0x), p(y, T_ny), p(x, T_ny), p(y, T_0x))$$

*for all $n \in (0, \infty) \cup \{0\}$, then there exists a common fixed point of the family $(T_n : n \in (0, \infty) \cup \{0\})$.*

*Proof.* Putting $T_1 = T_0$ and $T_2 = T_n$ for all $n \in N$ in Theorem 3.3 then there exists a common fixed point of the family $(T_n : n \in (0, \infty) \cup \{0\})$. □

## 4. Conclusion and future work

Fuzzy sets and mappings play an important role in the fuzzification of systems. In particular, in the recent years the fixed point theory for fuzzy mappings has been developed largely. We generalize, extend and unify several known results of metric spaces, into a weaker and generalize setting of quasi-pseudo metric space and quasi metric space for fuzzy mappings. We use a more generalize contractive condition than the existing ones, also we prove our results in quasi-pseudo metric space, quasi metric space and so as to obtain better results under weaker conditions. We conclude this paper with an open problem: Is it possible to prove the results of this paper in the setting of $b$-metric and partial metric spaces?

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